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Inference on the Mean Vectors of Two Multivariate Student-t Populations

by

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Abstract

The paper considers inference on the mean vectors of two multivariate Student-t populations. The maximum likelihood estimators of the means as well as the scaled covariance matrix have been derived. To test the equality of the two mean vectors the likelihood ratio test statistic has been obtained. The power function of the test is derived and the distribution of the test statistic is discussed.

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1 INTRODUCTION

In recent years there has been growing concern about the use of the normal distribution as a sole model for the distribution of errors. At the same time the Student-t model has been receiving increasing appreciation in the robust procedures. Fisher (1956, p. 133) warned the over-sensitivity of inference based on the normal model with slight deviation from its assumption. Prucha and Kelejian (1984) categorically disapproved the normal distribution due to the fact that it (i) is generally very sensitive to deviations from its assumption, (ii) places too much weight on 'outliers' (iii) fails to utilize sample information beyond the first two moments, and (iv) appeals to the central limit theorem at most approximately, not exactly, normal. Among the other criticisms of the normal distribution is that it fails to model the heavier or fait-tailed distributions, a common feature in financial and stock market data.

Fisher (1960, p. 46) analyzed Darwin's data (see also Box and Tiao, 1990, p. 153) by using a non-normal model, although initially he applied the normal model. Fraser and Fick (1975) analyzed the same data by a Student-t model. Fraser (1979, p. 37) illustrated that the Student-t distribution is robust and that normal distribution is extremely short-tailed. The use of the Student-t distribution as a 'better option', than the normal distribution has been discussed by Fraser (1979, p. 41) as analysis based on the former model is applicable to the latter, but not the vice-versa. This can be appreciated due to the fact that the normal distribution is a (limiting) special case of the Student-t distribution. Moreover, the Student-t distribution is a 'more typical' member of the elliptically contoured family of distributions (cf. Anderson, 1990). Zellner (1976) confirms that the assumption of Student-t model is a wider assumption than the normal model. Anderson (1993) and Khan (1992) suggest that unlike the normal model, the Student-t model can handle dependent but uncorrelated variables. Khan and Saleh (1996 and 1998) study the preliminary test and shrinkage estimators for the multivariate Student-t model.

In this paper we propose the likelihood theory based inference for the mean vectors of two multivariate Student-t populations. First, the unrestricted maximum likelihood estimator (mle) of the mean vectors is obtained. Then the restricted mle of the mean vector and the scale covariance matrix is derived. Finally, the likelihood ratio test statistic for testing the equality of mean vectors is found. The power function of the test is also provided.

The multivariate Student-t population and the method for sampling from this population are discussed in Section 2. In the next section the mle of the mean vector and scaled covariance matrix are given. Section 4 provides the restricted mle of the combined mean vector and the scaled covariance matrix. The likelihood ratio test statistic is obtained in Section 5. The power function of the test is derived in Section 6.

2 THE MULTIVARIATE STUDENT-t MODEL

The multivariate Student-t distribution as a function of the normal and Wishard (1928) distribution has been derived by Cornish (1953). It is well known (cf Khan and Saleh, 1998, for instance) that the multivariate Student-t distribution can be expressed as the mixture distribution of multivariate normal and inverted gamma distributions. In this paper we apply this approach for the derivation of the main results. Similar approach has been used in the econometric literature for statistical inference of linear models with Student-t errors. Zellner (1976) provided both Bayesian and non-Bayesian analyses, Ullah and Walsh (1984) compared different types of tests and discussed the optimality of likelihood ratio test and Giles (1993) studied the preliminary test estimator for multiple regression model with Student-t errors.

