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# On a problem of Wenzel

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**Abstract**

In this note, we introduce a weak form of equational compactness, which we called $x_0$-compactness, together with some of its basic properties. We then use it to settle in the negative a problem of G. H. Wenzel. We also prove the adjacent result that, over a wide class of commutative rings, if every module is $x_0$-compact, then the ring has finite representation type.

Let $\mathfrak{A}$ be a universal algebra. Following Haley [5], by an *$x_0$-system* over $\mathfrak{A}$ we mean a system $\sum$ of equations over $\mathfrak{A}$ with constants in $\mathfrak{A}$ and which is the union of finite subsystems $\sum_i$ $(i \in I)$ such that for a fixed variable $x_0$, the only variable appearing in both $\sum_i$ and $\sum_j$ $(i \neq j)$ is $x_0$. A fundamental theorem of Mycielski and Ryll-Nardzewski [8] asserts that $\mathfrak{A}$ is equationally compact if and only if every finitely solvable $x_0$-system over $\mathfrak{A}$ is solvable in $\mathfrak{A}$ (see also Gruson and Jensen [4] or Zimmermann [10] for a module-theoretic version of this result). Although Haley showed that replacing $x_0$-systems by the more restrictive *strong $x_0$-systems* does not ensure equational compactness of algebras in general (see [5]), an earlier result of Balcerzyk in[1] states that an abelian group $\mathfrak{G}$ is equationally compact whenever every finitely solvable ($x_0$-) system of the form $x_0 + r_n x_n = a_n$ $(n \in \mathbb{N})$ over $\mathfrak{G}$ is solvable in $\mathfrak{G}$ (so, in this particular case, one can even restrict the finite subsytems $\sum_i$ $(i \in I)$ above to consist of one equation only). In this connection, Wenzel poses in [9] the problem as to whether such particular forms exist for arbitrary modules. One main objective of this note is to prove that Wenzel's problem has a negative answer.

Throughout this paper, $R$ is an associative ring with 1 and all modules are left unital

$R$–modules. For any set $I$ and any module $M$, $|I|$ and $M^{(I)}$ denote respectively the cardinality of $I$ and the direct sum of $|I|$ copies of $M$. Also, for a property $p$ of modules, we say that a module $M$ is $\sum$-$p$ if any direct sum of copies of $M$ has the property $p$. Recall that a module $M$ is *equationally compact*, or, equivalently, *algebraically compact,* if every finitely solvable system of linear equations over $M$ is solvable in $M$. We first need the following weaker form of algebraic compactness

**Definition.** An $R$-module $M$ is *(countably) $x_0$-compact* if every finitely solvable (countable) system over $M$ of the form

$$x + r_j x_j = a_j \quad (j \in J, \ \ r_j \in R, \ a_j \in M)$$

is solvable.

**Remark** It is clear that, as in the case of algebraic compactness, $x_0$-compactness is inherited by direct products and summands, and that the two notions coincide for abelian groups (by the cited result of Balcerzyk).

The study of $x_0$-compactness is worthwhile since it is motivated, in addition to Wenzel's problem, by several results. For example, if we define the $R$-adic topology on an $R$-module $M$ to be the topology with the set $\{rM \, : \, r \in R\backslash\{0\}\}$ as a base of open neibourhoods of 0, then algebraically compact modules are necessarily complete in their $R$-adic topology (Fuchs [3]). A closer look at the proof of this result in [3] shows that this is indeed true for $x_0$-compact modules. Similarly, in an important result of Jensen and Zimmermann-Huisgen [6] on ultraproducts over commutative artinian rings (see also [7] for the case of commutative perfect rings), one can replace algebraic compactness by $x_0$-compactness. However, as we shall later show, $x_0$-compact modules are not necessarily algebraically compact.

**Proposition 1.** *An $R$ -module $M$ is $x_0$-compact if, and only if, every family of cosets of subgroups of $M$ of the form $\{a_j + r_jM\}_{j\in J}$ , where $r_j \in R$ , $a_j \in M$, with the finite intersection property, has a non-empty intersection.*

Proof. Assume that $M$ is $x_0$-compact, and let $\{a_j + r_jM\}_{j\in J}$ be a family as in the statement of the proposition, with the finite intersection property. For each finite subset $L$ of $J$, there exists $b_L \in \bigcap_{j\in L} (a_j + r_jM)$, *i.e.* $b_L + r_jc_{jL} = a_j$ $(j \in J)$ for some $c_{jL}$ in $M$. The system $x + r_jx_j = a_j$ is therefore finitely solvable, and hence solvable in $M$ by $x = b$, $x_j = c_j$ say. It is now clear that $b \in \bigcap_{j\in J} (a_j + r_jM)$. The converse is proved in a similar way.

**Proposition 2.** *Modules over rings with a finite lattice of right ideals (e.g. finite rings) are $x_0$-compact.*

Proof. If $R$ is such a ring, then for each $R$ -module $M$, there are only finitely many subgroups of the form $rM = (rR)M$ ( $r \in R$). So if a family of cosets of subgroups of $M$ of the form $\{a_j + r_jM\}_{j\in J}$ , where $r_j \in R$ , $a_j \in M$, has the finite intersection property, then it must have a non-empty intersection.

Now, by [2], the finite ring $R$ of $2 \times 2$ lower triangular matrices over $\mathbb{Z}/p^n\mathbb{Z}$, where $p$ is prime and $n \geq 4$, has infinite representation type, and so its left and its right pure global dimensions cannot vanish simultaneously. We therefore infer that there exists a countable $\sum$-$x_0$-compact $R$-module $M$ which is not algebraically compact. This settles in the negative Wenzel's question. Let us mention another point. The module $M$ above is pure in its pure-injective envelope $PE(M)$, which is $\sum$-$x_0$-compact, but clearly $M$ is not a direct summand of $PE(M)$. This illustrates that a basic property of algebraic compactness, namely that pure submodules of $\sum$-algebraically compact modules are direct summands, is not shared by $x_0$-compactness. (It is however true that pure submodules of $\sum$-$x_0$-compact modules are $\sum$-$x_0$-compact.)

