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Abstract

In this paper we have converted the Laplace transform to an integral equation of the first kind of convolution type, which is an ill–posed problem. We used the maximum likelihood method to solve it. The method is applied to several test problems taken from [6, 10, 11]. It gives a good approximation to the true solution. We have described the merits of the proposed method over moments method which is also a statistical method given in [18]. The results of the proposed method are shown in the respective table and diagrams.

AMS(MOS) Subject classification: 65R20, 65R30

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1. INTRODUCTION.

Inverse problems pertain to situations where one is interested in making inferences about a phenomenon from partial or incomplete information. Accordingly, statistical estimation and model-building are both inverse problems. In modern science, there is an increasingly important class of inverse problems which are not amenable to classical statistical estimation procedures and such problems are termed as ill–posed.

The notion of ill–posedness is usually attributed to Hadamard [13]. A modern treatment of the concept appears in Tikhonov and Arsenin [26]. In an ill–posed inverse problem, a classical least squares, minimum distance or cross–validation solution may not be uniquely defined. Moreover, the sensitivity of such solutions to slight perturbations in the data is often unacceptably large.

Ill–posed inverse problems have become a recurrent theme in modern sciences, see, for example, crystallography (Grunbaum [12]), geophysics (Aki and Richards [1]), medical electrocardiograms (Franzone et al. [9]), meteorology (Smith [24]), radio astronomy (Jaynes []14]), reservoir engineering (Kravaris and Seinfeld [15]) and tomography (Vardi et al. [28]). Corresponding to this broad spectrum of fields of application, there is a wide literature on different kinds of inversion algorithms, that is techniques for solving the inverse problems.

The basic principle common to all such methods is as follows: seek a solution that is consistent both with the observed data and prior notions about the physical behavior of the phenomenon under study. Different practical problems have led to unique strategies for implementation of this principle such as the method of regularization (Tikhonov and Arsenin [26]), maximum entropy (Jaynes [14], Mead [18]), quasi-reversibility (Lattes and Lions [16]) and cross-validation (Wahba [29]).

Laplace transform inversion is also an ill–posed inverse problem. There are many problems whose solution may be found in terms of Laplace transform which, however, is too complicated for inversion using different methods. However, no single method gives optimum results for all purposes and all occasions. For a detailed bibliography, the reader is referred to (Piessens, and Piessens and Branders, [21, 22]).

The problem of the recovery of a real function $f(t)$, $t \geq 0$, given its Laplace transform

$$\int_0^\infty e^{-st} f(t)dt = g(s) \tag{1.1}$$

for real values of s, is an ill–posed problem and, therefore, affected by numerical instability.

The ill–posedness of Laplace transform inversion in the case, where $f \in L^2(R_+)$ and $g(s)$ is known for all real and positive values of s, can be investigated by means of the Mellin transform (McWhirter [17]). In practice, however, $g(s)$ is known only in a finite set of points. The case of an infinite set of equidistant points was investigated by (Papoulis [20]). Several methods and a comparison is given in (Davies [7]) and (Talbot [25]).

The previous methods do not include regularization techniques. Regularization methods have been discussed by (Varah [27]) and (Essah and Delves [8]). Regularization by means of truncated singular function expansion is investigated by (Bertero [4]); other methods are also available in the literature for the numerical evaluation of the Laplace transform inversion which have been described by

(Norden [19]) and (Salzer [23]).

2. FREDHOLM EQUATION OF CONVOLUTION TYPE.

We shall convert the Laplace transform into the first kind integral equation of convolution type with the following substitution in equation (1.1)

$$s = a^x \text{ and } t = a^{-y} \text{ where } a > 1. \tag{2.1}$$

Then

$$g(a^x) = \int_{-\infty}^{\infty} \log a \, e^{-a^{x-y}} f(a^{-y}) a^{-y} dy. \tag{2.2}$$

Multiplying both sides of (2.2) by a^x, we obtain the convolution equation

$$\int_{-\infty}^{\infty} K(x-y)F(y)dy = G(x), \qquad -\infty \le x \le \infty \tag{2.3}$$

where

$$\left.\begin{aligned} G(x) &= a^x g(a^x) = sg(s) \\ K(x) &= \log a \, a^x \, e^{-a^x} = \log a \, se^{-s} \\ F(y) &= f(a^{-y}) = f(t) \end{aligned}\right\}. \tag{2.4}$$

Equation (2.3) occurs widely in applied sciences. K and G are known kernel and data functions, respectively, and F is to be found. We shall assume that F, G and K lie in suitable function spaces, such as $L_2(R)$, so that their Fourier transform (FTs) exists. ($^\wedge$ denotes FTs and $^\vee$ denotes inverse FTs).

3. DESCRIPTION OF THE PROPOSED METHOD.

We assume that the support of each function F, G and K is essentially finite and contained within the interval $[0, T]$, where the period $T = Nh$, N is the number of grid points and h is the spacing.

Let T_N denote the space of trigonometric polynomials of degree at most N and period T. We shall look for filtered solution of (2.3) within the space T_N for the following reasons:

(i) The discretization error in the convolution may be made precisely zero at the grid points.

(ii) Fast Fourier Transform (FFT) routines are easily employed in the solution procedure.

(iii) The adoption of T_N as an approximation function space is itself a regularizing feature.

Let G and K be given at N equally–spaced points $x_n = nh,\ \ n = 0, 1, 2, \ldots, N-1$, with spacing $h = T/N$. