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Abstract

In this paper we have converted the Laplace transform to an integral equation of the first kind of convolution type, which is an ill–posed problem and used the spline regularization method to solve it. Inversion of perturbed Laplace transforms also plays an important role in system theory. The method is applied to several test examples taken from [1, 2, 3, 8, 11, 22]. It gives a good approximation to the true solution and compares well with the methods discussed in [1, 2, 3, 8, 11, 22]. The results are shown in Table 1 and respective diagrams.

AMS(MOS) Subject Classification: 65R20, 65R30

Key Words Inversion of Laplace transform, ill–posed problem, convolution equation, cross-validation, spline regularization, filter function, system theory.

1. Introduction.

Noisy Laplace transforms arise in a wide variety of practical problems. They are frequently used in system theory and linear dynamical systems. They are also used in statistics where a sample is drawn from a cumulative distribution function G, which is an unknown mixture of exponential distributions and hence can be written as

$$G(t) = \int_0^{\infty} (1 - e^{-st}) f(s) ds, \quad t \in (0, \infty),$$

where f is a probability density function with support in $(0, \infty)$.

Ostrowsky et al. [13] introduced the exponential sampling technique for inversion of the Laplace transform in photon correlation spectroscopy. There are many problems whose solution may be found in terms of a Laplace transform which, however, is too complicated for inversion using different methods. However, no single method gives optimum results for all purposes and all occasions.

For a detailed bibliography, the reader should consult Piessens [16] and Piessens and Branders [17]. The problem of the recovery of a real function $f(t), \quad t \geq 0$, given its Laplace transform

$$\int_0^{\infty} e^{-st} f(t) dt = g(s) \tag{1.1}$$

for real values of s.

The Laplace transform inversion is an ill–posed problem and, therefore, affected by numerical instability. The ill–posedness of Laplace transform inversion in the case where $f \in L^2(R_+)$ and $g(s)$ is known for all real and positive values of s, can be investigated by means of the Melline transform [11]. In practice, however, $g(s)$ is known only in a finite set of points. The case of an infinite set of equidistant points was investigated by Papoulis [14]. Several methods and a comparison is given in Davies [5] and Talbot [20].

The previous methods do not include regularization techniques. Regularization methods have been discussed by Varah [22], Essah and Delves [6] and Chauveau [3] and Brianzi [2]. Regularization by means of truncated singular function expansion is investigated by Brianzi in [2]. Other methods for numerical evaluation of the Laplace transform inversion have been described by Linz [10], Norden [12] and Salzer [18]. A maximum likelihood approach is established by Jewell [9].

2. Description of the Method

In (1.1) given $g(s)$, $s \geq 0$ we wish to find $f(t)$, $t \geq 0$ and $f(t) = 0$ for $t < 0$, so that (1.1) holds. Frequently, $g(s)$ is measured at certain points. We assume $g(s)$ is given analytically with known $f(t)$, so that we can measure the error in the numerical solution. In order to convert the Laplace transform into the first kind integral equation of convolution type, we make the following substitution in equation (1.1).

$$s = a^x \text{ and } t = a^{-y} \text{ where } a > 1. \tag{2.1}$$

Then

$$g(a^x) = \int_{-\infty}^{\infty} (\log a) e^{-a^{x-y}} f(a^{-y}) a^{-y} dy. \tag{2.2}$$

Multiplying both sides of (2.2) by a^x we obtain the convolution equation

$$\int_{-\infty}^{\infty} K(x-y)F(y)dy = G(x), \quad -\infty \leq x \leq \infty \tag{2.3}$$

where

$$\left.\begin{array}{l} G(x) = a^x g(a^x) = sg(s) \\ K(x) = (\log a) a^x e^{-a^x} = (\log a) s e^{-s} \\ F(y) = f(a^{-y}) = f(t) \end{array}\right\} \tag{2.4}$$

In order that we can apply our deconvolution method to equation (2.3), it is necessary that $G(x)$ has essentially compact support, i.e. $G(x) \to 0$ as $x \to \pm\infty$ or $|\lambda - G(x)| \to 0$ as $x \to \pm\infty$ where $\lambda = \max|G(x)|$ as $x \to \pm\infty$ which is a property, we demand from our data function $G(x)$.

2.2 Tikhonov Regularization Using Cardinal Cubic B–splines

Let $B_j(H;x)$ be the N–th order cardinal B–spline (N even) with knots $\left(j - \frac{N}{2}\right) H, \ldots, \left(j + \frac{N}{2}\right) H$ i.e. $B_i(H;x) = Q_N\left(\frac{x}{H} - j + \frac{N}{2}\right)$ where

$$Q_N(x) = \frac{1}{(N-1)!}\sum_i^N (-1)^i \begin{pmatrix} N \\ i \end{pmatrix} (x-i)^{N-1} \tag{2.5}$$

In addition let $MH = T$ where $M \leq N$ is an integral power of 2. We assume that $B_i(H;x)$ is periodically continued outside the interval $(0,T)$ with period T. Then $B_j(H,x)$ has a Fourier series

$$B_j(H;x) = \sum_{q=-\infty}^{\infty} \hat{B}_{j,q} \exp(i\omega_q x) \tag{2.6}$$

where

$$\hat{B}_{j,q} = \int_0^T B_j(H;x) \exp(-i\omega_q x) \tag{2.7}$$

and $\omega_q = \frac{2\pi q}{T}$.

