King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 221

June 1997

An Analytical Inversion of the Laplace Transform

Amin Boumenir, Abdulaziz Al-Shuaibi

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

An Analytical Inversion of the Laplace Transform

Abstract

By comparing the Laplace transform $\mathcal{L}$ with the differential operator D, that is comparing two self-adjoint operators, we shall obtain a formula for the inverse Laplace transform $\mathcal{L}^{-1} = \frac{1}{\pi}\mathcal{L}V^{-1}\cos(\pi D)V$, where V is a change of variable. This will lead to the deduction of an explicit spectral representation of $\mathcal{L}$. Some applications are discussed.

1. Introduction

There is now an extensive but scattered literature on methods of inverting the Laplace transform. An extensive bibliography up to 1975 has been compiled by Piessen [1]. Bellman's book [2] shows a wide range of applications of numerical inversion but is limited in its coverage of inversion methods. Krylov and Skoblya [3] cover the theoretical basis of a number of inversion methods but do not consider implementation or present numerical results. The survey paper by Davies and Martin [4], which tests 14 inversion procedures on a set of 16 transforms, is a major contribution. However, only few authors had approached the inversion of the Laplace transform by considering its spectrum and spectral measure from an operator-theoretic point of view [5], [6] , [7]. This approach is going to be our main concern.

It is known that the range of the Laplace transform is in the space of analytic functions. Therefore the domain of the inverse Laplace transform is contained in the set of entire functions on the right half plane. Recall that operators acting on the space of entire functions can be represented by differential operators of infinite order. One way of verifying this result for the inverse Laplace transform is to use methods of spectral theory. For this we shall restrict the Laplace transform to an operator mapping $L^2[0,\infty)$ into $L^2[0,\infty)$.

$$\mathcal{L} : L^2[0,\infty) \longrightarrow L^2[0,\infty)$$

$$y(x) \quad \longrightarrow \quad \mathcal{L}(y(x))(s) \equiv \int_0^\infty e^{-sx} y(x) dx. \tag{1}$$

This is an integral operator with a symmetric kernel, and we shall try to show that it is a selfadjoint operator in $L^2[0,\infty)$. It is clear that

$$\mathcal{L}^2(y(x))(s) = \int_0^\infty \frac{y(x)}{s+x} dx. \tag{2}$$

Thus $\mathcal{L}^2$ can be seen as composition of projections and the Hilbert transform. We now use comparison techniques to find a spectral representation of the Laplace operator. To this end consider the following transformation V,

$$\begin{aligned} V : L^2[0,\infty) \quad &\longrightarrow \quad L^2(-\infty,\infty) \\ y \quad &\longrightarrow \quad Vy(x) = e^{x/2} y(e^x). \end{aligned} \tag{3}$$

It is easy to show that V is a unitary transformation, i.e.,

$$VV^* = I, \qquad \text{and} \quad V^{-1} = V^*$$

and the inverse operator is defined by

$$\begin{aligned} V^{-1} : L^2(-\infty,\infty) \quad &\longrightarrow \quad L[0,\infty) \\ g(x) \quad &\longrightarrow \quad V^{-1}g(x) = \frac{1}{\sqrt{x}} g(Lnx). \end{aligned}$$

The study of $\mathcal{L}^2$, involves the operator defined by $A \equiv V\mathcal{L}^2V^{-1}$, that is

$$\begin{aligned} A : \quad & L^2(-\infty,\infty) \longrightarrow L^2(-\infty,\infty) \\ & y(x) \rightarrow Ay(x) \equiv k * y = \int_{-\infty}^\infty k(x-\eta) y(\eta) d\eta \end{aligned} \tag{4}$$

where, $k(x) = \frac{1}{2\cosh\frac{x}{2}}$. A is obviously an integral operator of the convolution type.

We shall denote the Fourier transform $\wedge$ and its inverse by $\wedge^{-1}$

$$\wedge : f \longrightarrow \hat{f}(\lambda) \equiv \int_{-\infty}^\infty f(x) e^{i\lambda x} dx$$

$$\wedge^{-1} : \hat{f}(\lambda) \longrightarrow f(x) \equiv \int_{-\infty}^\infty \hat{f}(\lambda) e^{-i\lambda x} d\frac{\lambda}{2\pi}.$$

Theorem 1.1: *The operator A defined on $L^2(-\infty,\infty)$ by (4) is a bounded selfadjoint operator with $\|A\| = \pi$.*