Let $X_{11}, X_{12}, \ldots, X_{1n_1}$ be a random sample of size n_1 from a p-dimensional multivariate normal population with unknown mean vector μ_1 and covariance matrix $\tau^2\Sigma$. Similarly, let $X_{21}, X_{22}, \ldots, X_{2n_2}$ be another random sample of size n_2 from a multivariate normal population of the same dimension but with unknown mean vector μ_2 and covariance matrix $\tau^2\Sigma$. Then the joint distribution of the two random samples can be expressed as

$$f(X_1, X_2; \mu_1, \mu_2, \tau^2\Sigma) = (2\pi)^{-\frac{p(n_1+n_2)}{2}} (\tau^2)^{-\frac{p(n_1+n_2)}{2}} |\Sigma|^{-\frac{n_1+n_2}{2}} e^{-\frac{1}{2\tau^2}(Q_1+Q_2)} \tag{2.1}$$

where $Q_i = \sum_{j=1}^{n_i} (X_{ij} - \mu_i)'\Sigma^{-1}(X_{ij} - \mu_i)$ and X_i is a $p \times n_i$-dimensional matrix of the i-th sample for $i = 1, 2$.

Now assume that τ follows an inverted gamma (IG) distribution with shape ν and scale 1, so that its pdf becomes

$$f(\tau) = \frac{2}{\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{\nu}{2}\right)^{\nu/2} \tau^{-(\nu+1)} e^{-\frac{\nu}{2\tau^2}}. \tag{2.2}$$

Then it is well known that the mixture distribution of X_1, X_2 and τ is defined as

$$f(X_1, X_2, \mu_1, \mu_2, \Sigma) = \int_\tau f(X_1, X_2; \mu_1, \mu_2, \tau^2\Sigma) f(\tau) \theta\tau. \tag{2.3}$$

The completion of the integral yields the following density function

$$f(X_1, X_2; \mu_1, \mu_2, \Sigma) = K(\nu, n_1, n_2) |\Sigma|^{-\frac{n_1+n_2}{2}} \left[1 + \frac{1}{\nu}(Q_1 + Q_2)\right]^{-\frac{\nu+p(n_1+n_2)}{2}} \tag{2.4}$$

where $K(\nu, n_1, n_2) = \dfrac{\Gamma\left(\frac{\nu+p(n_1+n_2)}{2}\right)}{(\pi\nu)^{\frac{p(n_1+n_2)}{2}} \Gamma\left(\frac{\nu}{2}\right)}$.

Thus X_1 and X_2 jointly follow a multivariate Student-t distribution with shape ν, mean vector (μ_1, μ_2) and scaled covariance matrix Σ. Note that the elements of X_1 and X_2 are dependent but uncorrelated, and that $\mathrm{Cov}(X_1, X_2) = \frac{\nu}{\nu-2} \begin{bmatrix} \Sigma & I_p \\ I_p & \Sigma \end{bmatrix}$

where I_p is the identity matrix of order p, since $\text{Cov}(X_i) = \frac{\nu}{\nu-2}\Sigma$ for $i = 1, 2$. Let $\Omega = \{\mu_1, \mu_2, \Sigma, \nu\}$ denote the parameter space.

For unknown ν and Σ, we make inference on the mean vectors and covariance matrix of X_1 and X_2. In particular, we derive the maximum likelihood estimators of μ_1, μ_2 and Σ. For the estimation of ν one may refer to Singh (1988). Also, we develop a likelihood ratio test to test the null hypothesis of equality of mean vectors. The power function of the test is also provided.

3 THE MAXIMUM LIKELIHOOD ESTIMATION

The likelihood function of the samples can be given by

$$L(\mu_1, \mu_2, \Sigma) = K(\nu, n_1, \nu_2)|\Sigma|^{-\frac{n_1+n_2}{2}} \left[1 + \frac{1}{\nu}(Q_1 + Q_2)\right]^{-\frac{\nu+p(n_1+n_2)}{2}}. \tag{3.1}$$

