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NECESSARY AND SUFFICIENT CONDITIONS FOR REGULARITY AND STABILITY OF INTERVAL MATRICES

By

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Abstract

Verifiable necessary and sufficient conditions for regularity of an interval matrix are given. Based on these conditions an algorithm for checking the regularity of interval matrices is developed. It is shown that this algorithm can be modified to determine the stability of a class of interval matrices.

Key words: Interval matrices, Interval Mathematics, Regularity of interval matrices, Stability of interval matrices.

1.1 Introduction

An interval matrix is a matrix whose entries are intervals,e.g., a real matrix whose entries are not known exactly but rather to lie within certain closed intervals or a matrix whose entries are continuous functions of a parameter $x \in [a, b]$. For $n \times n$ real matrices $B = [b_{ij}]$ and $C = [c_{ij}]$ with $b_{ij} \leq c_{ij}, 1 \leq i, j \leq n$, an interval matrix denoted by $A^I = [B, C]$, is defined to be the set of real matrices

$$A^I = \{A = [a_{ij}] : b_{ij} \leq a_{ij} \leq c_{ij}\ , i, j = 1, 2, ..., n\}. \tag{1.1}$$

An interval matrix A^I is said to be regular if each $A \in A^I$ is nonsingular. An $n \times n$ matrix A is said to be stable if $Re[\lambda] < 0$ for all eigenvalues λ of A, and an interval matrix A^I is said to be stable if each $A \in A^I$ is stable. A^I is said to be symmetric if both B and C are symmetric. Note that a symmetric interval matrix may contain nonsymmetric matrices.

A system of linear interval equations is a system of linear equations in which the coefficients of variables and right-hand-side constants are not determined exactly but are known only to lie within certain closed intervals (obtained for example as a result of roundoff, truncation or data errors). Thus a system of n linear interval equations is a system of n linear equations in n variables with coefficients of variables and right-hand-side constants being intervals. Such a system of linear interval equations represents a family of linear systems of equations which can be obtained from it by fixing coefficients and right-hand values in the prescribed intervals. The matrix of coefficients of a system of linear interval equations will be an interval matrix A^I and such a system has a unique solution under the assumption that A^I is regular. Thus to check the solvability of such a system one has to determine whether the coefficients interval matrix A^I is regular.

Stability of interval matrices is also of great current interest. This has been recently studied in robust control theory due to its close relationship to stability of a linear time-invariant system

$$\dot{X}\ (t) = AX(t) \tag{1.2}$$

under data perturbation. Most of the existing results for stability of interval matrices constitute sufficient conditions and some of them provide also necessary conditions. The few results which give necessary and sufficient stability conditions involve criteria which are not practical to check or involve too many computations e.g., [2]. In [2], Wang, Michel, and Liu established necessary and sufficient conditions using Lyapunov Theorem and developed an algorithm to verify the stability of interval matrices using the Lyapunov equation. For an $n \times n$ matrix, to solve the Lyapunov equation, one has to solve $\frac{n(n-1)}{2} \times \frac{n(n-1)}{2}$ systems of equations.

In [3], it was shown that regularity and stability of interval matrices are closely related. Among other things, some necessary and sufficient conditions for the stability of symmetric interval matrices were developed in [3]. However, for checking regularity of an interval matrix only sufficient conditions are found in literature. Some results from [3] are used in the proofs of some theorems of this paper.

This paper is organized as follows. In Section 2, some notation and results are presented. In section 3, necessary and sufficient conditions for the regularity of interval matrix are given. Using these conditions some necessary and sufficient conditions for the stability of symmetric interval matrices are driven. These conditions are shown to be only sufficient for the stability of a general interval matrix. In Section 4, based on the necessary and sufficient conditions obtained in Section 3, we develop some algorithms for checking regularity and stability of interval matrices, demonstrate their effectiveness by applying them to some numerical examples and give some concluding remarks. In verifying the regularity and stability of an interval matrix, systems of $n \times n$ equations, compared to $\frac{n(n-1)}{2} \times \frac{n(n-1)}{2}$ as in the case of the algorithm developed in [2], need to be solved.

