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ABSTRACT. The existence of distinct N-phase wave solutions of an integrable system of coupled non-linear Schroedinger (CNLS) equations which possess finite genus spectral curves ("tori") with the same genus (thus the same number of phases) but *different* monodromy ("twist") is demonstrated. This feature of the solution space is new for the coupled system and is not present in the scalar NLS case[5], for the KdV equation [3], the Sine-Gordon equation [6], or any integrable p.d.e. with a second order Lax operator. This is a first step towards studying the saturation of the instabilities of the coupled system [14][7][2], generalizing the exact solution methods developed for the scalar NLS [12][13], and applying algebraic methods to soliton equations having third order Lax operators[4].

1. THE INTEGRABLE SYSTEM

1.1. Definitions. The following coupled non-linear Schroedinger (CNLS) system is considered:

$$\begin{aligned} ip_t + 2p_{xx} - p(pr + qs) &= 0 \\ iq_t + 2q_{xx} - q(pr + qs) &= 0 \\ -ir_t + 2r_{xx} - r(pr + qs) &= 0 \\ -is_t + 2s_{xx} - s(pr + qs) &= 0. \end{aligned} \tag{1}$$

The system (1) is integrable in the sense that it is equivalent to the compatibility condition of the following Lax pair [8][9][1]:

$$\begin{aligned} Q_x &= [L_1, Q] \\ Q_t &= [L_2, Q] \end{aligned} \tag{2}$$

where Q, L_1, L_2, are 3×3 matrices that depend on a formal parameter E. The matrices L_1 and L_2 are given by

$$\begin{aligned} L_1 &= EA_0 + A_1 \\ L_2 &= E^2A_0 + EA_1 + A_2 \end{aligned}$$

with

$$\begin{aligned} A_0 &= \frac{i}{3}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ A_1 &= \frac{1}{2}\begin{pmatrix} 0 & p & q \\ r & 0 & 0 \\ s & 0 & 0 \end{pmatrix} \\ A_2 &= -\frac{i}{4}\begin{pmatrix} pr+qs & -2p_x & -2q_x \\ 2r_x & -pr & -qr \\ 2s_x & -ps & -qs \end{pmatrix} \end{aligned} \tag{3}$$

and $[,]$ is the usual commutator. This is the "squared eigenfunction" formulation of the Lax pair in which Q corresponds to squared eigenfunctions of a vector function Lax pair in which E is the eigenvalue. However the key to studying N-phase or quasiperiodic solutions of integrable equations is the study of solutions Q in a graded loop algebra of 3×3 traceless matrices graded by the parameter E. Finding a solution Q of the Lax pair (2) is equivalent to finding a solution in p, q, r, s of the CNLS system (1). The N-phase wave solutions correspond to Q being a finite expansion in E within a graded loop algebra.

The CNLS system (1) specializes to a pair of coupled non-linear Schroedinger equations if $r = -\sigma p^*$ and $s = -\sigma q^*$:

$$\begin{aligned} ip_t + 2p_{xx} + \sigma p\left(|p|^2 + |q|^2\right) &= 0 \\ iq_t + 2q_{xx} + \sigma q\left(|p|^2 + |q|^2\right) &= 0, \end{aligned}$$

which, in turn, is a special integrable case of a family of coupled equations that arises in the study of the propagation of orthogonal components p and q of an electric field in a glass fiber [11]. The defocussing $(\sigma = -1)$ and focussing $(\sigma = +1)$ cases are distinguished by $(\sigma = \pm 1)$.

1.2. Hierarchy of Commuting Time Flows. A hierarchy of commuting time flows exists for the CNLS system (1) when Q solves the equation

$$Q_{T_K} = [L_K, Q]$$

where

$$L_K = A_0E^K + A_1E^{K-1} + \cdots + A_K$$

for $K \geq 1$ and $T_1 = x$ and $T_2 = t$. In order for the compatibility of the x-flow and the T_K-flow for Q to be equivalent to a flow of the potentials p, q, r, s, it must be that

L_K satisfies, for every order in E greater than zero,

$$\partial_x L_K = [L_1, L_K]$$

and in this way the A_j are determined for $j = 0, 1, 2, ..., K-1, K$.