Since $B_j(H;x)$ is simply a translation of $B_0(H;x)$ by an amount jH, we have

$$\hat{B}_{j,q} = \hat{B}_{0q} \exp(-i\omega_q H)$$

where

$$\hat{B}_{0,q} = H \left[\frac{\sin\frac{\omega_q H}{2}}{\frac{\omega_1 H}{2}}\right]^4 \tag{2.8}$$

Now we shall approximate the convolution equation (2.3) by

$$\int_0^T K_N(x-y)F_M(y)dy = G_N(x) \tag{2.9}$$

where we assume that F, G and K have essentially finite support in $[0,T)$, F_M is a cubic spline ($N=4$) of the form

$$F_M(x) = \sum_{j=1}^{M-1} \alpha_j B_j(H;x), \quad M \leq N \tag{2.10}$$

and K_N can be obtained from $K(x-y)$ in (2.4).

The real M–dimensional vector

$$\underline{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{M-1})^T$$

of unknown coefficients needs to be determined.

The spline in equation (2.10) has the Fourier series

$$F_M(x) = \sum_{q=-\infty}^{\infty} \hat{F}_{M,q} \exp(i\omega_q x) \tag{2.11}$$

where

$$\left.\begin{aligned} \hat{F}_{M,q} &= \sum_{j=0}^{M-1} \alpha_j \hat{B}_{jq} \\ &= \hat{B}_{0,q} \sum_{j=0}^{M-1} \alpha_j \exp\left(-\frac{2\pi i}{M} j_q\right) \\ &= \sqrt{M}\hat{B}_{0,q}\hat{\alpha}_s,\ s \approx q\ (\bmod M). \end{aligned}\right\} . \tag{2.12}$$

Here

$$\underline{\hat{\alpha}} = \psi_M^H \underline{\alpha}$$

where the symbol 'H' denotes the Hermitian transpose.

We find it advantageous to determine $\underline{\hat{\alpha}}$ rather than $\underline{\alpha}$, because of the simple properties available in discrete Fourier spaces. The vector $\underline{\alpha}$ in equation (2.10) may then be determined from the inverse M–dimensional FFT (Fast Fourier Transform)

$$\underline{\alpha} = \psi_M \underline{\hat{\alpha}} \tag{2.14}$$

where ψ is the unitary matrix with elements

$$\psi_{rs} = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi i}{N} rs\right), \quad r, s = 0, 1, 2, \ldots, N-1$$

2.2 P–th Order Tikhonov Regularization [21].

Consider the smoothing functional

$$C(F_M; \lambda) = C(\underline{\alpha}, \lambda) = \|K_N * F_M - G_N\|^2 + \lambda \|F_M^{(P)}\|^2 \tag{2.15}$$

where $F_M^{(P)}$ is the p–th order derivative.

Using Plancherel's theorem we have

$$\|K_N * F_M - G_N\|_2^2 = \frac{1}{N^2} \sum_{q=-N/2}^{N/2} |\hat{K}_{N,q} \hat{F}_{M,q} - \hat{G}_{N,q}|^2.$$

Hence using equation (2.12) we have

$$\|K_N * F_M - G_N\|_2^2 = \frac{1}{N^2} \sum_{q=-1/2N}^{1/2N} [(\sqrt{M}\hat{B}_{0,q}\hat{K}_{N,q}\hat{\alpha}_s - \hat{G}_{N,q}))\sqrt{M}\hat{B}_{0,q}\overline{\hat{K}}_{N,q}\overline{\hat{\alpha}}_s - \overline{\hat{G}}_{N,q})] \tag{2.16}$$

where $s \equiv q \pmod{M}$.

Also, Plancherel's theorem applied to the regularizing functional in equation

(2.15) gives

$$\begin{aligned} \|F_M^{(P)}\| &= \sum_{q=-\infty}^{\infty} \omega_q^{2p}|\hat{F}_{M,q}|^2 = 2\sum_{q=1}^{\infty}\omega_q^{2p}|\hat{F}_{M,q}|^2 \\ &= 2M\sum_{q=1}^{\infty}\omega_q^{2p}\hat{B}_{0,q}^2|\hat{\alpha}_s|^2 \quad \text{where } s \equiv q \pmod{M} \end{aligned} \tag{2.17}$$

The simplification of expression (2.17) requires the use of an attenuation factor τ_q. For cubic cardinal splines it is shown by Stoer [19] and Gautschi [7] that

$$\tau_q = \left[\frac{\sin\frac{\pi q}{M}}{\frac{\pi q}{M}}\right]^4 \frac{3}{1+2\cos\left(\frac{\pi q}{M}\right)} \tag{2.18}$$

In expression (2.17) we wish to arrange the summation over q to summation over s where $s \equiv q(\bmod M)$.

Define the matrix

$$W^{(i)} = \begin{bmatrix} \text{diag} & \sqrt{M} & \hat{B}_{0,s} & \hat{K}_{N,s} \\ \cdots & \cdots & \cdots & \cdots \\ \text{diag} & \sqrt{M} & \hat{B}_{0,M-s} & \overline{\hat{K}}_{N,M-s} \end{bmatrix} \quad \begin{matrix} \text{order } N\times \\ s = 0,1,\ldots,M-1 \end{matrix} . \tag{2.19}$$