Proof: It is easy to show that the Fourier transform of k is

$$\hat{k}(\lambda) = \frac{\pi}{\cosh \pi\lambda}. \tag{5}$$

and so $\hat{k}(\lambda)$ is a bounded function. Using the Parseval relation $\|f\| = \|\hat{f}\|$ and the fact that $\sup_{\lambda\in\mathcal{R}} |\hat{k}(\lambda)| \le \pi$ we have for any $f \in L^2(-\infty,\infty)$;

$$\|Af\| = \|k * f\| = \|\hat{k}\hat{f}\| \le \sup_{\lambda\in\mathcal{R}} |\hat{k}(\lambda)| \|\hat{f}\| \le \pi \|\hat{f}\| = \pi \|f\|.$$

Hence; $D_A = L^2(-\infty,\infty)$, and $\|A\| \le \pi$. In fact the equality holds. This is shown by choosing

$$f_n \text{ such that } \hat{f}_n(\lambda) = \begin{cases} \sqrt{n} & \frac{-1}{2n} < \lambda < \frac{1}{2n} \\ 0 & \text{otherwise .} \end{cases}$$

Then

$$\int_{-\infty}^{\infty} \left[\hat{f}_n(\lambda)\right]^2 d\lambda = 1 \text{ and } \operatorname{supp} \hat{f}_n(\lambda) \to \{0\}.$$

Hence,

$$\lim_{n\to\infty} \|Af_n\| = \lim_{n\to\infty} \sqrt{\int [\hat{k}(\lambda)]^2 [\hat{f}_n(\lambda)]^2 d\lambda} = \lim_{n\to\infty} \sqrt{[\hat{k}(0)]^2} \equiv |\hat{k}(0)| = \pi.$$

Since A is bounded and symmetric, it is a selfadjoint operator in $L^2(-\infty,\infty)$.

Corollary 1.1 *The operator $\mathcal{L}^2$ is a bounded selfadjoint operator and,*

$$\|\mathcal{L}^2\| = \pi.$$

Proof: Follows from the fact that A is a bounded selfadjoint operator (Theorem 1.1) and $\mathcal{L}^2$ is unitarily equivalent to A.

2. The Spectral Representation of $\mathcal{L}^2$:

In order to find the spectral function of $\mathcal{L}^2$, we only need to find the spectral function of A. It is known that the spectrum is invariant under a unitary transformation of the operator.

By using Parseval relation for the Fourier transform

$$(Af,\psi) \equiv (k*f,\psi) \equiv (\widehat{k*f},\hat{\psi}) \equiv (\hat{k}(\lambda)\hat{f}(\lambda),\hat{\psi}(\lambda))$$

where, $\hat{f}(\lambda) \equiv \int_{-\infty}^{+\infty} f(x)\, e^{i\lambda x} dx$. From this we deduce,

$$(Af,\psi) = \int_{-\infty}^{\infty} \hat{k}(\lambda)\hat{f}(\lambda)\,\overline{\hat{\psi}(\lambda)}\, d\frac{\lambda}{2\pi}. \tag{6}$$

Let $a(\lambda)$ be the inverse of $\hat{k}(\lambda)$ defined by

$$a(\lambda) \equiv -\frac{1}{\pi} ln[\frac{\pi}{\lambda} + \sqrt{\frac{\pi^2}{\lambda^2} - 1}].$$

Clearly

$$\frac{\pi}{\cosh \pi a(\lambda)} = \lambda, \quad a(\lambda) < 0 \quad \lambda > 0.$$

We have from (6)

$$\begin{aligned} (Af,\psi) &= \int_0^{\pi} \lambda \hat{f}(a(\lambda)) \cdot \overline{\hat{\psi}(a(\lambda))}\, d\frac{a(\lambda)}{2\pi} + \int_{\pi}^{0} \lambda \hat{f}(-a(\lambda))\overline{\hat{\psi}(-a(\lambda))}\, d\frac{-a(\lambda)}{2\pi} \\ (Af,\psi) &= \int_0^{\pi} \lambda \left[\left\{\hat{f}(a(\lambda) \cdot \overline{\hat{\psi}(a(\lambda))}\right\} + \left\{\hat{f}(-a(\lambda)) \cdot \overline{\hat{\psi}(-a(\lambda))}\right\}\right] d\frac{a(\lambda)}{2\pi}. \end{aligned}$$

The above equation can be written in the matrix form as

$$(Af,\psi) = \int_0^{\pi} \lambda \left[\hat{f}(a(\lambda)),\ \hat{f}(-a(\lambda))\right] \begin{bmatrix} d\frac{a(\lambda)}{2\pi} & 0 \\ 0 & d\frac{a(\lambda)}{2\pi} \end{bmatrix} \overline{\begin{bmatrix} \hat{\psi}(a(\lambda)) \\ \hat{\psi}(-a(\lambda)) \end{bmatrix}}. \tag{7}$$

This only means that $\begin{bmatrix} e^{ixa(\lambda)} \\ e^{-ixa(\lambda)} \end{bmatrix}$ is the eigenfunctional of $\mathcal{L}^2$, so the multiplicity is two, and the associated spectral matrix is $\begin{bmatrix} d\frac{a(\lambda)}{2\pi} & 0 \\ 0 & d\frac{a(\lambda)}{2\pi} \end{bmatrix}$, (cf. [8]).