The zero-th order term in the above condition produces the higher time flow equation for p, q, r, s :

$$\partial_{T_K} L_1 = \{\partial_x L_K - [L_1, L_K]\}_0$$

where $\{\}_0$ indicates that the zero-th order term in the E expansion is taken.

Thus the hierarchy of higher time flows is generated from a solution of the x-flow of the Lax pair (2) in the form of a Laurent series in E :

$$Q = Q_0 + Q_1 E^{-1} + Q_2 E^{-2} + Q_3 E^{-3} + \cdots \tag{4}$$

so that $L_K = \{E^K Q\}_+$ where the notation indicates that all the terms in the series with non-negative E exponent are taken.

- The time flows in the hierarchy will be partial differential equations if and only if the entries in the Q_i are all differential polynomials in p, q, r, s. Otherwise integro-differential equations will arise.

Time independent solutions of the higher time flow equations are equivalent to solutions Q of the Lax pair which truncate at finite order in the Laurent series (4). These are the N-phase solutions of the CNLS system (1). The N-phase waves satisfy an additional differential constraint, namely the time-independent equation of one of the higher time flows, and are equivalent to polynomial solutions of Q with respect to the parameter E. The characteristic polynomial of Q is an integral of the flow and is a desingularization of the spectral curve of the solutions if they are considered potentials in the vector function formulation of the Lax pair. The study of such Q is the goal of the next section.

The explicit solution for an N-phase wave can be obtained in terms of theta functions when the flows are linearized on the Jacobian of the characteristic polynomial of Q, anticipating the generalization of the results obtained for KdV and scalar NLS and other integrable equations with second order Lax operators. The work of McKean [10] on the Boussinesq equation is the singular reference in this regard for Lax operators of order greater than 2. Although the explicit construction is not attempted here, the groundwork is established for carrying out such a construction. An unusual technical feature of the spectral curves ("twisted tori") and the solution space of N-phase waves is elucidated, viz. *there are distinct N-phase waves with topologically equivalent spectral curves ("tori") but possessing different monodromy ("twist").*

2. The Loop Algebra

2.1. Definitions. The basic tool for studying N-phase solutions of the infinite hierarchy of the CNLS system is a loop algebra of formal series of 3×3 matrices graded by the spectral parameter E, following the techniques of Flaschka, Newell, and Ratiu [4], in the case of the 2×2 AKNS system (i.e. the scalar NLS case.)

Definition 1. *Let $\mathfrak{G}$ be the ring of formal series of the type $Q = \sum_{n=0}^{\infty} Q_n E^{-n}$ where $Q_n \in sl(3,\mathfrak{B})$, for all n, and $Q_0 \in sl(3,\mathbf{C})$, and where $\mathfrak{B}$ is the polynomial ring over the complex field $\mathbf{C}$ generated by the symbols p, q, r, s and equipped with the derivation ∂.*

Notice that if $Q_0 \in GL(3, \mathbf{C})$ then $\mathfrak{G}$ will be an algebra, or a graded loop algebra. In particular if $Q \in \mathfrak{G}$ then $Q^{-1} \in \mathfrak{G}$ also, each term in Q^{-1} being a finite sum of finite products of Q_0^{-1} and the Q_i.

- The above definition can be extended by equipping $\mathfrak{B}$ with an anti-derivation ∂^{-1}. If we define $\partial^{-1} 0 = 0$ then the anti-derivation is well-defined and we denote the extended rings by $\mathfrak{B}^{+}$ and $\mathfrak{G}^{+}$.

If $Q \in \mathfrak{G}$ then $\det Q = \sum_{n=-\infty}^{0} c_{-n} E^{n}$ where each $c_n \in \mathfrak{B}$ comes from finite sums and products of entries from finitely many terms in Q. Moreover the usual product rule for determinants holds for elements of $\mathfrak{G}$.