From the property $\hat{K}_{N,q} = \overline{\hat{K}}_{N,N-q}$ of discrete FTs, it then follows that expression (2.16) simplifies to

$$\|K * F_M - G_N\|^2 = \|W^{(1)}\underline{\hat{\alpha}} - \underline{\hat{G}}_N\|_2^2 \tag{2.20}$$

and (2.17) can be written as

$$\|F_M^{(p)}\|^2 = 2M\sum_{s=1}^{M-1}\left\{|\hat{\alpha}_s\|^2\sum_{n=0}^{\infty}\omega_{Mn+s}^{2p}\hat{B}_{0,Mn+s}^2\right\} \tag{2.21}$$

$$= 2M\sum_{s=1}^{M-1}\tau_s|\hat{\alpha}_s|^2 \tag{2.22}$$

where

$$\tau_s = \sum_{n=0}^{\infty}\omega_{Mn+s}^{2p}\hat{B}_{0,Mn+s}^2 \tag{2.23}$$

$$= (2\pi)^{2p} \sum_{n=0}^{\infty} (Mn+s)^{2p} H^2 \left[\frac{\sin \pi \frac{(Mn+s)}{M}}{\frac{\pi(Mn+s)}{M}} \right] \tag{2.24}$$

$$\tau_s = (2\pi)^{2p} H^2 s^8 \left[\frac{\sin \frac{\pi s}{M}}{\pi s/M} \right]^8 \sum_{n=0}^{\infty} \quad (Mn+s)^{2p-8} \tag{2.25}$$

$$= (2\pi)^{2p} s^8 \hat{B}_{0,s}^2 \sum_{n=0}^{\infty} (Mn+s)^{2p-8} \tag{2.26}$$

since $\hat{\alpha}_s = \overline{\alpha}_{M-s}$, equation (2.21) further simplifies to

$$\|F_M^{(p)}\|_2^2 \sum_{s=1}^{1/2M} (\tau_s + \tau_{M-s}) |\hat{\alpha}_s|^2. \tag{2.27}$$

In particular when $p = 2$ (the order of regularization from (2.23), it follows that

$$\tau_s = (2\pi)^4 s^4 \hat{B}_{0s}^2 \sum_{n=0}^{\infty} \left(\frac{s}{Mn+s} \right)^4$$

while

$$\tau_{M-s} = (2\pi)^4 s^4 \hat{B}_{0s}^2 \left(\frac{s}{Mn-s} \right)^4$$

so that

$$\tau_s + \tau_{M-s} = (2\pi)^4 s^4 \hat{B}_{0,s}^2 \sum_{n=-\infty}^{\infty} \left(\frac{s}{Mn+s} \right)^4$$

$$= \frac{(2\pi)^4 s^4 \hat{B}_{0,s}^2 \left[1 + 2\cos^2\left(\frac{\pi s}{M}\right) \right]}{3 \left[\frac{\sin \frac{\pi s}{M}}{\pi s/M} \right]^4} \qquad \text{See Pennisi [15])}$$

$$\tau_s + \tau_{M-s} = \frac{16}{3} M^2 \sin^4\left(\frac{\pi s}{M}\right) \left[1 + 2\cos^2\left(\frac{\pi s}{M}\right) \right]$$

Defining the $M \times m$ matrix

$$W^{(2)} = \text{diag} \left\{ [M(\tau_s + \tau_{M-s})]^{1/2} \right\} \tag{2.28}$$

it follows from (2.24) that

$$\|F_M^{(p)}\|_2^2 = \|W^{(2)} \underline{\hat{\alpha}}\|^2. \tag{2.29}$$

Thus from equations (2.20) and (2.27) we may express the smoothing functional (2.15) as

$$C(\underline{\alpha}, \lambda) = \|W^{(1)}\underline{\hat{\alpha}} - \underline{\hat{G}}_N\|_2^2 + \lambda\|W^{(2)}\underline{\hat{\alpha}}\|_2^2. \tag{2.30}$$

The minimizer of (2.28) is clearly

$$\underline{\hat{\alpha}} = (W + \lambda V)^{-1} W^{(1)H} \underline{\hat{G}}_N \tag{2.31}$$

where

$$\left.\begin{array}{rcl} W & = & W^{(1)H}W^{(1)} \\ V & = & W^{(2)H}W^{(2)} \end{array}\right\} \tag{2.32}$$

It is not necessary to invert the matrix $W + \lambda V$ directly because it is diagonal. From equations (2.19), (2.26), (2.29) and (2.30), it follows that

$$\begin{aligned} \hat{\alpha}_s &= \frac{1}{\sqrt{M}} \frac{\hat{B}_{0,s}\overline{\hat{K}}_{N,s}\hat{G}_{N,s} + \hat{B}_{),M-s}\hat{K}_{N,M-s}\overline{\hat{G}}_{N,M=s}}{\left[\hat{B}_{0,s}^2\left\{|\hat{K}_{N,s}|^2 + \left(\frac{s}{M-s}\right)^8 |\hat{K}_{N,M-s}|^2\right\} + N^2\lambda(\tau_s + \tau_{M-s})\right]} \\ \hat{\alpha}_s &= \frac{1}{\sqrt{M}} \frac{\hat{B}_{),s}\left[\overline{\hat{K}}_{N,s}\hat{G}_{N,s} + \left(\frac{s}{M-s}\right)^4 \hat{K}_{N,M-s}\overline{\hat{G}}_{N,M-s}\right]}{\hat{B}_{0,s}^2\left[|\hat{K}_{N,s}|^2 + \left(\frac{s}{M-s}\right)^8 |\hat{K}_{N,M-s}|^2\right] + N^2\lambda(\tau_s + \tau_{M-s})} \end{aligned} \tag{2.33}$$

since

$$\hat{B}_{0,M-s} = \left(\frac{s}{Mn - s}\right)^4 \hat{B}_{0,s} \tag{2.34}$$

we can easily verify that $\hat{\alpha}_s = \overline{\hat{\alpha}}_{M-s}$ so that the inverse FFT gives $\underline{\alpha} = \psi_M \underline{\hat{\alpha}}$ is a real vector as required.