Proposition 2.1: *The spectral function of A is* $\begin{bmatrix} d\frac{a(\lambda)}{2\pi} & 0 \\ 0 & d\frac{a(\lambda)}{2\pi} \end{bmatrix}$ *and the multiplicity is two and* $\sigma = \text{supp}\, da(\lambda) = [0,\pi]$.

Now we use our relation $A = V\mathcal{L}^2V^{-1}$ to deduce the spectral representation for $\mathcal{L}^2$

$$(\mathcal{L}^2 f,\psi) = (V^{-1}AVf,\ \psi) = (AVf,\ V\psi) = (\lambda\widehat{Vf},\widehat{V\psi}).$$

Define the transform $\overset{\wedge_2}{}$;

$$\overset{\wedge_2}{f}(\lambda) = \int_0^\infty f(x) x^{-\frac{1}{2}+ia(\lambda)}\, dx.$$

Then, we obtain the following relation between $\overset{\wedge_2}{}$ and $\wedge$ by using the isometry V

$$\overset{\wedge_2}{f}(\lambda) = \int_0^\infty f(x) x^{-\frac{1}{2}+ia(\lambda))}\, dx = \int_{-\infty}^\infty Vf(x)\, e^{ixa(\lambda)}\, dx = \overset{\wedge}{Vf}(a(\lambda)). \tag{8}$$

Hence, equation (7) implies;

$$\begin{aligned}
(\mathcal{L}^2 f, \psi) &= (AVf, V\psi) \\
&= \int_0^\pi \left[\overset{\wedge}{Vf}(a(\lambda)), \overset{\wedge}{Vf}(-a(\lambda))\right] \begin{bmatrix} d\frac{a(\lambda)}{2\pi} & 0 \\ 0 & d\frac{a(\lambda)}{2\pi} \end{bmatrix} \overline{\begin{bmatrix} \overset{\wedge}{V\psi}(a(\lambda)) \\ \overset{\wedge}{V\psi}(-a(\lambda)) \end{bmatrix}} \\
&= \int_0^\pi \left[\overset{\wedge_2}{f}(\lambda), \overline{\overset{\wedge_2}{f}}(\lambda)\right] \begin{bmatrix} d\frac{a(\lambda)}{2\pi} & 0 \\ 0 & d\frac{a(\lambda)}{2\pi} \end{bmatrix} \overline{\begin{bmatrix} \overset{\wedge_2}{\psi}(\lambda) \\ \overline{\overset{\wedge_2}{\psi}(\lambda)} \end{bmatrix}}
\end{aligned}$$

This simply means that our spectrum is continuous of multiplicity two covering $[0, \pi]$, which is the support of the spectral matrix function $\mu(\lambda) = \begin{bmatrix} \frac{a(\lambda)}{2\pi} & 0 \\ 0 & \frac{a(\lambda)}{2\pi} \end{bmatrix}$, and the corresponding eigenfunctionals are $\begin{bmatrix} x^{-\frac{1}{2}+ia(\lambda)} \\ x^{-\frac{1}{2}-ia(\lambda)} \end{bmatrix}$.

3 The Spectral Resolution of $\mathcal{L}$:

In what follows we shall consider the square root $\mathcal{L}^2$. For simplicity set $w(\lambda) \equiv -\frac{1}{2} + ia(\lambda^2)$.

Now, with the help of the eigenfunctionals of $\mathcal{L}^2$ we shall construct the eigenfunctionals of $\mathcal{L}$ as follows: it is easy to see that the following

$$y(x, \lambda) = \Gamma(1+w) x^{\overline{w}} + s(\lambda)|\Gamma(1+w)| x^{w} \tag{9}$$

satisfies $\mathcal{L}y = \lambda y$ where, $s(\lambda) = \begin{cases} +1 & \lambda > 0 \\ -1 & \lambda < 0 \end{cases}$ and $\lambda = s(\lambda)|\Gamma(1 + w(\lambda))|$.