The problem is to classify those $Q \in \mathfrak{G}^{+}$ which solve the equation $\partial Q = [L_1, Q]$, and then to examine the subset of such solutions which truncate to finite order in E. Such "polynomial ansatz" Q correspond, equivalently, to N-phase wave solutions of the hierarchy or time independent solutions of different time flows in the hierarchy.

- Previous experience with 2×2 AKNS systems [4] suggests that $Q \in \mathfrak{G}$, viz. all the entries of Q are necessarily differential polynomials of p, q, r, s, however this is **not** true in the 3×3 case. In general $Q \in \mathfrak{G}^{+}$, which leads directly to the existence of different N-phase waves having spectral curves of the same genus but *different* monodromy, as will be demonstrated in the next section.

With the correct choice of $Q_0 = A_0$ it will be shown that in fact $Q \in \mathfrak{G}$, and the corresponding N-phase waves satisfy genuine partial differential constraints (without integrals).

- *It is suspected that the N-phase waves having integro-differential constraints ($Q \in \mathfrak{G}^{+}$) are special reductions of the more usual N-phase waves with differential constraints ($Q \in \mathfrak{G}$); this has been shown to be so in the simplest case, however a general proof has not been obtained.*

2.2. Generating Solutions. A series of lemmas is now given that will enable us to determine all those solutions Q of the Lax pair which are free of any intregral expressions and to examine the spectral curves with regard to genus and monodromy.

Lemma 2. *$Q \in \mathfrak{G}^+$ and $\partial Q = [L_1, Q]$ only if $\partial Q_0 = 0$ and $[A_0, Q_0] = 0$.*

The proof of this proposition is a simple calculation, moreover it shows that the entries of Q_0 are constants. The restriction of zero trace Q_0 is no loss of generality since the identity matrix commutes with all matrices.

Lemma 3. *$Q \in \mathfrak{G}^+$ and $\partial Q = [L_1, Q]$ if and only if the entries of Q_{n+1} are given recursively in terms of the entries of Q_n, denoted by q_{ij}^n for $n > 0$, by the formulae:*

$$\begin{aligned}
&\textit{(off-block entries)} \\
q_{12}^{n+1} &= i\partial q_{12}^n - \frac{i}{2}p(q_{22}^n - q_{11}^n) - \frac{i}{2}qq_{32}^n \qquad (5)\\
q_{13}^{n+1} &= i\partial q_{13}^n - \frac{i}{2}q(q_{33}^n - q_{11}^n) - \frac{i}{2}pq_{23}^n \\
q_{21}^{n+1} &= -i\partial q_{21}^n - \frac{i}{2}r(q_{22}^n - q_{11}^n) - \frac{i}{2}sq_{23}^n \\
q_{31}^{n+1} &= -i\partial q_{31}^n - \frac{i}{2}s(q_{33}^n - q_{11}^n) - \frac{i}{2}rq_{32}^n \\
&\textit{(block entries)} \\
q_{11}^{n+1} &= -q_{22}^{n+1} - q_{33}^{n+1} \\
q_{22}^{n+1} &= \frac{1}{2}\partial^{-1}(-pq_{21}^{n+1} + rq_{12}^{n+1}) + \alpha \\
q_{33}^{n+1} &= \frac{1}{2}\partial^{-1}(-qq_{31}^{n+1} + sq_{13}^{n+1}) + \delta \\
q_{23}^{n+1} &= \frac{1}{2}\partial^{-1}(-qq_{21}^{n+1} + rq_{13}^{n+1}) + \beta \\
q_{32}^{n+1} &= \frac{1}{2}\partial^{-1}(-pq_{31}^{n+1} + sq_{12}^{n+1}) + \gamma
\end{aligned}$$

The proof is an immediate calculation using the definition of L_1. The constants $\alpha, \beta, \gamma, \delta$ are constants of integration belonging to the kernal of ∂. There is no loss of generality in assuming that $Q_n \in sl(3, \mathfrak{B}^+)$ since Q is arbitrary up to addition of constant multiples of the identity matrix.