Now note the following representation of Q_i,

$$\begin{aligned} Q_i &= \sum_{j=1}^{n_i}(X_{ij} - \mu_i)'\Sigma^{-1}(X_{ij} - \mu_i) \\ &= tr\left\{\Sigma^{-1}\sum_{j=1}^{n_i}(X_{ij} - \mu_i)(X_{ij} - \mu_i)'\right\} \\ &= tr\left\{\Sigma^{-1}S_i\right\} + n_i(\overline{X}_i - \mu_i)'\Sigma^{-1}(\overline{X}_i - \mu_i) \end{aligned} \tag{3.2}$$

where $S_i = \sum_{j=1}^{n_i}(X_{ij} - \overline{X}_i)(X_{ij} - \overline{X}_i)'$ is the sample sum of squares and products matrix in which $\overline{X}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} X_{ij}$, and

$$\sum_{j=1}^{n_i}(X_{ij} - \mu_i)(X_{ij} - \mu_i)' = S_i + n_i(\overline{X}_i - \mu_i)(\overline{X}_i - \mu_i)' \tag{3.3}$$

for $i = 1$ and 2. Then the log-likelihood function can be written as

$$\ln L(\cdot) \propto -\frac{n_1 + n_2}{2}\ln|\Sigma| - \frac{\nu + p(n_1 + n_2)}{2} \times$$

$$\ln\left[1+\frac{1}{\nu}\left\{tr\Sigma^{-1}(S_1+S_2)\right\}+\sum_{i=1}^{2}n_i(\overline{X}_i-\mu_i)'\Sigma^{-1}(\overline{X}_i-\mu_i)\right]. \quad (3.4)$$

Now, applying the principle of the maximum likelihood method we have

$$\frac{\partial \ln L(\cdot)}{\partial \mu_i}=-\frac{\nu+p(n_1+n_2)}{2}\left[1+\frac{1}{\nu}\left\{tr\Sigma^{-1}(S_1+S_2)\right\}\right.$$
$$\left.+\sum_{i=1}^{2}n_i(\overline{X}_i-\mu_i)'\Sigma^{-1}(\overline{X}_i-\mu_i)\right]\times\left[-n_i(\overline{X}_i-\mu_i)\Sigma^{-1}\right]=0, \quad (3.5)$$

which implies that $(\overline{X}_i-\mu_i)=0$. Therefore, the maximum likelihood estimator (mle) for μ_i is obtained as

$$\tilde{\mu}_i=\overline{X}_i, \qquad \text{for} \quad i=1,2. \quad (3.6)$$

To find the mle of Σ, let us consider the following representation. First write,

$$\Sigma^{-1}=(\sigma_{\ell\ell'})^{-1}=\sigma^{\ell\ell'} \quad \text{for} \quad \ell,\ell'=1,2,\ldots,p; \quad (3.7)$$

then $|\Sigma|^{-1}=|\sigma^{\ell\ell'}|$. Now letting $S=S_1+S_2$, we can write

$$tr\Sigma^{-1}(S_1+S_2)=tr\Sigma^{-1}S=\sum_{\ell=1}^{p}\sum_{\ell'=1}^{p}\sigma^{\ell\ell'}s_{\ell\ell'} \quad (3.8)$$

where $S=(s_{\ell\ell'})$ is the matrix of the sum of the sum of squares and product matrices. Then the log-likelihood function can be expressed as

$$\ln L(\cdot)\propto\frac{n_1+n_2}{2}\ln|(\sigma^{\ell\ell'})|-\frac{\nu+p(n_1+n_2)}{2}\ln\left[1+\frac{1}{\nu}\sum_{\ell=1}^{p}\sum_{\ell'=1}^{p}\sigma^{\ell\ell'}s_{\ell\ell'}\right] \quad (3.9)$$

after setting the mle of μ_i for itself. Taking the derivative of (3.9) with respect to $\sigma^{\ell\ell'}$ we get

$$\frac{\partial \ln L(\cdot)}{\partial \sigma^{\ell\ell'}}=\frac{n_1+n_2}{2}\left[\frac{1}{|(\sigma^{\ell\ell'})|}\frac{\partial|(\sigma^{\ell\ell'})|}{\partial\sigma^{\ell\ell'}}\right]-\frac{\nu+p(n_1+n_2)}{2}\left[1+\frac{1}{\nu}\sum_{\ell}\sum_{\ell'}\sigma^{\ell\ell'}s_{\ell\ell'}\right]^{-1}\frac{s_{\ell\ell'}}{\nu}. \quad (3.10)$$