Define the transform $\overset{\wedge_1}{} : \overset{\wedge_1}{f}(\lambda) = \int_0^\infty f(x)y(x,\lambda)dx$ and recall that $\hat{f}^2 = \int_0^\infty f(x)x^w dx$. Observe that

$$\overset{\wedge_1}{f}(\lambda) = \Gamma(1+w)\cdot \overline{\overset{\wedge_2}{f}}(\lambda^2) + s(\lambda)|\Gamma(1+w)|\,\overset{\wedge_2}{f}(\lambda^2). \tag{10}$$

It is clear that the multiplicity of the spectrum of $\mathcal{L}$ is either 1 or two. We claim that equation (10) defines a complete system of eigenfunctionals and so the multiplicity is only one. Recall that a system of eigenfunctionals is complete if

$$\hat{f}^1(\lambda) = 0 \text{ for } \lambda \in \sigma \quad \Longrightarrow \quad f = 0 \text{ in } L^2[0,\infty).$$

Since both $\lambda \in \sigma$ and $-\lambda \in \sigma$, we obtain the following system

$$\overset{\wedge_1}{f}(\lambda) = 0 = \Gamma(1+w)\cdot \overline{\overset{\wedge_2}{f}}(\lambda^2) + |\Gamma(1+w)|\,\overset{\wedge_2}{f}(\lambda^2) \tag{11}$$

$$\overset{\wedge_1}{f}(-\lambda) = 0 = \Gamma(1+w)\cdot \overline{\overset{\wedge_2}{f}}(\lambda^2) - |\Gamma(1+w)|\,\overset{\wedge_2}{f}(\lambda^2) \tag{12}$$

The determinant of the system being non-zero means that $\overline{\overset{\wedge_2}{f}}(\lambda^2) = 0$ and $\overset{\wedge_2}{f}(\lambda^2) = 0$. That is $\overline{\overset{\wedge_2}{f}} = 0$.

This transition formula (11), together with Parseval relation, implies for all $f, \psi \in L^2[0,+\infty)$.

$$(\mathcal{L}f, \mathcal{L}\psi) = (\mathcal{L}^2 f, \psi).$$

For the left-hand-side use the Parseval equality associated with the operator $\mathcal{L}$ and for the right-hand-side use the parseval associated with $\mathcal{L}^2$ to obtain

$$\int_\sigma \lambda^2\,\overset{\wedge_1}{f}(\lambda)\overline{\overset{\wedge_1}{\psi}}(\lambda)d\rho(\lambda) = \int_0^\pi \lambda\left[\overset{\wedge_2}{f}(\lambda)\overline{\overset{\wedge_2}{\psi}}(\lambda) + \overline{\overset{\wedge_2}{f}}(\lambda)\,\overset{\wedge_2}{\psi}(\lambda)\right] d\frac{a(\lambda)}{2\pi}. \tag{13}$$

Now, we use this equation to evaluate our spectral function ρ of $\mathcal{L}$ and recall that

$$|\Gamma(1+w)|^2 = \lambda^2.$$

We now compute the left-hand-side of equation (14) using equation (11)

$$\int_\sigma \lambda^2\,\overset{\wedge_1}{f}(\lambda)\,\overset{\wedge_1}{\psi}(\lambda)d\rho(\lambda) = \int_\sigma \lambda^2\left\{\Gamma(1+w)\overline{\overset{\wedge_2}{f}}(\lambda^2) + s(\lambda)|\Gamma(1+w)|\,\overset{\wedge_2}{f}(\lambda^2)\right\}\cdot$$

$$
\begin{aligned}
\cdot & \left\{\overline{\Gamma(1+w)}\,\overset{\wedge_2}{\psi}(\lambda^2)+s(\lambda)|\Gamma(1+w)|\overline{\overset{\wedge_2}{\psi}(\lambda^2)}\right\} d\rho(\lambda) \\
= & \int_\sigma \lambda^4 \left\{\overline{\overset{\wedge_2}{f}(\lambda^2)}\,\overset{\wedge_2}{\psi}(\lambda^2)+\overset{\wedge_2}{f}(\lambda^2)\,\overline{\overset{\wedge_2}{\psi}(\lambda^2)}\right\} d\rho(\lambda) \\
+ & \int_\sigma S(\lambda)\left\{\overline{\Gamma(w+1)}|\Gamma(1+w)|\,\overset{\wedge_2}{f}(\lambda^2)\,\overset{\wedge_2}{\psi}(\lambda^2)+\right. \\
+ & \left.\Gamma(1+w)|\Gamma(1+w)|\overline{\overset{\wedge_2}{f}(\lambda^2)}\,\overline{\overset{\wedge_2}{\psi}(\lambda^2)}\right\} d\rho(\lambda)
\end{aligned}
$$