- The recursion relation for the "block" entries involves ∂^{-1} so, in general, $Q \in \mathfrak{G}^+$.

Lemma 4. *There exists a unique invertible $V \in \mathfrak{G}^+$ of the form*

$$V = I + \sum_{n=1}^{\infty} V_n E^{-n}$$

which solves

$$\partial V = [A_0, V]\, E + A_1 V$$

and satisfies the condition that $V = I$ when $p = q = r = s = 0$. The form of A_0 and A_1 is given in Equation (3).

Here I is 3×3 identity matrix. The proof of the proposition is obtained by writing down recursion relations for the elements of V_{n+1} in terms of V_n for $n > 0$ and $V_0 = I$, in a similar fashion to the previous recursion relations for Q. In the proposition, that particular V with constants of integration equal to zero was chosen for definiteness.The purpose of defining this $V \in \mathfrak{G}^+$ is in order to use V to characterize solutions Q according to the following theorem.

Theorem 5. *$Q \in \mathfrak{G}^+$ is a solution of $\partial Q = [L_1, Q]$ if and only if $Q = VXV^{-1}$ where $X \in \mathfrak{G}$ satisfies $[A_0, X] = 0$ and $\partial X = 0$.*

The proof of the sufficiency of the condition is a direct calculation of ∂Q making use of the equation satisfied by V , the commutation of A_0 and X, and the fact that both A_0 and X are in the kernel of the derivation ∂.

The necessity of the condition follows by repeated application of the linearity of the equation satisfied by Q and Proposition 1 and the sufficiency part of the Theorem. In this way Q can be shown to possess the required form.

Remark 1. *The element X in the above theorem has the form*

$$X = X_0 + X_1 E^{-1} + X_2 E^{-2} + X_3 E^{-3} + \cdots \tag{6}$$

where each X_j is a 3×3 matrix of "block" type, i.e. it commutes with A_0, and whose entries are complex constants which are precisely the constants of integration of recursion relations (5) *. X_0 is the "seed" term of the recursion relations.*

The above remark leads immediately to the following corollary.

Corollary 6. *If $Q \in \mathfrak{G}^+$ is a solution of $\partial Q = [L_1, Q]$ then $\partial \det(\lambda I - Q) = 0$.*

Thus the characteristic polynomial (possibly of infinite degree) of Q is an integral of the x-flow. Moreover if it is assumed that it is an integral of all the flows simultaneously than Q can be constructed explicitly in terms of theta functions based on experience with integrable equations with second order Lax pairs such as the scalar NLS and the KdV, however this is not attempted here.

We are now able to characterize precisely which choices of the "seed" term X_0 in the recursion relations (5) and the constants of integration X_j in the series of constant matrices (6) which determine the solution Q of the x-flow of the Lax pair, according to the previous theorem, will produce a solution Q which has only differential polynomial entries. *By the linearity of the Lax pair equations for Q, we need only to give the necessary and sufficient conditions on $Q_0 = X_0$ that will produce $Q \in \mathfrak{G}$.*

The necessary condition is a simple calculation using the recursion relations of Equations (5).

Proposition 7. *If $Q \in \mathfrak{G}$ is a solution of $\partial Q = [L_1, Q]$ with p and q independent and r and s independent then Q_0 must be proportional to the matrix A_0.*

It is far from obvious that the choice $Q_0 = A_0$, (and all other constants of integration set to zero) is sufficient to ensure that the recursion relations will always produce pure differential polynomials of p, q, r, s, but the following theorem shows that it is in fact true.

Theorem 8. *Let $Q \in \mathfrak{G}^+$ be the unique solution of $\partial Q = [L_1, Q]$ defined by $Q_0 = A_0$ and the remaining Q_n are given by the recursion relations (5) in which the constants of integration are all set to zero. Then $Q = VA_0V^{-1} \in \mathfrak{G}$.*

The fact that the Q under consideration is equal to VA_0V^{-1} follows from the uniqueness of the recursion relations once the constants of integrations are all set to zero, and the fact that V was defined so that all the constants of integration were zero.

