King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 223

June 1997

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On the Uniqueness of the Solution of a Class of Functional Differential Equations.

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Abstract

With aid of a transformation, the uniqueness of the solution of a class of nonlinear functional differential equations is resolved.

Keywords— functional differential equation (fde), transformation.

1 Introduction

Of interest is the following problem: Find a continuous function $f : [0, \infty) \to [0, \infty)$ satisfying $f(0) = 0$ and the functional differential equation (fde)

$$f'(t) = \frac{1}{f(f(t))} \tag{1.1}$$

for $t > 0$ and show that no other such function exists. This problem was proposed by Y.–F.S. Pétermann in the American Mathematical Monthly, Vol. 104, No. 2, Feb. 1997, page 168. The functional differential equation (1.1) is an advance equation when $f(t) > t$. It is a retarded functional differential equation (rfde) when $f(t) < t$. Much research work has been devoted to rfde because they describe what are known as delay systems in practical applications. The reader is referred to [4] and references therein for a comprehensive and up to date account of the established theory of rfde.

In [1,2], [3], the first author introduced a technique for transforming a linear, autonomous, rfde into an ordinary differential equation (ode). This allowed the establishment of several control theoretic results for linear time invariant delay systems. The purpose of this article is to show that even though the above equation is nonlinear, the general idea of transforming such an equation into an ordinary differential is still valid. A more difficult problem is to delineate the most general class of fde which will be amenable to such a transformation.

As a step in the direction of resolving this question, we shall consider a class of functional differential equations given by

$$f'(t) = \frac{u(t)}{g(f(f(t)))}$$

where g and u are subject to appropriate hypotheses so that this new fde subsumes (1.1). A transformation which reduces the new fde to an ordinary differential equation (ode) will be given. The utility of such a transformation will be demonstrated by using it to resolve the question of uniqueness of the solution of the new equation. From here, the resolution of the uniqueness of (1.1) follows as a special case. Once the uniqueness of the solution of (1.1) is established, the problem of existence can be solved constructively.

2 Result

Theorem 2.1 *Let $\mathbb{R}_+ = [0,\infty)$ and $C(\mathbb{R}_+;\mathbb{R}_+)$ denote the space of non-negative continuous functions from $\mathbb{R}_+$ to $\mathbb{R}_+$. Assume that $g \in C(\mathbb{R}_+;\mathbb{R}_+)$ is not identically zero and is non-decreasing. Then the functional differential equation,*

$$\begin{aligned} f'(t) &= \frac{u(t)}{g(f(f(t)))} \qquad t > 0 \\ f(0) &= 0 \end{aligned} \tag{2.1}$$

where $u : [0,\infty) \to (0,\infty)$, has no more than one solution in $\mathrm{C}(\mathbb{R}_+;\mathbb{R}_+)$.

Proof. We observe that the graph of a solution, $f(\cdot)$, of (2.1) passes through the origin and is increasing since $f'(t) = \dfrac{u(t)}{g(f(f(t)))} > 0$. Let $f_1, f_2 \in C(\mathbb{R}_+; \mathbb{R}_+)$ be two solutions of (2.1). Define

$$\overline{h}(t) = (f_1 \vee f_2)(t), \quad \underline{h}(t) = (f_1 \wedge f_2)(t)$$

where $(f_1 \vee f_2)(t) = \max(f_1(t), f_2(t))$ and $(f_1 \wedge f_2)(t) = \min(f_1(t), f_2(t))$. Observe that $\overline{h}(0) = \underline{h}(0) = 0$ since $f_1(0) = 0 = f_2(0)$. Furthermore, for $t_2 \geq t_1$,

$$\begin{aligned} \overline{h}(t_2) &= (f_1 \vee f_2)(t) \\ &= \max(f_1(t_2), f_2(t_2)) \\ &\geq \max(f_1(t_1), f_2(t_1)) \\ &= \overline{h}(t_1) \end{aligned}$$

so that $\overline{h}(t)$ is monotonic increasing and hence differentiable almost everywhere. Similarly,