Since the second integrand is an odd function, we only need to assume that the $\rho(\lambda)$ is an odd function for the above expression to reduce to

$$
\int_\sigma \lambda^2\,\overset{\wedge_1}{f}(\lambda)\,\overset{\wedge_1}{\psi}(\lambda)d\rho(\lambda)=\int_\sigma \lambda^4 \left\{\overline{\overset{\wedge_2}{f}(\lambda^2)}\,\overset{\wedge_2}{\psi}(\lambda^2)+\overset{\wedge_2}{f}(\lambda^2)\,\overline{\overset{\wedge_2}{\psi}(\lambda^2)}\right\} d\rho(\lambda)
$$

and use equation (14) to obtain

$$
\begin{aligned}
\int_\sigma \lambda^4\{\overline{\overset{\wedge_2}{f}(\lambda^2)}\,\overset{\wedge_2}{\psi}(\lambda^2) \;+\; \overset{\wedge_2}{f}(\lambda^2)\,\overline{\overset{\wedge_2}{\psi}(\lambda^2)}\}d\rho(\lambda) &= \\
\int_0^\pi \lambda\left[\overline{\overset{\wedge_2}{f}(\lambda)}\,\overset{\wedge_2}{\psi}(\lambda)+\overset{\wedge_2}{f}(\lambda)\,\overline{\overset{\wedge_2}{\psi}(\lambda)}\right] d\frac{a(\lambda)}{2\pi} & \\
2\int_{\sigma+} \lambda^4\{\overline{\overset{\wedge_2}{f}(\lambda^2)}\,\overset{\wedge_2}{\psi}(\lambda^2) \;+\; \overset{\wedge_2}{f}(\lambda^2)\,\overline{\overset{\wedge_2}{\psi}(\lambda^2)}\}d\rho(\lambda) &= \\
\int_0^\pi \lambda\left[\overline{\overset{\wedge_2}{f}(\lambda)}\,\overset{\wedge_2}{\psi}(\lambda)+\overset{\wedge_2}{f}(\lambda)\,\overline{\overset{\wedge_2}{\psi}(\lambda)}\right] d\frac{a(\lambda)}{2\pi} &
\end{aligned}
$$

where $\sigma+ \equiv \sigma \cap [0,+\infty)$. Therefore we need to have

$$
2\lambda^2 d\rho(\sqrt{\lambda}) = \lambda d\frac{a(\lambda)}{2\pi} \qquad \lambda \geq 0
$$

, from which we deduce that

$$
\frac{\rho'(\sqrt{\lambda})\lambda}{\sqrt{\lambda}} = \frac{a'(\lambda)}{2\pi} = \frac{1}{2\pi\lambda\sqrt{\pi^2-\lambda^2}}
$$

and conclude that

$$
\rho'(\lambda) = \frac{1}{2\pi\lambda^3\sqrt{\pi^2-\lambda^4}} \qquad 0 \leq \lambda \leq \sqrt{\pi}.
$$

To recover all of $\rho(\lambda)$ on σ, we need to recall that $\rho(\lambda)$ is an odd function

$$
\rho'(\lambda) = \frac{1}{2\pi\lambda^3 s(\lambda)\sqrt{\pi^2-\lambda^4}} \qquad -\sqrt{\pi} \leq \lambda \leq \sqrt{\pi}
$$

where $s(\lambda) = \begin{cases} +1 & \lambda > 0 \\ -1 & \lambda < 0 \end{cases}$.

This formula coincides with the one found in [5].

4 Construction of a Real Inversion Formula by Using Pseudo-Differential Operators

Let f and its Laplace transform be in $L^2[0,\infty)$, V be the change of variables defined in (3), and t $D = \frac{d}{dx}$. Then the following theorem furnishes a real inversion formula for $\mathcal{L}^{-1}$.

Theorem 4.1: $\mathcal{L}^{-1} = \frac{1}{\pi}\mathcal{L}V^{-1}\cos(\pi D)V$.

Proof: It is known that convolution operators can be represented as differential operators. Indeed from (4) and (5)

$$Af(x) \equiv k * f(x).$$

So by taking the Fourier transform

$$\begin{aligned}
\widehat{Af}(\lambda) &= \hat{k}(\lambda)\hat{f}(\lambda) \\
&= \frac{\pi}{\cosh \pi\lambda}\hat{f}(\lambda) \\
\widehat{A^{-1}f}(\lambda) &= \frac{\cosh \pi\lambda}{\pi}\hat{f}(\lambda) \\
&= \frac{\widehat{\cosh(\pi \frac{-id}{dx})} f}{\pi}(\lambda) \\
A^{-1}f &= \frac{1}{\pi}\cos(\pi\frac{d}{dx})f(x).
\end{aligned}$$

Thus

$$\begin{aligned}
\mathcal{L}^{-2} &= V^{-1}A^{-1}V = V^{-1}\frac{1}{\pi}\cos(\pi D)\ V \\
\mathcal{L}^{-1} &= \frac{1}{\pi}\mathcal{L}V^{-1}\cos(\pi D)V
\end{aligned} \tag{14}$$

where $D \equiv \frac{d}{dx}$.