1.2 Auxiliary Results

For a bounded linear operator T on a normed linear space X, $\rho(T)$ denotes the set of all numbers λ in the complex plane for which $R(\lambda; T) \equiv (\lambda I - T)^{-1}$ exist as a bounded operator. $\rho(T)$ is called the resolvent of T. The complement of $\rho(T)$ denoted by $\sigma(T)$, is called the spectrum of T. In the case when $X = R^n$, $\rho(T)$ consists of all eigenvalues of T.

Let

$$[B, C] = \{ A = [a_{ij}] \, : \, b_{ij} \leq a_{ij} \leq c_{ij} \, , 1 \leq i, j \leq n \}$$

be an interval matrix. If $[B_o, C_o]$ is another interval matrix such that $[B_o, C_o] \subseteq [B, C]$, then $[B_o, C_o]$ is called a subinterval matrix of $[B, C]$. Let $A^m = \frac{1}{2}(B + C)$ and $\Delta = \frac{1}{2}(C - B)$, then $[B, C]$ can also be written as $[A^m - \Delta, \; A^m + \Delta]$.

Now we state some results from [5] which are used in the proofs of the main results of Section 3. The proofs of the results stated in this section can be found in [5].

Lemma 1 *Let $\ell\,(X)$ be the set of bounded linear operators on a normed linear space X. Then the set G of elements in $\ell\,(X)$ which have inverses in $\ell\,(X)$ is an open set with respect to the uniform topology on $\ell\,(X)$. Furthermore for $AA \in G$ the sphere*

$$\{B : |A - B| < \left|A^{-1}\right|^{-1}\}, \tag{1.3}$$

is contained in G and the inverse of an element B of this sphere is given by

$$B^{-1} = A^{-1} \sum_{n=0}^{\infty} [(A - B)A^{-1}]^n. \tag{1.4}$$

Corollary 2 *Let $T,\ T_1$ be in $\ell\,(X)$, $\lambda \in \rho\,(T)$ and $|T - T_1| < |R\,(\lambda; T)|^{-1}$. Then $\lambda \in \rho\,(T_1)$ and*

$$R\,(\lambda; T_1) = R\,(\lambda; T) \sum_{n=0}^{\infty} [(T_1 - T)\, R\,(\lambda; T)]^n\,.$$

Remark 1 *For $n \times n$ real matrices A and A_1 it follows from the above corollary that if $\lambda \notin \sigma(A)$ and $\|A - A_1\| < \|(\lambda I - A)^{-1}\|^{-1}$ then $\lambda \notin \sigma(A_1)$ and hence if A is nonsingular and $\|A - A_1\| < \|A^{-1}\|^{-1}$ then A_1 is nonsingular.*

1.3 Main Results

In this section we establish necessary and sufficient conditions for the regularity of an interval matrix $A^I = [B, C]$. Based on these conditions we derive some necessary and sufficient conditions for the stability of a symmetric interval matrix and (only) sufficient conditions for the stability of a general interval matrix.

First we establish some sufficient conditions for regularity of A^I under the following assumptions:

$(H.1)$ $A^m = \frac{1}{2}(B + C)$ is nonsingular,

$(H.2)$ $\|C - B\|_\infty < \frac{2}{\|(A^m)^{-1}\|_\infty}$.

Lemma 3 *If the interval matrix $[B, C]$ satisfy assumptions $(H.1)$ and $(H.2)$, then $[B, C]$ is regular.*

Proof. To show that $[B, C]$ is regular, we need to show that each $A \in [B, C]$ is nonsingular. Let $A \in [B, C]$ be any matrix and $\Delta A = A - A^m$. Then $\Delta A = [\Delta a_{ij}]$ satisfies

$$|\Delta a_{ij}| = \left| a_{ij} - \frac{1}{2}(b_{ij} + c_{ij}) \right| \leq \frac{1}{2}(c_{ij} - b_{ij}), \quad 1 \leq i, j \leq n,$$

which in turn implies that

$$\|A - A^m\|_\infty = \|\Delta A\|_\infty \leq \frac{1}{2} \|C - B\|_\infty < \frac{1}{\|(A^m)^{-1}\|_\infty}. \tag{1.5}$$

It follows from (1.5) and Remark 1 of Section 2 that A is nonsingular. ■

In developing our necessary condition for regularity of an interval matrix A^I we will need the following lemma.