Setting the right hand side of (3.10) to 0, and simplifying the expression, we obtain

$$(n_1+n_2)\sigma_{\ell\ell'}=\frac{\{\nu+p(n_1+n_2)\}s_{\ell\ell'}}{\nu+\sum_{\ell}\sum_{\ell'}\sigma^{\ell\ell'}s_{\ell\ell'}} \quad (3.11)$$

which leads to the solution $\tilde{\sigma}_{\ell\ell'} = \dfrac{1}{n_1+n_2} s_{\ell\ell'}$. Therefore, the mle for Σ becomes

$$\tilde{\Sigma} = \frac{1}{n_1+n_2} S. \tag{3.12}$$

It can be seen that $\tilde{\Sigma}$ is a biased estimator of Σ, and it is not difficult to show that $\tilde{\Sigma}^* = \dfrac{1}{n_1+n_2-2} S$ is an unbiased estimator of Σ. As in the case of univariate estimation, the mle's of μ_i and Σ have intuitive appeal as a result of being appropriate sample counterparts. Also note that neither $\hat{\mu}_i$ nor $\tilde{\Sigma}$ depends on ν.

4 THE RESTRICTED ESTIMATORS

In the next section we wish to develop a likelihood ratio test for testing $H_0 : \mu_1 = \mu_2$ against $H_1 : \mu_1 \neq \mu_2$. Here we derive the restricted maximum likelihood estimators of μ_i's and Σ. Under the null hypothesis, we write $\mu_1 = \mu_2 = \mu$, and hence the restricted likelihood function becomes

$$L(\mu, \Sigma) = K(\cdot)|\Sigma|^{-\frac{n_1+n_2}{2}} \left[1 + \frac{1}{\nu}(D_1 + D_2)\right]^{-\frac{\nu+p((n_1+n_2)}{2}} \tag{4.1}$$

where

$$\begin{aligned} D_i &= \sum_{j=1}^{n_i} (X_{ij} - \mu)'\Sigma^{-1}(X_{ij} - \mu) \\ &= tr\Sigma^{-1} \sum_{j=1}^{n_i} (X_{ij} - \mu)(X_{ij} - \mu)' \\ &= tr\Sigma^{-1} R_i + n_i(\overline{X} - \mu)'\Sigma^{-1}(\overline{X} - \mu) \end{aligned} \tag{4.2}$$

in which $\overline{X} = \dfrac{1}{n_1+n_2} \sum_{i=1}^{2} \sum_{j=1}^{n_i} X_{ij}$ and

$$R_i = \sum_{j=1}^{n_i} (X_{ij} - \overline{X})(X_{ij} - \overline{X})' \quad \text{for} \quad i = 1, 2. \tag{4.3}$$

Then

$$D = D_1 + D_2 = tr\Sigma^{-1} R + \sum_{i=1}^{2} n_i(\overline{X} - \mu)'\Sigma^{-1}(\overline{X} - \mu) \tag{4.4}$$

with $R = R_1 + R_2$, and using the representation of Σ as in (3.7), the (restricted) log-likelihood function under H_0 is written as

$$\ln L(\cdot) \propto \frac{n_1+n_2}{2}\ln|(\sigma^{\ell\ell'})| - \frac{\nu+p(n_1+n_2)}{2}\ln\left[1+\frac{1}{\nu}\sum_{\ell}\sum_{\ell'}\sigma^{\ell\ell'}s_{\ell\ell'} + \sum_{i=1}^{2} n_i(\overline{X}-\mu)'\Sigma^{-1}(\overline{X}-\mu)\right]. \quad (4.5)$$