The analyticity of F helps a great deal in utilizing the formula of Theorem 4.1. That is, one can always expand F in terms of its Fourier series in a small neighborhood of the positive real axis. An application of the theorem is given in the following section.

5 Regularization Via Truncation

Direct application of Theorem 4.1 is very difficult. Hence, we need to expand $F(s)$ in terms of its Fourier series, which in turn can only be done if $F(s)$ is periodical, otherwise we need to truncate F and expand safely in a finite interval $\ell \leq s \leq L$.

Now, let us discuss the error we commit by truncating F.

Assumption 5.1: Let $\tilde{\mathcal{R}}$ denote the interval $\mathcal{R} - [Ln\,\ell, Ln\,L]$ and define the error function E_L

$$E_{L,\ell}(\lambda) = \int_{\tilde{\mathcal{R}}} e^{i\lambda x}\, e^{x/2}\, F(e^x)dx.$$

Furthermore, suppose that for some $\alpha > 0$, $\beta > 0$ and $\epsilon > 0$

$$|E_{L,\ell}(\lambda)| \leq \ell^{\beta} \frac{e^{-\pi(1+\epsilon)|\lambda|}}{L^{\alpha}}.$$

Proposition 5.1: *Let $F = \mathcal{L}f$ and suppose we want to recover f from its Laplace transform using the formula $\mathcal{L}^{-1} = \frac{1}{\pi}\mathcal{L}V^{-1}\cos(\pi D)V$, and replacing F by its truncated approximation F_T,*

$$\begin{cases} F_T(s) &= F(s) \quad s \in [\ell, L] \\ &= 0 \qquad \text{elsewhere}\ . \end{cases}$$

Furthermore, suppose that Assumption 5.1 is satisfied. Then the error E_f we commit in recovering f by using F_T is of order $O(L^{-\alpha})\ O\big(\beta^{\ell}\big)$.

Proof: Let f_T and E_f denote the approximation of f using F_T and the error committed respectively. Then,

$$\begin{aligned}
\|E_f\|_2^2 &= \|f - f_T\|_2^2 = \left\|\frac{1}{\pi}\mathcal{L}V\cos(\pi D)\left(\widehat{VF} - \widehat{VF_T}\right)\right\|_2^2 \\
&\leq \frac{1}{\pi}\|\cosh \pi\lambda\,(VF - VF_T)\|_2^2 \\
&= \frac{1}{\pi}\left\|\cosh \pi\lambda\left(\int_{-\infty}^{\infty} e^{ix\lambda}e^{x/2}F(e^x) - \int_{Ln\,\ell}^{Ln\,L} e^{ix\lambda}e^{x/2}F(e^x)\right)dx\right\|_2^2 \\
&= \frac{1}{\pi}\|\cosh \pi\lambda E_{L,\ell}(\lambda)\|_2^2 \\
&\leq \frac{1}{\pi}\left\|\cosh \pi\lambda\, \ell^{\beta}\, e^{-\pi(1+\epsilon)|\lambda|}\right\|_2^2 \\
&= \frac{1}{4\pi}O\big(L^{-\alpha}\big)\ O\big(\ell^{\beta}\big)
\end{aligned}$$

$$\rightarrow \; 0 \qquad \text{as } \ell \rightarrow 0 \text{ and } L \rightarrow \infty.$$

Application 1: We shall work out an example on the computation of the inverse Laplace of a given function by using equation (14). In case $e^{\frac{x}{2}}F(e^x)$ is continuous, it can also be approximated by polynomials over a finite interval. Let

$$F(x) \equiv \sum_{n\leq 0} a_n \frac{(Ln(x))^n}{\sqrt{x}} \quad \text{where } \; n \leq 0 \quad \text{and} \quad 1 \leq x \leq L.$$

Clearly $\mathcal{L}^{-1}F(x) = \mathcal{L}V^{-1}\frac{1}{\pi}\cos(\pi D)VF = \sum a_n \mathcal{L}V^{-1}\frac{1}{\pi}\cos(\pi D)x^n$. We need to evaluate

$$\cos(\pi D)x^n = \sum_{0\leq 2k} \frac{(-1)^k(\pi)^{2k}n!}{2k!(n-2k)!}x^{n-2k}$$