Lemma 4 *Suppose that $[B, C]$ is regular. Then there exists a constant $r > 0$ such that for any subinterval matrix $[B_o, C_o] \subseteq [B, C]$, $[B_o, C_o]$ satisfy assumptions $(H.1)$ and $(H.2)$ as long as $\|C_o - B_o\|_\infty < r$.*

Proof. By supposition $[B, C]$ is regular, therefore each $A \in [B, C]$ is nonsingular. Thus for each $A \in [B, C]$ there exist a non-singular matrix $D = D(A)$ such that

$$AD = I.$$

Since $[B, C]$ is a compact set in $R^{n\times n}$ and since every continuous function on a compact set assumes its minimum value, there exist a constant $r > 0$ such that

$$r \leq \frac{1}{\|D(A)\|_\infty} < \frac{2}{\|D(A)\|_\infty} \quad \text{for all } A \in [B, C].$$

Any $[B_o, C_o] \subseteq [B, C]$, with $\|C_o - B_o\|_\infty < r$ satisfies

$$\|C_o - B_o\|_\infty < \frac{2}{\|D\|_\infty}, \tag{1.6}$$

where $D = D(A_o^m)$, $A_o^m = \frac{1}{2}(B_o + C_o)$.

The regularity of $[B, C]$ together with (1.6) imply that $[B_o, C_o]$ satisfy assumptions $(H.1)$ and $(H.2)$. ■

Using Lemmas 3 and 4 we now prove the main results of this paper. Theorem 5 provides a necessary and sufficient condition for the regularity of an interval matrix. Theorem 6 gives necessary and sufficient conditions for the stability of a symmetric interval matrix. Theorem 7 gives a sufficient condition for the stability of any interval matrix.

Theorem 5 *An interval matrix $[B, C]$ is regular if and only if it contains at least one nonsingular matrix and there are finitely many subinterval matrices $[B_i, C_i] \subseteq [B, C]$, $1 \leq i \leq k$, such that*

$$[B, C] = \bigcup_{i=1}^{k} [B_i, C_i], \tag{1.7}$$

and for each $1 \leq i \leq k$, $[B_i, C_i]$ satisfies assumptions $(H.1)$ and $(H.2)$.

Proof. (Sufficiency) Assume that $[B_i, C_i]$ satisfies assumptions $(H.1)$ and $(H.2)$ for each $1 \leq i \leq k$. Then by Lemma 3, $[B_i, C_i]$ is regular for each $1 \leq i \leq k$ and (1.7) implies that $[B, C]$ is regular.

(Necessity) Given that $[B, C]$ is regular, then by Lemma 4, there exist a constant $r > 0$ such that any subinterval matrix $[B_o, C_o] \subseteq [B, C]$, satisfies assumptions $(H.1)$ and $(H.2)$ as long as $\|C_o - B_o\|_\infty < r$. Since $[B, C]$ is a hyperrectangle in R^{n^2}, we can subdivide it into a finite number of hyperrectangles $[B_i, C_i]$, $1 \leq i \leq k$ such that $\|C_i - B_i\|_\infty < r$ for each $1 \leq i \leq k$ (Notice that each $A \in R^{n \times n}$ satisfies $\|A\|_\infty \leq n \max_{1 \leq i \leq k} |a_{ij}|$). Therefore by Lemma 4, all the subinterval matrices $[B_i, C_i]$, $1 \leq i \leq k$ satisfy assumptions $(H.1)$ and $(H.2)$. ■

The proof of the following result uses Theorem 8 of [3].

Theorem 6 *A symmetric interval matrix $[B, C]$ is stable if and only if it contains at least one stable matrix and there are finitely many subinterval matrices $[B_i, C_i]$, $1 \leq i \leq k$ such that*

$$[B, C] = \bigcup_{i=1}^{k} [B_i, C_i]$$

and for each $1 \le i \le k$, $[B_i, C_i]$ *satisfies assumptions* $(H.1)$ *and* $(H.2)$.

Proof. (Sufficiency) Since every stable matrix is nonsingular, it follows that if $[B, C]$ contains a stable matrix and $[B_i, C_i]$ satisfies assumptions $(H.1)$ and $(H.2)$ for each $1 \le i \le k$. Then $[B, C]$ is regular. Hence by Theorem 8 of [3] it follows that $[B, C]$ is stable.