The important part of the theorem is that the solution so defined is an element of $\mathfrak{G}$, viz. all the entries of $Q = VA_0V^{-1}$ are differential polynomials of p, q, r, s. The proof is a generalization of an idea in Flaschka, Newell, and Ratiu [4] for the 2×2 AKNS case. However, unlike the 2×2 case, the relation

$$\det(\lambda I - VA_0V^{-1}) = \det(\lambda I - A_0) \tag{7}$$

does not lead to explicit expressions for the entries of VA_0V^{-1} to all orders. Instead, explicit calculation is first needed to show that

$$VA_0V^{-1} = A_0 + A_1E^{-1} + A_2E^{-2} + \cdots \tag{8}$$

where $A_0, A_1, A_2 \in sl(3, \mathfrak{B})$ are as previously defined by equations (3). With this information, an induction argument is now needed to show that $VA_0V^{-1} \in \mathfrak{G}$ to all orders in E. In particular, the $i-j$th entry of VA_0V^{-1} can be written as

$$q_{ij} = q_{ij}^0 + q_{ij}^1 E^{-1} + q_{ij}^2 E^{-2} + \cdots$$

and we know from Equation (7) that

$$\lambda^3 + \frac{1}{3}\lambda - \frac{2i}{27} \equiv \lambda^3 - \sigma_1\lambda^2 + \sigma_2\lambda - \sigma_3$$

where

$$\begin{aligned}
\sigma_1 &= q_{11} + q_{22} + q_{33}, \\
\sigma_2 &= q_{11}q_{22} + q_{11}q_{33} + q_{22}q_{33} - q_{21}q_{12} \\
&\quad -q_{31}q_{13} - q_{23}q_{32}, \\
\sigma_3 &= q_{11}q_{22}q_{33} + q_{12}q_{13}q_{23} + q_{13}q_{21}q_{32} \\
&\quad -q_{11}q_{32}q_{23} - q_{12}q_{21}q_{33} - q_{13}q_{31}q_{22}.
\end{aligned}$$

Now assume that for $0 \le k \le n$ and $n \ge 2$ we know that $q_{ij}^k \in \mathfrak{B}$. This is certainly true for $n = 2$. By the recursion relations (5) we know that the off-block terms $q_{12}^{n+1}, q_{21}^{n+1}, q_{13}^{n+1}, q_{31}^{n+1} \in \mathfrak{B}$ also. If we consider the terms at order E^{-n-3} we are led to the following identity:

$$\begin{aligned}
0 \equiv\ & ((\frac{3}{2} - 3i)qs - (\frac{3}{4} + 3i)pr)q_{22}^{n+1} + \\
& ((\frac{3}{2} - 3i)pr - (\frac{3}{4} + 3i)qs)q_{33}^{n+1} + \\
& (\frac{3}{4} - \frac{i}{4})psq_{23}^{n+1} + (\frac{3}{4} - \frac{i}{4})qrq_{32}^{n+1} + b
\end{aligned} \tag{9}$$

where $b \in \mathfrak{B}$ is known in terms of expressions already assumed to be in $\mathfrak{B}$. By the inductive hypothesis the terms $q_{22}^{n+1}, q_{33}^{n+1}, q_{23}^{n+1}, q_{32}^{n+1}$ can only be of the form $u + \partial^{-1}v$ where $u \in \mathfrak{B}$ and $\partial^{-1}v$ is a genuine integral term which cannot be simplified any further. However integral terms of this type could not cancel each other out of the above identity (9), and therefore it must be that the block terms $q_{22}^{n+1}, q_{33}^{n+1}, q_{23}^{n+1}, q_{32}^{n+1} \in \mathfrak{B}$. Moreover $q_{11}^{n+1} = -q_{22}^{n+1} - q_{33}^{n+1} \in \mathfrak{B}$, also, so only pure differential polynomials can be generated. The inductive step is now complete.