Therefore

$$V^{-1}\frac{1}{\pi}\cos(\pi D)x^n = \sum_{0\leq 2k} \frac{1}{\pi}\frac{(-1)^k(\pi)^{2k}n!}{2k!(n-2k)!}x^{-1/2}(Ln\, x)^{n-2k}.$$

Thus

$$\mathcal{L}V^{-1}\frac{1}{\pi}\cos(\pi D)x^n = \sum_{0\leq 2k} \frac{1}{\pi}\frac{(-1)^k(\pi)^{2k}n!}{2k!(n-2k)!}\mathcal{L}\left[x^{-1/2}(Ln\, x)^{n-2k}\right]$$

and so

$$f(x) = \mathcal{L}^{-1}F(x) = \sum_{n\leq 0} a_n \sum_{0\leq 2k} \frac{1}{\pi}\frac{(-1)^k(\pi)^{2k}n!}{2k!(n-2k)!}\mathcal{L}\left[x^{-1/2}(Ln\, x)^{n-2k}\right].$$

Now, we need to evaluate $\mathcal{L}\left[x^{-1/2}(Ln(x))^n\right]$. For this, we change variables in the definition of the gamma function by putting $t = sx$

$$\begin{aligned}\Gamma(z) &= \int_0^\infty e^{-t}t^{z-1}dt \\ &= s^z\int_0^\infty e^{-sx}x^{z-1}dx.\end{aligned}$$

Putting $x = e^y$, we have

$$s^{-z}\Gamma(z) = \int_{-\infty}^\infty e^{-se^y}e^{zy}dy.$$

Similarly putting $x = e^y$

$$\begin{aligned}h(s) &= \mathcal{L}\left(x^{-1/2}(Ln\, x)^n\right) = \int_0^\infty e^{-sx}x^{-1/2}(Ln\,(x))^n dx \\ &= \int_{-\infty}^\infty e^{-se^y}e^{(\frac{1}{2}+n)y}\,dy.\end{aligned}$$

Hence

$$h(s) = \frac{\Gamma(\frac{1}{2}+n)}{s^{\frac{1}{2}+n}}.$$

Finally, we get

$$f(x) = \mathcal{L}^{-1}F(x) = \sum_{n\leq 0} a_n \sum_{2k\geq 0} \frac{1}{\pi}\frac{(-1)^k(\pi)^{2k} n!\Gamma(\frac{1}{2}+n-2k)}{(2k)!(n-2k)!x^{\frac{1}{2}+n-2k}}.$$

Example: Consider $F(s) = \frac{1}{\sqrt{s}}$, which is the Laplace transform of $f(t) = \frac{1}{\sqrt{\pi t}}$. Using application 1, we have

$$e^{x/2}F(e^x) = 1.$$

Then,

$$\begin{aligned} f(t) &= \mathcal{L}V^{-1}\frac{\cos(\pi D)}{\pi}VF \\ &= \mathcal{L}V^{-1}\frac{\cos(\pi D)}{\pi}(1) \\ &= \mathcal{L}\left(\frac{1}{\pi\sqrt{x}}\right) \\ &= \frac{1}{\sqrt{\pi t}}. \end{aligned}$$

Application 2: Let $(\mathcal{L}f)(x) \equiv F(x)$ where $F(x)$ is defined by

$$F(x) = \frac{1}{\sqrt{x}}\sum_{n\geq 0} a_n \cos(n \ln x\frac{\pi}{L}) \quad 1 \leq x \leq e^L$$

where $\int_1^{e^L} |F(\eta)|^2 d\eta < \infty$. Clearly from (5.9)

$$\mathcal{L}^{-1}F(x) = \mathcal{L}V^{-1}\frac{1}{\pi}\cos(\pi D)VF$$

where

$$VF = e^{\frac{x}{2}}F(e^x) = \sum_{n\geq 0} a_n \cos(nx\frac{\pi}{L}).$$

We deduce that

$$\mathcal{L}^{-1}F(x) = \mathcal{L}\frac{1}{\pi}V^{-1}\cos(\pi D)\sum a_n \cos(nx\frac{\pi}{L}). \tag{15}$$