(Necessity) If $[B, C]$ is stable then it is regular and proof follows form Theorem 5.

Remark 2 *: Given any interval matrix* $A^I = [B, C]$ *we can construct a symmetric interval matrix* $A_s^I = [B', C']$ *, where* ·

$$B' = \frac{1}{2}(B + B^T) \quad \text{and} \quad C' = \frac{1}{2}(C + C^T) \tag{1.8}$$

Using a result given in [3] which states that the stability of the symmetric interval matrix A_s^I *implies the stability of* A^I, *the following result follows from Theorem 6.*

Theorem 7 *If* $A_s^I = [B', C']$ *contains at least one stable matrix and there are finitely many subinterval matrices* $[B_i', C_i']$, $1 \le i \le k$, *such that*

$$[B', C'] = \bigcup_{i=1}^{k} [B_i', C_i']$$

and each of $[B_i', C_i']$ *satisfy assumptions* $(H.1)$ *and* $(H.2)$ *then* A^I *is stable.* ■

1.4 Algorithms and Examples

In this section we develop three algorithms which are based on Theorem 5, 6 and 7 for testing regularity and stability of interval matrices. We demonstrate the effectiveness of our algorithms by applying them to some numerical examples.

The first algorithm is designed to determine the regularity of a general interval matrix $A^I = [B, C]$. In this algorithm we first determine the nonsingularity of the midpoint matrix $A^m = \frac{1}{2}(B + C)$. If A^m is singular, the algorithm terminates with the result that $A^I = [B, C]$ is not regular. If A^m is non-singular, then we find the inverse of A^m and verify wether assumptions $(H.1)$ and $(H.2)$ are satisfied. If these assumptions are satisfied, then the algorithm terminates with the result that A^I is regular. Otherwise, we divide the interval matrix $[B, C]$ into two subinterval matrices and repeat the above process for each subinterval matrix of $[B, C]$. The algorithm continues until each subinterval matrix of $[B, C]$ is determined to be regular or one of the subinterval matrices is determined to be not regular in this manner.

Using the same algorithm with some modifications we write a second algorithm to check the regularity of a given symmetric interval matrix A^I and stability of one point matrix in A^I to verify the stability A^I.

Our third algorithm constructs a symmetric interval matrix A^I_s from a given interval matrix A^I and then, using the second algorithm, checks the stability of A^I_s.

The first algorithm answers.

Problem 1 Given an interval matrix $A^I = [B, C]$ with $B = [b_{ij}]$ and $C = [c_{ij}]$, determine its regularity under the assumptions that B and C are nonsingular.

Algorithm 1

1. Initialization: $B_1 = [b^1_{ij}] = B,\ C_1 = [c^1_{ij}] = C$ and $K1 = \{1\}$.

2. Let $K = K1$.

3. For every $k \in K$, compute $A^m_k = \frac{1}{2}(B_k + C_k) = [a^k_{ij}]_{n \times n}$ and determine wether

A_k^m is singular. If for any k, A_k^m is singular, terminate the algorithm with the message that $[B, C]$ is not regular.

4. If for every k, A_k^m is non-singular, find the inverse V_k of A_k^m.

5. For every $k \in K$, compute $P_k = C_k - B_k = [c_{ij}^k]_{n \times n}$, $\alpha_k = \|P_k\|_\infty$ and $\beta_k = \frac{2}{\|V_k\|_\infty}$.

6. If for every $k \in K$, $\alpha_k < \beta_k$, the interval matrix is regular. Stop. Otherwise, determine $K_o = \{k \in K : \alpha_k \geq \beta_k\}$.

7. For every $k \in K_o$, find the maximal element p_{rs}^k of the matrix P_k and partition $[B_k, C_k]$ into two interval matrices $[D_k, E_k]$ and $[F_k, G_k]$ where $D_k = B_k$, $G_k = C_k$, $E_k = [e_{ij}^k]$, and $F_k = [f_{ij}^k]$ with

$$e_{ij}^k = \begin{cases} a_{ij}^k & if\ i = r \text{ and} j = s, \\ c_{ij}^k & \text{otherwise,} \end{cases}$$

and

$$f_{ij} = \begin{cases} a_{ij}^k & if\ \ i = r \text{ and } j = s, \\ b_{ij}^k & \text{otherwise.} \end{cases}$$

8. Relabel the set $\{[D_k, E_k], [F_k, G_k], k \in K_o\}$ using $\{[B_k, C_k], k \in K1\}$ where $K1 = \{1, 2, ..., N\}$, N is the number of subinterval matrices in step 7.