2.3 The Filter for Cardinal B–Spline Regularization.

The Fourier coefficients of the regularized (filtered) solution $F_M \in B_M(0,T)$ clearly depends on λ through equations (2.12), (2.13) and (2.31). In equation

(2.31), we denote the dependence of $\hat{\alpha}_s$ on λ by writing $\hat{\alpha}_s = \hat{\alpha}_s(\lambda)$. Thus the Fourier coefficients of the filtered solution are

$$\hat{F}_{M,q}(\lambda) = \sqrt{M}\hat{B}_{0,q}\hat{\alpha}_s(\lambda), \quad s \equiv q(\bmod M)$$

whereas those of the unregularized (unfiltered) solution is

$$\hat{F}_{0,q}(0) = \sqrt{M}\hat{B}_{0,q}\hat{\alpha}_s(0)$$

Clearly the underlying filter $Z_{q;\lambda}$ must satisfy

$$\hat{F}_{M,q}(\lambda) = Z_{q;\lambda}\hat{F}_{M,q}(0)$$

so that we can deduce

$$Z_{q;\lambda} = \frac{\hat{\alpha}_s(\alpha)}{\hat{\alpha}_s(0)} \tag{2.35}$$

$$Z_{q;s} = \frac{\hat{B}^2_{0,s}\left[|\hat{K}_{N,s}|^2 + \left(\frac{s}{Mn-s}\right)^8 |\hat{K}_{N,M-s}|^2\right]}{\hat{B}^2_{0,s}\left[|\hat{K}_{N,s}|^2 + \left(\frac{s}{Mn-s}\right)^8 |\hat{K}_{N,M-s}|^2\right] + N^2\lambda(\tau_s + \tau_{M-s})} \tag{2.36}$$

The filter will of course apply to every Fourier coefficient $q = 0, \pm 1, \pm 2, \ldots,$ but will have only M possible values depending upon q modulo M. The regularization parameter λ is still to be determined.

2.4 Determination of Regularization Parameter λ.

Let the filtered solution $F_M \in B_M(0,T)$, which minimizes $\|K_N * F_M - G_N\|_2^2 + \lambda\|F_M''\|_2^2$ be given by (we have $p = 2$)

$$F_M(x) = \sum_{q=-\infty}^{\infty} \hat{F}_{M,q} \exp(i\omega_q x) \tag{2.37}$$

Consider

$$\begin{aligned}\hat{G}_{N,\lambda,q} &= \hat{K}_{N,q}\hat{F}_{M,q}, \quad q = 0,1,\ldots,N-1 \\ &= \begin{cases} \sqrt{M}\hat{B}_{0,1}\hat{K}_{N,q}\hat{\alpha}_s, & s \equiv q (\text{mod } M) \\ 0, & \text{otherwise} \end{cases} \quad \text{for } q = 0,1,2,\ldots N-1 \end{aligned} \tag{2.38}$$

We now introduce the $N \times N$ influence matrix

$$A(\lambda) = \psi_N \tilde{A}(\lambda)\psi_N^H$$

where

$$\underline{\hat{G}}_{N,\lambda} = \tilde{A}(\lambda)\underline{G}_N \tag{2.39}$$

$\hat{A}(\lambda)$ is block–diagonal with the following structure

$$\hat{A}(\lambda) = \left[\begin{array}{c|c} \text{diag } \underline{a}_1 & \text{diag } \underline{a}_2 \\ \hline \text{diag } \underline{a}_3 & \text{diag } \underline{a}_4 \end{array}\right] \tag{2.40}$$

where $\underline{a}_k \in C^M, \quad K = 1,2,3,4$ and

$$a_{1,s} = \begin{cases} \dfrac{\sqrt{M}(\hat{B}_{0,s})^2|\hat{K}_s|^2}{D_s}, & s = 0 \\ \dfrac{\sqrt{M}(\hat{B}_{0,s})^2|\hat{K}_s|^2}{2D_s}, & 1 \le s \le M-1 \end{cases}$$

$$a_{2,s} = \begin{cases} 0, & s = 0 \\ \dfrac{\sqrt{M}(\hat{B}_{0,s})^2\left(\frac{s}{M-s}\right)^4}{\hat{K}_s}\overline{\hat{K}}_{M+s}, & 1 \le s \le M-1 \\ \dfrac{\sqrt{M}\hat{K}_{M+s}\hat{B}_{0,M+s}\hat{B}_{0,s}\overline{\hat{K}}_s}{D_s}, & s = 0 \\ \dfrac{\sqrt{M}\hat{K}_{M+s}\hat{B}_{0,M+s}\hat{B}_{0,s}\overline{\hat{K}}_s}{2D_s}, & 1 \le s \le M-1 \end{cases}$$

$$a_{4,s} = \begin{cases} 0, & s = 0 \\ \dfrac{\sqrt{M}\hat{B}_{0,M+s}\hat{B}_{0,s}\left(\frac{s}{M+s}\right)^4|\hat{K}_{M+s}|^2}{2D_s}, & 1 \le s \le M-1 \end{cases}$$

where

$$D_s = M\hat{B}_{0,s}^2\left[|\hat{K}_s|^2 + \left(\frac{s}{M-s}\right)^8|\hat{K}_{M-s}|^2\right] + \lambda N^2(\tau_s + \tau_{M+s}).$$

For simplicity of notation we have written $\hat{K}_s$ for $\hat{K}_{N,s}$ in $a_{1,s}, a_{2,s}, a_{3,s}, a_{4,s}$ and D_s. The optimal λ as defined by GCV method may be found in Wahba [23]. Now minimizing the expression

$$V(\lambda) = \frac{\frac{1}{N}\|I - \hat{A}(\lambda)\underline{\hat{G}}_N\|_2^2}{\left[\frac{1}{N}\text{ Trace}(I - \hat{A}(\lambda))\right]^2} \tag{2.41}$$

which from equation (2.38) can be written as

$$V(\lambda) = \frac{\frac{1}{N}\left\{\sum_{s=0}^{M-1} |(1 - a_{1,s})\hat{G}_s - a_{2,s}\overline{\hat{G}}_{M-s}|^2 + \sum_{s=0}^{M-1} |(1 - a_{4,s})\overline{\hat{G}}_{M-s} - a_{3,s}\hat{G}_s|^2\right\}}{\left[1 - \frac{1}{N}\sum_{s=0}^{M-1}(a_{1,s} + a_{4,s})\right]^2}. \tag{2.42}$$

In order to minimize $V(\lambda)$ in equation (2.40) we have used a subroutine which uses a quadratic interpolation technique to obtain a minimum.