Note that the explicit calculation of Equation(8) was necessary since the entries of q_{ij}^2 were used explicitly in the identity (9) .

3. Twisted Tori

It is now possible to classify solutions $Q \in \mathfrak{G}^+$ of the x-flow equation $Q_x = [L_1, Q]$ which truncate at finite order according to their characteristic polynomials or "spectral curves". In particular it will be shown that there exist distinct N-phase solutions that possess trigonal spectral curves of the same genus (and so the curves are topologically equivalent and the solutions have the same number of phases) but different monodromy (the topology of pairs of sheets in the realization of the curve as a three sheeted covering of the Riemann sphere is different and so the interaction of the phases is different.)

Definition 9. *An N-phase wave solution in p, q, r, s for the CNLS system (1) is a quasiperiodic solution generated by a polynomial ansatz for the solution Q of the Lax pair (2). In particular, an N-phase wave solution is generated when the solution $Q \in \mathfrak{G}^+$ of the x-flow equation $\partial Q = [L_1, Q]$ is truncated at a finite order in E, so that QE^K is a polynomial of degree K in E, for some positive integer K which, in general, is not equal to N, the number of phases in the solution. This ansatz forces the potentials p, q, r, s to obey an integro-differential constraint that comes from setting the off-block terms at level $K+1$ in the recursion relations (5) identically equal to zero:*

$$
\begin{aligned}
0 &\equiv q_{12}^{K+1} = i\partial q_{12}^K - \frac{i}{2}p(q_{22}^K - q_{11}^K) - \frac{i}{2}qq_{32}^K \\
0 &\equiv q_{13}^{K+1} = i\partial q_{13}^K - \frac{i}{2}q(q_{33}^K - q_{11}^K) - \frac{i}{2}pq_{23}^K \\
0 &\equiv q_{21}^{K+1} = -i\partial q_{21}^K - \frac{i}{2}r(q_{22}^K - q_{11}^K) - \frac{i}{2}sq_{23}^K \\
0 &\equiv q_{31}^{K+1} = -i\partial q_{31}^K - \frac{i}{2}s(q_{33}^K - q_{11}^K) - \frac{i}{2}rq_{32}^K
\end{aligned}
\tag{10}
$$

The number of phases in the solution will be fixed by the genus of the characteristic polynomial of QE^K. The precise form of the linearization of the flows on the Jacobian of the characteristic polynomial will introduce some trivial exponential phases.

Moreover, it is known from Theorem 5 that $Q = VXV^{-1}$ so the constraints (10) force the constants in X to be dependent. In particular, if

$$X = X_0 + X_1E^{-1} + X_2E^{-2} + X_3E^{-3} + \cdots$$

then X_j for $j = 0, 1, 2, ..., K$ are free parameters which determine the constrained system (10) which defines the N-phase wave. The entries of X_j for $j = K+1, K+2, ...$ are integro-differential expressions $\mathcal{F}_j(p, q, r, s)$ such that $\partial_x \mathcal{F}_j(p, q, r, s) = 0$, and so are actually interdependent integrals of the constrained system (10) determined by the initial conditions.

Lemma 10. *Let $Q \in \mathfrak{G}^+$ be an N-phase wave solution of $\partial Q = [L_1, Q]$ such that QE^K is a polynomial of degree K in E. Then*

$$\lambda^3 + A(E)\lambda + B(E) = \det(\lambda I - E^K Q) = \det(\lambda I - E^K X) \tag{11}$$

where

$$X = X_0 + X_1 E^{-1} + X_2 E^{-2} + \cdots$$

and

$$X_j = \begin{pmatrix} -\alpha_j - \delta_j & 0 & 0 \\ 0 & \alpha_j & \beta_j \\ 0 & \gamma_j & \delta_j \end{pmatrix}$$

with constant elements $\alpha_j, \beta_j, \gamma_j, \delta_j \in \mathbf{C}$. Moreover $A(E)$ and $B(E)$ are polynomials in E with coefficients in $a_j, b_j \in \mathfrak{B}^+$,

$$\begin{aligned} A(E) &= a_0 E^{2K} + a_1 E^{2K-1} + \cdots + a_{2K}, \\ B(E) &= b_0 E^{3K} + b_1 E^{3K-1} + \cdots + b_{3K}. \end{aligned}$$