9. Go to step 2.

The second algorithm answers.

Problem 2 Given a symmetric interval matrix $A^I = [B, C]$ with $B = [b_{ij}]$ and $C = [c_{ij}]$, determine its stability under the assumptions that B and C are stable.

Algorithm 2

1. Check the stability of any point matrix in $A^I = [B, C]$, e.g., $A^m = \frac{1}{2}(B+C)$. If the real part of any eigenvalue of A^m is positive terminate the algorithm with the result that $[B, C]$ is not stable, Otherwise go to step 2.

2. Check the regularity of symmetric interval matrix $[B, C]$, using algorithm 1 with step 7 being replaced by

   Step 7′. For every $k \in K_o$, find the maximal element p^k_{rs} of the matrix P_k and partition $[B_k, C_k]$ into two interval matrices $[D_k, E_k]$ and $[F_k, G_k]$ where $D_k = B_k$, $G_k = C_k$, $E_k = [e^k_{ij}]$, and $F_k = [f^k_{ij}]$ with

$$e^k_{ij} = \begin{cases} a^k_{ij} & \text{if } i = r \text{ and } j = s \text{ or if } i = s \text{ and } j = r, \\ c^k_{ij} & \text{otherwise,} \end{cases}$$

   and

$$f_{ij} = \begin{cases} a^k_{ij} & \text{if } i = r \text{ and } j = s \text{ or if } i = s \text{ and } j = r, \\ b^k_{ij} & \text{otherwise.} \end{cases}$$

The third algorithm answers

Problem 3. Given an interval matrix $A^I = [B, C]$ with $B = [b_{ij}]$ and $C = [c_{ij}]$. Construct the corresponding symmetric interval matrix $A^I_s = [B', C']$ and check the stability of A^I.

Algorithm 3:

1. Compute $B' = \frac{1}{2}(B + B^T)$ and $C' = \frac{1}{2}(C + C^T)$.

2. Check the stability of $A^I_s = [B', C']$ using Algorithm 2.

3. If A^I_s is stable then A^I is also stable, else no conclusion.

Example 1: The interval matrix $A^I = [B, C]$, where B and C are given by

$$B = \begin{bmatrix} -3 & 4 & 4 & -1 \\ -4 & -4 & -4 & 1 \\ -5 & 2 & -5 & -1 \\ -1 & 0 & 1 & -4 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} -2 & 5 & 6 & 1.1 \\ -3 & -3 & -3 & 2 \\ -4 & 3 & -4 & 0 \\ 0.1 & 1 & 2 & -2.5 \end{bmatrix},$$

is determined to be regular using Algorithm 1, which determine the regularity of A^I by checking only 11 matrices A_k^m.

Example 2: The interval matrix $A^I = [B, C]$, where B and C are given by

$$B = \begin{bmatrix} 1 & 2 & 0 & 1 \\ -3 & 4 & -2 & 4 \\ 2 & 7 & 1.5 & 1 \\ 5 & 8 & 2 & 6 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 3 & 4 & 2 & 3 \\ -1 & 5 & 0 & 6 \\ 4 & 8 & 2.5 & 2 \\ 7 & 10 & 4 & 8 \end{bmatrix},$$

is determined to be not regular using Algorithm 1, which determine the nonregularity of A^I by checking 4098 matrices A_k^m and gives a singular matrix

$$A_o = \begin{bmatrix} 1.5 & 3.5 & 0.5 & 2.5 \\ -1.5 & 4.5 & -1.5 & 4.5 \\ 2.5 & 7.5 & 2 & 1.5 \\ 6.5 & 9.5 & 2.5 & 7 \end{bmatrix}.$$