3. Addition of Random Noise to the Data Functions.

In solving the problems (1–5), we have considered the data functions contaminated by varying amounts of random noise. To generate sequences of random errors of the form $\epsilon_n, n = 0, 1, 2, \ldots, N-1$, we have used a subroutine which returns pseudo–random real numbers taken from a normal distribution of prescribed mean A and standard deviation B.

To mimic experimental errors we have

$$A = 0 \text{ and } B = \left(\frac{X}{100}\right)\max_{0\leq n\leq N-1}|G_n| \tag{3.1}$$

where X denotes a chosen percentage. In all our test problems we have taken $x = 0.7$ because Laplace transform inversion is a severely ill–posed problem. Thus the random error ϵ_n added to G_n does snot exceed $3X\%$ of the maximum value of $G(x)$.

4. Optimal Convergence.

Here we shall discuss the optimal convergence to the case of convolution equation (2.3).

$$\int_0^T K(x-y)F(y)dy = G(x),$$

in which the function K is periodic of period T. Now we expand K in a Fourier series

$$K(x-y) = \sum_{q=-\infty}^{\infty} \hat{K}_q \exp\left(\frac{2\pi i}{T}q(x-y)\right) \tag{4.1}$$

where $\hat{K}_q$ is the Fourier coefficient, i.e.

$$\hat{K}_q = \int_0^T K(y)\exp\left(\frac{2\pi i}{T}qy\right)dy = \overline{\hat{K}}_{N-q}. \tag{4.2}$$

We shall assume that

$$|\hat{K}_q| \simeq q^{-K}, \quad K > 1$$

and so the Fourier series is uniformly convergent.

5. Test Problems.

Problem (1).

This problem has been taken from Varah [22]

$$g(s) = \frac{1/2}{s(s+1/2)}$$

$$f(t) = 1 - e^{-t/2}$$

The optimal results are shown in diag (1), table 1 and Varah's results in diag (6).

Problem (2)

This problem has also been taken from Varah [22].

$$g(s) = \frac{2}{\left(s + \frac{1}{2}\right)^3}$$
$$f(t) = t^2\, e^{-t/2}$$

The optimal results are shown in diag (2), table 1 and Varah's results in diag (7).

Problem 3

This problem has been taken from McWhirter and Pike [11] and Brianzi [1, 2].

$$g(s) = \frac{1}{(s+1)^2}$$
$$f(t) = t\, e^{-t}$$

The optimal results are shown in diag (3), table 1 and McWhirter's results in diag (8).

Problem (4)

This problem has been taken from Chauveau [3]

$$g(s) = \frac{\lambda}{\lambda + s}$$
$$f(t) = \lambda\, e^{-\lambda t} \quad \text{for } \lambda = 5.0$$

The optimal results are shown in diag (4), table 1 and Chauaveau's results in diag (9).

Problem (5)

This problem has been taken from Cristina [4] and Giunta [8], and we made the problem more ill–posed by taking higher value of α and adding noise.

$$g(s) = \frac{\beta}{(s+\alpha)^2 + \beta^2}$$

$$f(t) = e^{-\alpha t} \sin \beta t, \qquad \text{for } \alpha = 3.0 \text{ and } \beta = 1.0$$

The optimal results ar shown in diag (5) and table 1.

6. Numerical Results.

In this section we tabulate the results of the above method applied to the test problems taken from [1, 2, 3, 8, 11, 22]. All data functions have the property $g(s) = 0(s^{-1})$ and 0.7% noise has been added apart from machine rounding error; only optimal results have been quoted in table 1 and demonstrated in the respective diagrams. In each of the test problems, 64 sample points are used to calculate the discrete Fourier coefficients.

In our numerical calculations, we need to choose two numbers $x_{\min}$ and $x_{\max}$. We find $x_{\min}$ and $x_{\max}$ as the smallest and largest solutions of the non–linear equation $|G(x)| = \epsilon$ where $\epsilon = 10^{-4}$. We may then pose the deconvolution problem (2.3) on the interval $[0, T]$, where $T = x_{\max} - x_{\min}$. Since the size of the essential support of $G(x)$ depends upon 'a', we have for a fixed number N of equidistant data points $\{x_n\}$, $h = T/N$, we have minimized (2.40), with respect to λ for values of $a > 1$ and compared the L_∞ error of the resulting solution with the values of the true solutions. We find the minimum value of $V(\lambda)$ which yields the L_∞ error of the regularized solution as the least.