It can be easily shown that

$$\frac{\partial \ln L(\cdot)}{\partial \mu'} = 0 \quad \text{implies} \quad \Sigma^{-1}(\overline{X}-\mu) = 0. \quad (4.6)$$

Thus the *restricted* maximum likelihood estimator of μ becomes

$$\hat{\mu} = \overline{X} = \frac{1}{n_1+n_2}\sum_{i=1}^{2}\sum_{j=1}^{n_i} X_{ij}. \quad (4.7)$$

Applying the same procedure as in the previous section, the restricted mle of Σ is found to be

$$\hat{\Sigma} = \frac{1}{n_1+n_2}R. \quad (4.8)$$

Once again $\hat{\Sigma}$ is a biased estimator of Σ, while $\hat{\Sigma}^* = \frac{1}{n_1+n_2-2}R$ is an unbiased estimator of Σ.

5 DERIVATION OF TEST

The maximum value of the likelihood function at $\mu_i = \tilde{\mu}_i$ and $\Sigma = \tilde{\Sigma}$, for an arbitrary ν, is given by

$$L(\tilde{\Omega}) = K(\cdot)|\tilde{\Sigma}|^{-\frac{n_1+n_2}{2}}\left[1+\frac{p}{\nu}\right]^{-\frac{\nu+p(n_1+n_2)}{2}}. \quad (5.1)$$

Therefore, the right hand side of (5.1) gives the global (unrestricted) maximum of the likelihood function over Ω . Similarly, the maximum value of the restricted likelihood function is found by setting $\mu_i = \hat{\mu}$ and $\Sigma = \hat{\Sigma}$ in the restricted likelihood function (4.1). This yields,

$$L(\hat{\Omega}) = K(\cdot)|\hat{\Sigma}|^{-\frac{n_1+n_2}{2}} \left[1 + \frac{p}{\nu}\right]^{-\frac{\nu+p(n_1+n_2)}{2}}. \tag{5.2}$$

Then the ratio of the maximized likelihoods is given by

$$\lambda = \frac{L(\hat{\Omega})}{L(\tilde{\Omega})} = \frac{|\hat{\Sigma}|^{-\frac{n_1+n_2}{2}}}{|\tilde{\Sigma}|^{-\frac{n_1+n_2}{2}}}, \tag{5.3}$$

that is,

$$\lambda^{\frac{2}{n_1+n_2}} = \frac{|\tilde{\Sigma}|}{|\hat{\Sigma}|}. \tag{5.4}$$

Now, $\hat{\Sigma}$ can be expressed as

$$\begin{aligned} \hat{\Sigma} &= \frac{1}{n_1+n_2} R = \frac{1}{n_1+n_2} \sum_{i=1}^{2} \sum_{j=1}^{n_i} (X_{ij} - \overline{X})(X_{ij} - \overline{X})' \\ &= \frac{1}{n_1+n_2} \sum_{i=1}^{2} \left[\sum_{j=1}^{n_i} (X_{ij} - \overline{X}_i)(X_{ij} - \overline{X}_i)' + n_i(\overline{X}_i - \overline{X})(\overline{X}_i - \overline{X}) \right] \\ &= \frac{1}{n_1+n_2} \left[S + \sum_{i=1}^{2} n_i(\overline{X}_i - \overline{X})(\overline{X}_i - \overline{X})' \right] \\ &= \frac{S}{n_1+n_2} + \frac{n_1 n_2}{(n_1+n_2)^2}(\overline{X}_2 - \overline{X}_1)(\overline{X}_2 - \overline{X}_1)'. \end{aligned} \tag{5.5}$$

Therefore,

$$\begin{aligned} \lambda^{\frac{2}{n_1+n_2}} &= \frac{|S|}{|S + \frac{n_1 n_2}{n_1+n_2}(\overline{X}_2 - \overline{X}_1)(\overline{X}_2 - \overline{X}_1)'|} \\ &= \frac{1}{1 + \frac{n_1 n_2}{n_1+n_2}(\overline{X}_2 - \overline{X}_1)' S^{-1} (\overline{X}_2 - \overline{X}_1)}. \end{aligned} \tag{5.6}$$

The statistic

$$T^2 = \frac{n_1 n_2}{n_1+n_2}(\overline{X}_2 - \overline{X}_1)' S^{-1} (\overline{X}_2 - \overline{X}_1) \tag{5.7}$$

follows a modified Hotelling's T^2 distribution (cf. Anderson, 1985, p. 109).