Although Wenzel's question remains open for arbitrary infinite rings, it seems worthwhile to conclude this note with an adjacent, positive result (Proposition 3). The following lemma says more than is required and is a straightforward adaptation of a well-known characterization of $\sum$-algebraically compact modules. (See [10]).

**Lemma.** *An R-module M is $\sum$-$x_0$-compact if and only if every descending chain of the form $r_1M \supseteq r_1M \cap r_2M \supseteq r_1M \cap r_2M \cap r_3M \supseteq \ldots$ ( $r_1, r_2, r_3, \ldots$ in R) is stationary. In particular, if the free module $R^{(\mathbb{N})}$ is $x_0$-compact, then R is left perfect.*

Proof. Suppose there exist $a_i \in \bigcap_{k=1}^{i} r_kM \backslash r_{i+1}M$ $(i \in \mathbb{N})$, then we can easily check that the following system over $M^{(\mathbb{N})}$, $x_0 + r_i x_i = (a_1, a_2, \ldots, a_i, 0, 0, \ldots)$ is finitely solvable in $M^{(\mathbb{N})}$, and therefore solvable in $M^{(\mathbb{N})}$ by $x_0 = (b_1, b_2, \ldots, b_N, 0, 0, \ldots)$, $x_i = (b_{i1}, b_{i2}, \ldots, b_{in_i}, 0, 0, \ldots)$, say. This yields $a_N \in r_{N+1}M$, which is impossible. For the converse, assume that every descending chain $r_1M \supseteq r_1M \cap r_2M \supseteq r_1M \cap r_2M \cap r_3M \supseteq \ldots$ is stationary, then it is easy to see that every family of subgroups of $M$ of the form $r_1M \cap r_2M \cap \ldots \cap r_nM$ has a minimal member. Suppose now that the family $\mathcal{F} = \{a_j + r_jM\}_{j \in J}$ has the finite intersection property, and order the set $\mathcal{L} = \{\bigcap_{j \in L} r_jM : L \text{ is a finite subset of } J\}$ by reverse inclusion. In view of the previous statement, an easy application of Zorn's lemma shows that $\mathcal{L}$ has a minimal member $\bigcap_{j \in L_0} r_jM$ for some finite subset $L_0$ of $J$, and clearly, $\bigcap_{j \in L_0} r_jM = \bigcap_{j \in J} r_jM$. It is easy to see that if $a \in \bigcap_{j \in L_0} (a_j + r_jM)$ and $b_j \in \bigcap_{j \in L_0} (a_j + r_jM) \cap (a_j + r_jM)$ $(j \in J)$, then $a - b_j \in r_jM$ and $b_j - a_j \in r_jM$, *i.e.* $a \in \bigcap_{j \in J} (a_j + r_jM)$. This proves, by Proposition 1, that $M$ is $x_0$-compact, and since for each index set $I$ and $r \in R$, $rM^{(I)} = (rM)^{(I)}$, it follows that $M$ is $\sum$-$x_0$-compact.

**Proposition 3.** *Let $R$ be a commutative ring, and suppose that every $R$-module is (countably) $\aleph_0$-compact. If each of the residue fields of the local ring factors of $R$ is infinite, then $R$ is an artinian principal ideal ring, and so every $R$-module is algebraically compact.*

Proof. Without loss of generality, we may clearly assume that $R$ is local with Jacobson radical $J$. We show that $R$ is a valuation ring. Let $H$ be an infinite subset of $R/J$ whose elements are distinct modulo $J$ and let $u, v \in R$. For each $h \in H$, set $r_h = u - hv$, $R_h = R/Rr_h$, $P = \prod_{h\in H} R_h$, $S = \bigoplus_{h\in H} R_h$. Denote by $f_h$ $(h \in H)$ the canonical decomposition $P \xrightarrow{proj} R_h \xrightarrow{incl} S$, let $a \in P$ be given by $a(h) = 1 + Rr_h$ $(h \in H)$ and consider the system (**1**) of equations

$$x + r_h y_h = f_h(av) \quad (h \in H)$$

with unknowns $x$, $(y_h)_{h\in H}$ . If (**1**)' is the system obtained from (1) by restricting $h$ to a finite subset $\{h_1, h_2, ..., h_n\}$ of $H$, and if $w_{ij}$ $(1 \leq i,\ j \leq n,\ i \neq j)$ are elements of $R$ with $(h_i - h_j)w_{ij} = 1$ (recall that $h_i - h_j$ is a unit of $R$), then $x = \sum_{j=1}^{n} f_{h_j}(av)$, $y_{h_i} = \sum_{\substack{j=1\\ j\neq i}}^{n} w_{ij} f_{h_j}(a)$ $(1 \leq i \leq n)$, is easily seen to be a solution of (**1**)' in $S$. Since $S$ is countably $\aleph_0$-compact, the system (1) is solvable by $b$, $(b_h)_{h\in H}$ in $S$, say. Now $b$, $(b_h)_{h\in H}$ have finite support, and so (using the fact that $R/J$ is infinite) there exist $h_0 \in H$ and $c \in R$ such that $v = cr_{h_0}$, *i.e.*, since $R$ is local, either $u \in Rv$ or $v \in Ru$, as required. Since $R$ is perfect, this implies that $R$ is an artinian principal ideal ring.

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