We would now like to evaluate the right hand side of equation (15). Recall that

$$\frac{1}{\pi}\cos(\pi D) \equiv \frac{1}{\pi}\sum_{k\geq 0}(-1)^k(\pi^{2k})\frac{D^{2k}}{2k!}$$

and $(-1)^k\pi^{2k}D^{2k}\cos(nx\frac{\pi}{L}) = (n\frac{\pi^2}{L})^{2k}\cos(nx\frac{\pi}{L})$. Hence

$$\begin{aligned}\frac{1}{\pi}\cos(\pi D)\cos(nx\frac{\pi}{L}) &= \frac{1}{\pi}\sum_{k\geq 0}\frac{1}{2k!}(n\frac{\pi^2}{L})^{2k}\cos(nx\frac{\pi}{L}) \\ &= \frac{1}{\pi}\cosh(\frac{\pi^2 n}{L})\cos(nx\frac{\pi}{L})\end{aligned}$$

and so

$$\frac{1}{\pi}V^{-1}\cos(\pi D)\cos(nx\frac{\pi}{L}) = \frac{1}{\pi}\frac{1}{\sqrt{x}}\cosh(\frac{\pi^2 n}{L})\cos(nlnx\frac{\pi}{L}).$$

The last remaining operation is taking the Laplace transform

$$\mathcal{L}\frac{1}{\pi}\frac{1}{\sqrt{x}}\cosh(\frac{\pi^2 n}{L})\cos(n\ln x\frac{\pi}{L}) = \frac{1}{\pi}\cosh(\frac{\pi^2 n}{L})\mathcal{L}\frac{1}{\sqrt{x}}\cos(n\ln x\frac{\pi}{L}).$$

Therefore

$$f(x) = \mathcal{L}^{-1}(F(x)) = \sum_{n\geq 0} a_n\cosh(\frac{\pi^{2n}}{L})Re\{\frac{\Gamma(\frac{1}{2}+in\frac{\pi}{L})}{x^{\frac{1}{2}+in\frac{\pi}{L}}}\}.$$

Application 3: For $|x|\leq L$, let $\int_{e^{-L}}^{e^L}|F(\eta)|^2 d\eta < \infty$. We can expand

$$e^{\frac{x}{2}}F(e^x) \equiv \sum c_n e^{inx\frac{\pi}{L}} \quad \text{where} \quad |x|\leq L.$$

Obviously $VF(x) \equiv \sum c_n e^{inx\frac{\pi}{L}}$ and

$$\frac{1}{\pi}\cos(\pi D)e^{inx\frac{\pi}{L}} = \frac{1}{\pi}\cosh(n\frac{\pi^2}{L})e^{inx\frac{\pi}{L}}.$$

Thus $V^{-1}\frac{1}{\pi}\cos\pi De^{inx\frac{\pi}{L}} = \frac{1}{\pi}\cosh(n\frac{\pi^2}{L})x^{-\frac{1}{2}+in\frac{\pi}{L}}$. All we need now is to apply Laplace transform to obtain

$$\begin{aligned}f(x)\equiv\mathcal{L}^{-1}F(x) &= \sum c_n\frac{1}{\pi}\cosh(n\frac{\pi^2}{L})\mathcal{L}x^{-\frac{1}{2}+in\frac{\pi}{L}} \\ &= \sum c_n\frac{1}{\pi}\cosh(n\frac{\pi^2}{L})\frac{\Gamma(\frac{1}{2}+in\frac{\pi}{L})}{x^{\frac{1}{2}+in\frac{\pi}{L}}}.\end{aligned}$$

Acknowledgment. The author gratefully acknowledges the support provided by King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia.

REFERENCES

[1] R. Piessens, A bibliography on numerical inversion of the Laplace transform and application, J. Comp. Appl. Math., 1(1975), 115–128.

[2] R. Bellman, R.E. Kalaba and J.A. Lockett, Numerical Inversion of the Laplace Transform, American Elsevier, New York, 1966.

[3] V.I. Krylov and N.S. Skoblya, A Handbook of Methods of Approximate Fourier Transformation and Inversion of the Laplace Transform, Mir Publishers, Moscow, 1977.

[4] B. Davies and B. Martin, Numerical inversion of the Laplace transform: a survey and comparison of methods, J. Comp. Phys. 33(1979), 1–32.

[5] A. Al-Shuaibi, On the inversion of the Laplace transform, Ph.D. thesis, King Fahd University of Petroleum and Minerals, Dhahran, 1992.

[6] J.G. MacWhirter and E.R. Pike, On numerical inversion of the Laplace transform and similar Fredholm integral equations of the first kind, J. Phys., A: Math. Gen., 11(1978), 1729–1745.

[7] H Pollard, Integral transforms, Duke Math. J., 13(1946), 307–330.

[8] A. Boumenir, A comparison theorem for self-adjoint operators, Proc. Amer. Math. Soc., Vol. III, number 1(1991), 161–175.

Amin Boumenir
KFUPM Box 1812
Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia

and

Abdulaziz Al–Shuaibi
KFUPM Box 449
Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran 31261
Saudi Arabia

(shb1961)