The above likelihood ratio λ is a monotone function of the T^2-statistic in (5.7), and hence a test for $H_0 : \mu_1 = \mu_2$ against $H_1 : \mu_1 \neq \mu_2$ can be based on the F-distribution.

Distribution of T^2-Statistic

From the definition of the Hotelling's T^2-distribution we know

$$T^{*^2} = \frac{n_1 n_2}{n_1 + n_2}(\overline{X}_2 - \overline{X}_1)^{-1} S^{*^{-1}}(\overline{X}_2 - \overline{X}_1) \qquad (5.8)$$

where $S^* = \frac{1}{n_1 + n_2 - 2}\sum_{i=1}^{2}\sum_{j=1}^{n_i}(X_{ij} - X_i)(X_{ij} - \overline{X}_i)' = \frac{S}{n_1 + n_2 - 2}$ follows a Hotelling's T^2-distribution with $(n_1 + n_2 - 2)$ d.f. Therefore,

$$\frac{T^{*^2}}{n_1 + n_2 - 2} = \frac{n_1 n_2}{n_1 + n_2}(\overline{X}_2 - \overline{X}_1)' S^{-1}(\overline{X}_2 - \overline{X}_1) = T^2 \qquad (5.9)$$

which is defined in (5.7). Then it can be easily shown that conditional on τ

$$T^2 \left(\frac{m}{p}\right) \sim F_{p,m}(\Delta_\tau), \qquad (5.10)$$

a non-central F-distribution with p and $m = (n_1 + n_2 - p - 1)$ df and non-centrality parameter $\Delta_\tau = (\mu_2 - \mu_1)'\frac{\Sigma^{-1}}{\tau^2}(\mu_2 - \mu_1) = \frac{\delta'\Sigma^{-1}\delta}{\tau^2}$. Note that for any arbitrary value of τ,

$$(\overline{X}_2 - \overline{X}_1) \sim N_p\left(\mu_2 - \mu_1, \sqrt{\frac{n_1 n_2}{n_1 + n_2}}\tau^2\Sigma\right) \quad \text{and} \quad S^* \sim W_p(n_1 + n_2 - 2, \tau^2\Sigma).$$

Finally, from (5.10) we can write

$$T^2 \sim \frac{p}{m}F_{p,m}(\Delta_\tau), \qquad (5.11)$$

and hence at the α-level of significance one may reject the null hypothesis if the observed value of T^2, T_0 (say), is greater than $\frac{p}{m}F_{p,m}(\alpha)$ where $F_{p,m}(\alpha)$ is the $(1-\alpha)$-th quantile of a central F-distribution with p and m d.f.

6 THE POWER FUNCTION

To derive the power function of the test, first note that for a given value of τ,

$$\begin{aligned}\pi(\delta|\tau) &= P\left(T^2 \geq \frac{p}{m}F_{p,m}(\Delta_\tau)\right) \\ &= P\left(F_{p,m}(\Delta_\tau) \leq \frac{m}{p}T^2\right) = G_{p,m}\left(\frac{m}{p}T^2;\Delta_\tau\right) \qquad (6.1)\end{aligned}$$

where $G_{p,m}\left(\frac{m}{p}T^2;\Delta_\tau\right)$ is the distribution function evaluated at $q = \frac{m}{p}T^2$ of a non-central F-distribution with p and m d.f. and non-centrality parameter $\Delta_\tau = \frac{\Delta}{\tau^2}$ with $\Delta = \delta'\Sigma^{-1}\delta$.