Remark 2. *The coefficients a_j and b_j of A and B in the characteristic polynomial of QE^K are integrals of the constrained system of the N-phase wave (10), viz. $\partial a_j = \partial b_j = 0$, and depend on the parameters of the system given by X_j for $j = 0, 1, 2, ..., K$ and the integrals of the motion X_j for $j = K+1, K+2, K+3, ...$ Notice that only a finite number of these integrals of motion are independent, the relations determining their interdependence come from the expansion of the characteristic polynomial in equation (11).*

Since the characteristic polynomial (11) is trigonal, its realization as a Riemann surface has three λ sheets covering the Riemann E sphere which are connected, at most, by $6K$ square root type branch points located at the $6K$ roots of the discriminant $4A^3 + 27B^2 = 0$,assuming that each root is distinct. Each branch point connects only two of the three sheets. The way in which the branch points are distributed between the three sheets of the spectral curve determines the *monodromy*. If each pair of sheets is connected by exactly $2K$ branch points (so that the number of branch points between any pair of sheets is the same), then we call this a *symmetric* monodromy. Otherwise the monodromy is called *asymmetric*.

Proposition 11. *There exist distinct N-phase wave solutions of the CNLS system (1) with polynomial ansatz of different degrees J and K whose trigonal spectral curves have the same genus G (and so the same number of phases) but different monodromy whenever $G + 2$ is a multiple of 6. In particular, there exist solutions for which $G = 3K - 2$ with symmetric monodromy and different solutions for which $G = 2J - 2$ with asymmetric monodromy.*

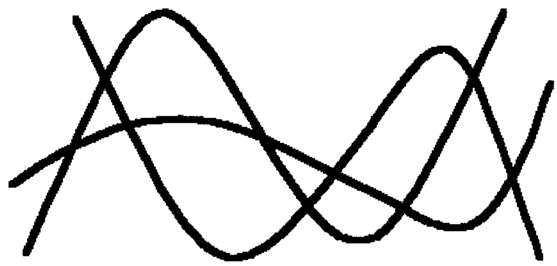

Figure 1: Curve with Symmetric Monodromy

The proof comes from choosing X in Equation (6) appropriately and making use of Lemma 10.

First we construct curves with **symmetric** monodromy by choosing

$$E^K X = X_0 E^K + X_1 E^{K-1} + \cdots + X_K + X_{K+1} E^{-1} + \cdots$$

where each X_j for $j = 0, 1, 2, ..., K$ has the form

$$X_j = \begin{pmatrix} -\alpha_j - \delta_j & 0 & 0 \\ 0 & \alpha_j & 0 \\ 0 & 0 & \delta_j \end{pmatrix}$$

and, in particular, $-\alpha_0 - \delta_0 \neq \alpha_0 \neq \delta_0$. Moreover we consider the limit where all the integrals of motion are set to zero, so that $X_j = 0$ for $j \geq K+1$. In this limit the characteristic polynomial of the N-phase wave is reducible and factors into three factors, each factor corresponding to a separate sheet of the Riemann surface:

$$\begin{aligned} \lambda - (-\alpha_0 - \delta_0)E^K - \cdots - (-\alpha_K - \delta_K) &= 0 \\ \lambda - \alpha_0 E^K - \cdots - \alpha_K &= 0 \\ \lambda - \delta_0 E^K - \cdots - \delta_K &= 0. \end{aligned}$$

When the integrals of motion are not exactly zero but are small in absolute value then there will exist an N-phase solution whose spectral curve is a perturbation of the reducible one which factors into the above three sheets. In particular, points of intersection of the above three sheets will split open into pairs of genuine square root