Table 1

Test Problem	a	T	h	λ	$V(\lambda)$	$\|F - F_\lambda\|_\infty$	diag
1	10.0	9.60	0.150	0.31×10^{-8}	0.56358	0.01	1
2	15.0	12.10	0.18906	0.101×10^{-10}	0.6207×10^{-8}	0.015	2
3	10.0	11.60	0.18125	0.11×10^{-8}	0.1167×10^{-8}	0.010	3
4	10.0	12.20	0.19062	0.1009×10^{-5}	0.1411×10^{2}	0.070	4
5	8.0	12.60	0.19688	0.11×10^{-8}	0.1013×10^{-9}	0.007	5

Conclusion

Our method worked very well over all the test problems and the results obtained are shown in diags (1–5) and table 1. As regards comparison with other available methods, the authors of the other methods obtained the results with clean data and we have added 0.7% noise in the data functions which make the problem more ill–posed. We have compared our results with others i.e. with Varah [22], McWhirter and Pike [11], Brianzi [1, 2], and Chauveau [3], which are quoted in diags (6–9). The results of problem (5) are not available to compare with.

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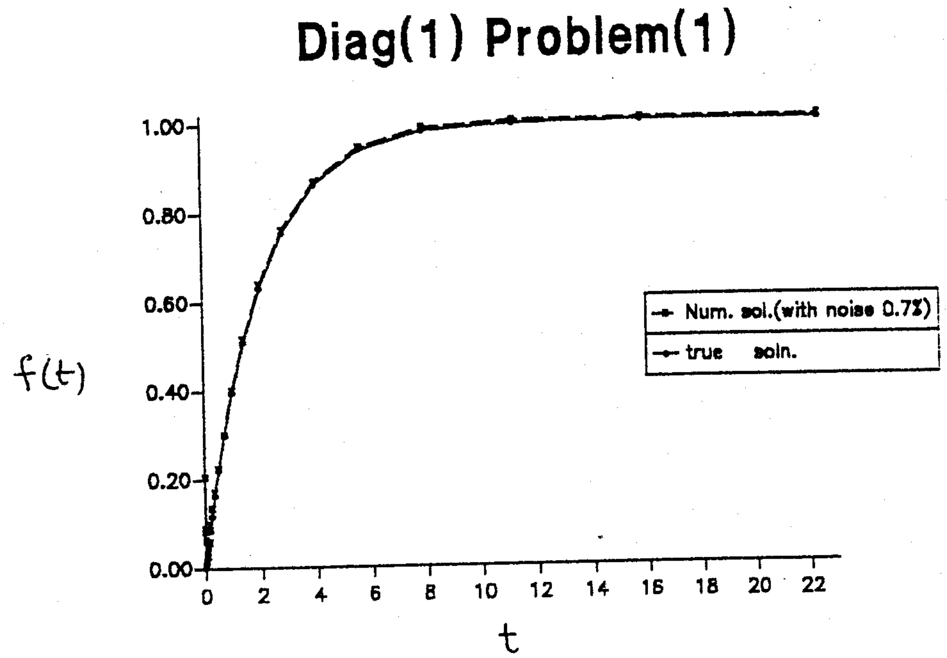
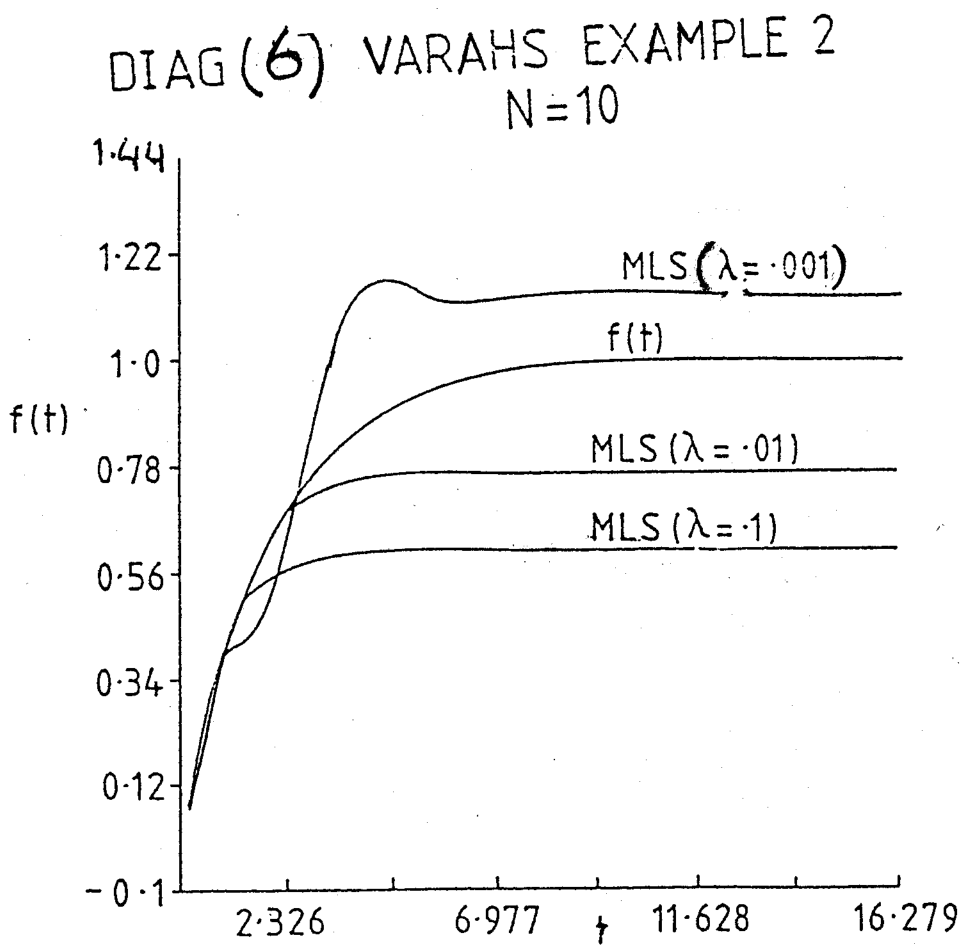
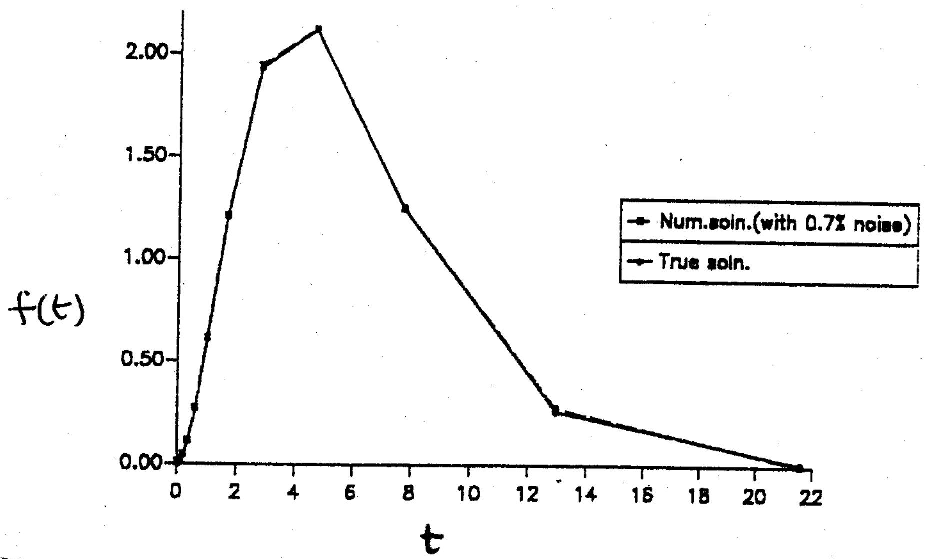