Example 3: Using Algorithm 2 it is shown that the symmetric interval matrix $A^I = [B, C]$, where B and C are given by

$$B = \begin{bmatrix} -3 & 0 & -0.5 & -1 \\ 0 & -4 & -1 & 0.5 \\ -0.5 & -1 & -5 & 0 \\ -1 & 0.5 & 0 & -4 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} -2 & 1 & 1 & 0.28 \\ 1 & -3 & 0 & 1.5 \\ 1 & 0 & -4 & 1 \\ 0.28 & 1.5 & 1 & -2.5 \end{bmatrix},$$

is regular and its mid point matrix

$$A^m = \begin{bmatrix} -2.5 & 0.5 & 0.25 & -0.36 \\ 0.5 & -3.5 & -0.5 & 1 \\ 0.25 & -0.5 & -4.5 & 0.5 \\ -0.36 & 1 & 0.5 & -3.25 \end{bmatrix}$$

is stable with maximum eigenvalue -2.2773 .It follows that the symmetric interval matrix A^I is stable.

Example 4: The symmetric interval matrix $A^I = [B, C]$, where B and C are given by

$$B = \begin{bmatrix} -2.149 & -3.72 & -11.48 \\ -2.048 & -4.59 & -8.61 \\ -1.947 & -5.46 & -5.740 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3.851 & 2.280 & -5.48 \\ 3.952 & 1.41 & -2.61 \\ 4.053 & 0.540 & 0.260 \end{bmatrix}$$

is not stable. It is found to be not regular using Algorithm 2 and A^I contains a singular point matrix $\begin{bmatrix} 0.851 & -0.72 & -8.48 \\ 0.952 & -1.59 & -5.61 \\ 1.053 & -2.46 & -2.74 \end{bmatrix}$.

Example 5: The symmetric interval matrix $A_s^I = [B', C']$, where B' and C' given by

$$B' = \begin{bmatrix} -3 & 0 & -0.5 & -1 \\ 0 & -4 & -1 & 0.5 \\ -0.5 & -1 & -5 & 0 \\ -1 & 0.5 & 0 & -4 \end{bmatrix} \quad \text{and} \quad C' = \begin{bmatrix} -2 & 1 & 1 & 0.6 \\ 1 & -3 & 0 & 1.5 \\ 1 & 0 & -4 & 1 \\ 0.6 & 1.5 & 1 & -2.5 \end{bmatrix}$$

is constructed from the interval matrix A^I given in Example 1 and is found to be stable using Algorithm3. Thus the interval matrix A^I given in Example 1 is also stable.

1.5 Concluding Remarks

In this paper we established new necessary and sufficient conditions for the regularity of interval matrices. Using these conditions and results given by Rohan in [3], we established necessary and sufficient conditions for the stability of symmetric interval matrices and a sufficient condition for the stability of a general interval matrix. Using the above mentioned results, we developed three algorithms. The first algorithm determines the regularity of a general interval matrix, the second one determines the stability of a symmetric interval matrix and the third one checks the stability of a general interval matrix. We demonstrated the applicability of our algorithms by means of five numerical examples.

REFERENCES

[1] J.D. Cobb, "The minimal dimension of stable faces required to guarantee stability of matrixplototype: D-stability", Proc. 27th IEEE Conf. Decision and Contr., Austin, TX, Dec. 1988, pp.119-120.

[2] K. Wang, A. Michel and D. Liu, "Necessary and Sufficient Conditions for the Hurwitz and Schur Stability of Interval Matrices", IEEE Trans. Automat. Contr.,vol. 39, No. 6, (1994), pp. 1251-1254.

[3] J. Rohan, "Positive Definiteness and Stability of Interval Matrices", SIAM J. Matrix Anal. Appl., vol. 15, (1994), pp. 175-184.

[4] S. Barnett and C. Storey, Matrix Methods in Stability Theory, (1970), Thomas Nelson and Sons Ltd, 36 Park Street London W1Y 4DE.

[5] N. Dunford and J.T. Schwartz, Linear Operators Part I, (1957), Intersciences Publishers INC., NEW YORK.

[6] J. Rohan, "Systems of Linear Interval Equations", Linear Algebra and its Applications, 126 (1989), pp. 39-78.

[7] M. Yousuf, "On the Marginal Stability of Interval Matrices", Master thesis, KFUPM, Dhahran 31261, Saudi Arabia.