$$\begin{aligned} \underline{h}(t_2) &= (f_1 \wedge f_2)(t) \\ &= \min(f_1(t_2), f_2(t_2)) \\ &\geq \min(f_1(t_1), f_2(t_1)) \\ &= \underline{h}(t_1) \end{aligned}$$

so that $\underline{h}(t)$ is also monotonic increasing and therefore differentiable almost everywhere. We now construct the transformation

$$z(t) = \int_0^{\overline{h}(t)} \overline{h}(\tau)d\tau - \int_0^{\underline{h}(t)} \underline{h}(\tau)d\tau \tag{2.2}$$

where $\overline{h}$ and $\underline{h}$ are as defined above. It follows immediately that $z(0) = 0$. Furthermore, since both f_1 and f_2 are solutions to the fde, it follows that

$$f_1'(t).g(f_1(f_1(t))) = u(t), \quad f_2'(t).g(f_2(f_2(t))) = u(t).$$

Employing *differentiation under the integral*, (2.2) gives, almost everywhere,

$$z'(t) = \overline{h}'(t).g(\overline{h}(\overline{h}(t))) - \underline{h}'(t).g(\underline{h}(\underline{h}(t))) = u(t) - u(t) = 0$$

We have thus reduced the nonlinear fde to the ode, $z'(t) = 0$. This, in turn, gives $z(t) = 0$. Substituting this result into (2.2) yields

$$\int_0^{\underline{h}(t)} g(\underline{h}(\tau))d\tau = \int_0^{\overline{h}(t)} g(\overline{h}(\tau))d\tau$$

Employing the full expressions for $\underline{h}(t)$ and $\overline{h}(t)$ in the last equation, we obtain

$$\begin{aligned}
&\int_0^{(f_1\wedge f_2)(t)} g\circ(f_1\wedge f_2)(\tau)d\tau \\
&\quad = \int_0^{(f_1\vee f_2)(t)} g\circ(f_1\vee f_2)(\tau)d\tau \\
&\quad = \int_0^{(f_1\wedge f_2)(t)} g\circ(f_1\vee f_2)(\tau)d\tau + \int_{(f_1\wedge f_2)(t)}^{(f_1\vee f_2)(t)} g\circ(f_1\vee f_2)(\tau)d\tau
\end{aligned}$$

or,

$$\begin{aligned}
0 \;=\; &\int_0^{(f_1\wedge f_2)(t)} \left[g\circ(f_1\vee f_2)(\tau) - g\circ(f_1\wedge f_2)(\tau)\right]d\tau \\
&+ \int_{(f_1\wedge f_2)(t)}^{(f_1\vee f_2)(t)} g\circ(f_1\vee f_2)(\tau)d\tau. \qquad (2.3)
\end{aligned}$$

Since $(f_1\vee f_2)(t) \geq (f_1\wedge f_2)(t) \geq 0$ and $g(\cdot)$ is non-decreasing, it follows that each of the integrals on the RHS of (2.3) is non-negative and must therefore vanish. We deduce from the first integral that $g\circ(f_1\vee f_2)(t) = g\circ(f_1\wedge f_2)(t)$ while the second integral yields

$$(f_1\vee f_2)(t) = (f_1\wedge f_2)(t)$$

which implies that $f_1(t) = f_2(t)$ to establish the uniqueness of the solution of (2.1) in the space $C(\mathbb{R}_+;\mathbb{R}_+)$.

Remark 2.1 *The uniqueness of the solution of the Pétermann problem follows directly by considering $g(t) = t$ and $u(t) = 1$. Furthermore, the existence of the solution can be demonstrated constructively. Indeed, substituting trial solution $f(t) = \beta t^\alpha$*

where $\alpha > 0$, $\beta > 0$ *into (1.1) yields* $\alpha = \frac{1}{2}(\sqrt{5} - 1)$, $\beta = \alpha^{-\left(\frac{1}{2+\alpha}\right)}$ *so that*

$$f(t) = \alpha^{-\left(\frac{1}{2+\alpha}\right)} t^{\alpha}, \qquad \alpha = \frac{1}{2}(\sqrt{5} - 1)$$

is the unique solution.

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