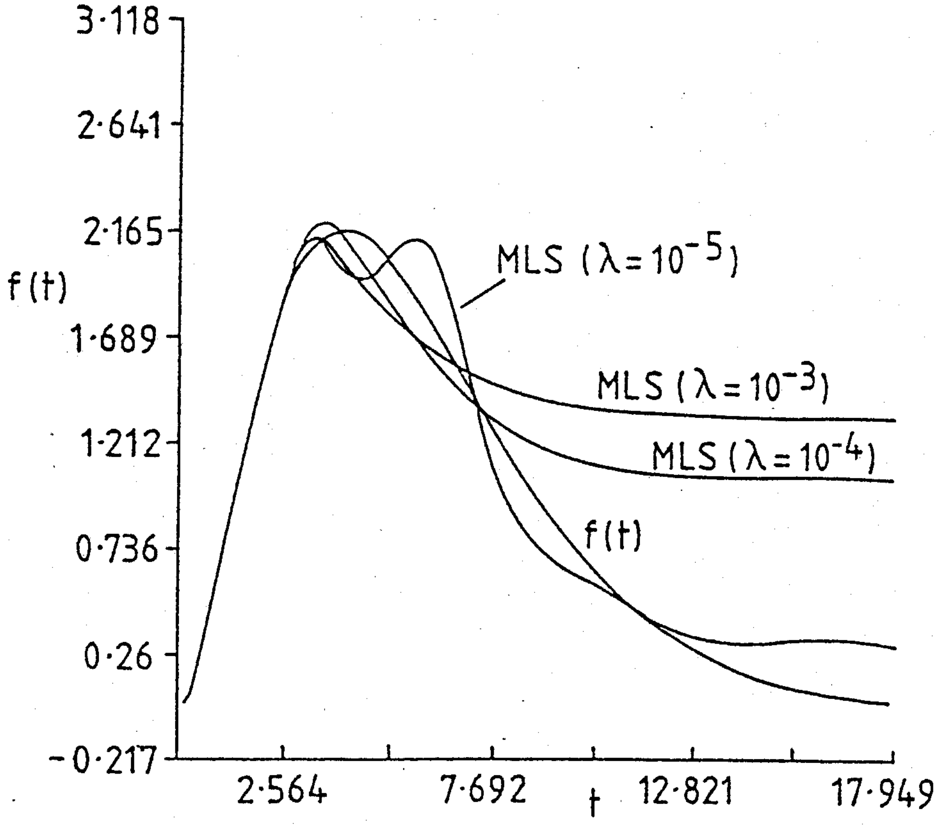


MLS solutions for Example 2.

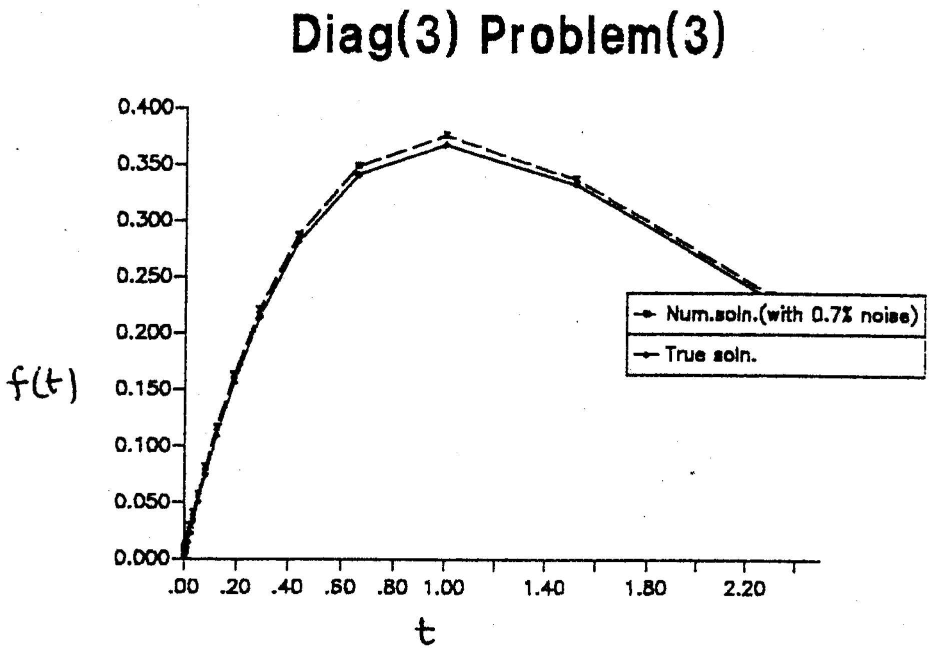
Diag(2) Problem(2)



DIAG (7) VARAH'S EXAMPLE 3
N = 20



MLS solutions for Example 3.



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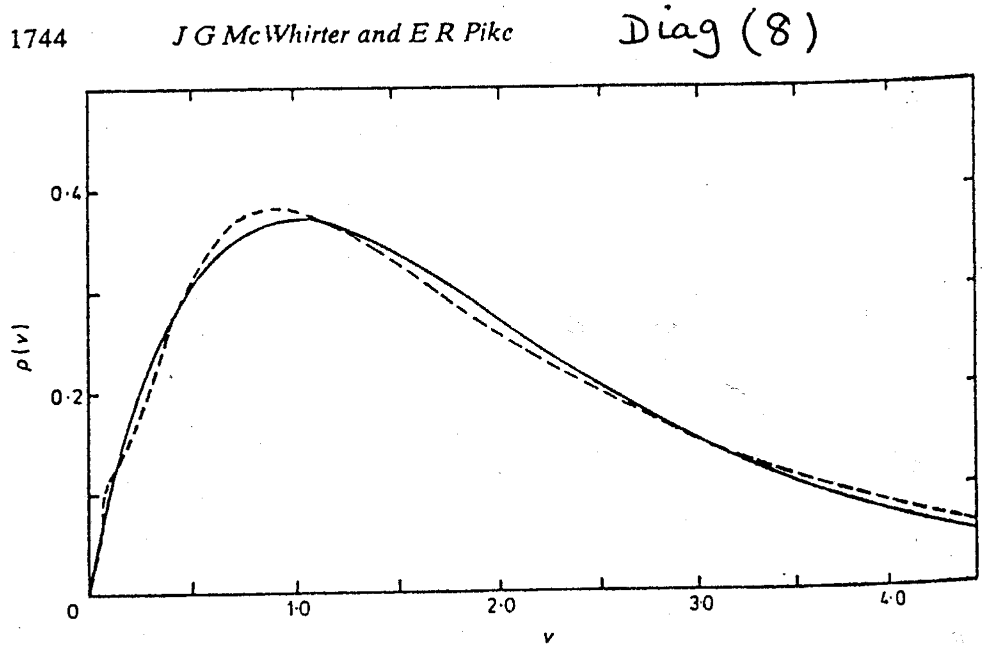
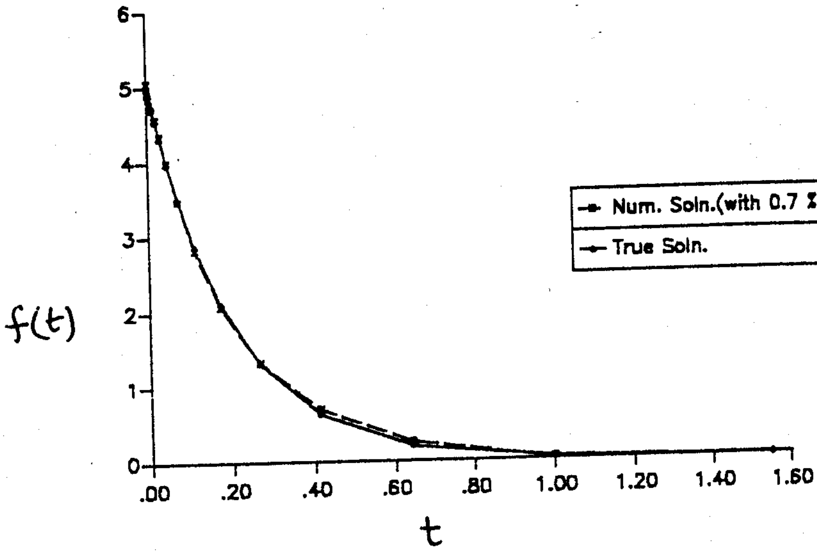


Diag(4) Problem(4).



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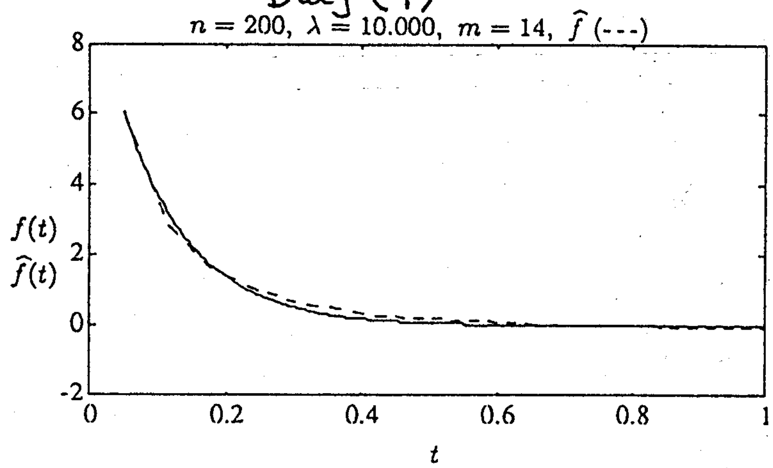
Diag (9)



Diag(5) Problem(5).