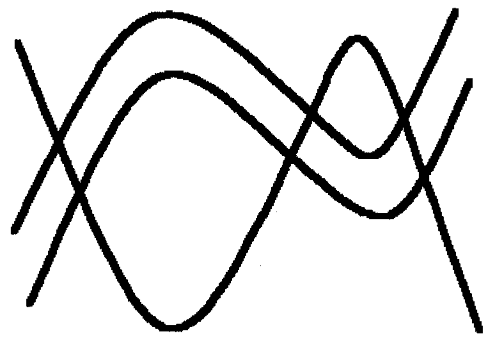

Figure 2: Curve with Asymmetric Monodromy

type branch points. If, for example, $K = 3, \alpha_0 > 0, \delta_0 > 0, \alpha_0 \neq \delta_0$ and the remaining α_j and δ_j are chosen arbitrarily then, in general, the three sheets of the factored curve will be three cubics, each pair of sheets intersecting exactly three times. See Figure (1). Under small perturbations of the integrals of motion the $3K$ intersections of the sheets will split open into $6K$ branch points with *symmetric* monodromy. The genus as given by the Riemann-Hurwitz relation will be $G = 3K - 2$.

On the other hand, we can construct curves with **asymmetric** monodromy by choosing X to be of the following form

$$\begin{aligned} X &= \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} (\alpha_0 E^J + \cdots + \alpha_{J-1} E^1) + \\ & \begin{pmatrix} -\alpha_J - \delta_J & 0 & 0 \\ 0 & \alpha_J & 0 \\ 0 & 0 & \delta_J \end{pmatrix} + X_{J+1} E^{-1} + \cdots \end{aligned}$$

where $\alpha_0 > 0$ and $\alpha_J \neq \delta_J$. When the limit $X_j = 0$ for $j \geq J+1$ is considered, then the three sheets factor into

$$\begin{aligned} \lambda - (-2\alpha_0 E^J - 2\alpha_1 E^{J-1} - \cdots - (\alpha_J + \delta_J)) &= 0 \\ \lambda - (\alpha_0 E^J + \alpha_1 E^{J-1} + \cdots \alpha_{J-1} E + \alpha_J) &= 0 \\ \lambda - (\alpha_0 E^J + \alpha_1 E^{J-1} + \cdots \alpha_{J-1} E + \delta_J) &= 0 \end{aligned}$$

Notice that two of the sheets are identical except for a shift in the last constant. Thus when $J = 3$, for example, there will be three cubics but one pair never intersects.

See Figure (2). In particular there are $2J$ intersections between the three sheets which, under perturbation of the initial conditions, split open into $4J$ square root type branch points. Thus the genus of this type of curve will be $G = 2J - 2$ but the monodromy is *asymmetric.*

Thus when $G = 2J - 2 = 3K - 2$, viz. when $G + 2$ is a multiple of 6, there will be two solutions having the same genus but different monodromy.

Note that Proposition 11 by no means exhausts all of the possible N-phase waves which have the same genus but different monodromy. However a complete classification of N-phase waves according to genus and monodromy has not been obtained.

4. Conclusion

The existence of N-phase solutions of the integrable CNLS system, with arbitrarily large number of phases, which have the same number of phases but distinctly different phase interactions has been demonstrated. In particular there are solutions whose spectral curves have the same genus but different monodromy. Thus the way in which the tori are "twisted" will play a significant role in the linearization of the flow on the Jacobian of the curve and the subsequent explicit construction of the solutions. This phenomenon is new to integrable systems with Lax pairs of order three and above; in the well studied cases of integrable equations with second order Lax pairs such as the KdV and the scalar NLS, the spectral curve is always hyperelliptic (two-sheeted) so issues of monodromy do not arise.

Moreover, the groundwork, for classifying N-phase waves according to genus and monodromy of the spectral curve and for explicitly constructing these waves, has been established.

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