Now taking expectation on (6.1) with respect to the inverted gamma distribution with shape ν and scale 1, we obtain the power function of the test for the Student-t model. Since

$$G_{p,m}(q;\Delta_\tau) = \sum_{r=0}^{\infty} \frac{e^{-\frac{\Delta}{2\tau^2}}\left(\frac{\Delta}{2\tau^2}\right)^r}{r!} \frac{\Gamma\left(\frac{p+m}{2}+r\right)}{\Gamma\left(\frac{p}{2}+r\right)\Gamma\left(\frac{m}{2}\right)} \int_{y=0}^{q} \frac{y^{r+\frac{p}{2}-1}}{(1+y)^{r+\frac{p+m}{2}}}dy \qquad (6.2)$$

yields

$$G_{p,m}(u;\Delta_\tau) = \sum_{r=0}^{\infty} \frac{e^{-\frac{\Delta}{2\tau^2}}\left(\frac{\Delta}{2\tau^2}\right)^r}{r!} \frac{\Gamma\left(\frac{p+m}{2}+r\right)}{\Gamma\left(\frac{p}{2}+r\right)\Gamma\left(\frac{m}{2}\right)} I_u\left(\frac{m}{2};\frac{p}{2}+r\right) \qquad (6.3)$$

where $I_u\left(\frac{m}{2};\frac{p}{2}+r\right)$ is the value of an incomplete beta function with arguments $\frac{m}{2}$ and $\frac{p}{2}+r$, and evaluated at $u = \frac{1}{1+q}$. Therefore,

$$\begin{aligned}G^{(1)}_{p,m}(u;\Delta^*) &= E_\tau\{G_{p,m}(u;\Delta_\tau)\} \\ &= \sum_{r=0}^{\infty} h_r(u,p,m)\frac{2}{\Gamma\left(\frac{\nu}{2}\right)}\left(\frac{\nu}{2}\right)^{\nu/2}\left(\frac{\Delta}{2}\right)^r \int_{\tau=0}^{\infty} e^{-\frac{(\nu+\Delta)}{2\tau^2}}\tau^{-(\nu+2r+1}d\tau \qquad (6.4)\end{aligned}$$

where

$$h_r(u,p,m) = \frac{\Gamma\left(\frac{p+m}{2}+r\right)I_u\left(\frac{m}{2};\frac{p}{2}+r\right)}{\Gamma\left(\frac{p}{2}+r\right)\Gamma\left(\frac{m}{2}\right)r!}, \qquad (6.5)$$

becomes

$$G^{(1)}_{p,m}(u;\Delta^*) = \sum_{r=0}^{\infty} h_r(u,p,m) \frac{\Gamma\left(\frac{\nu}{2}+r\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{\left(\frac{\Delta^*}{\nu-2}\right)^r}{\left(1+\frac{\Delta^*}{\nu-2}\right)^{\frac{\nu}{2}+r}} \tag{6.6}$$

in which $\Delta^* = \frac{\nu-2}{\nu}\Delta$.

Therefore, unlike the normal model, the power function of the test of equality of the mean vectors for the multivariate Student-t model cannot be expressed in terms of any standard non-central distribution. However, the power can be computed by using the above $G^{(1)}_{p,m}(u;\Delta^*)$ function for a given value of ν. If ν is not known it may be estimated by using the method moment technique (cf. Singh, 1988).

Under H_0, the value of Δ^* is zero, and by expanding the infinite sum of $G^{(1)}_{p,m}(u;\Delta^*)$ it can be shown that the power function for the null distribution becomes

$$G^{(1)}_{p,m}(u;0) = \frac{\Gamma\left(\frac{p+m}{2}\right)}{\Gamma\left(\frac{p}{2}+1\right)\Gamma\left(\frac{m}{2}\right)} I_u\left(\frac{m}{2};\frac{p}{2}\right). \tag{6.7}$$

This is obvious due to the fact that all the terms in the infinite sum of $G^{(1)}_{p,m}(u;\Delta^*)$ are zero except for the first term.

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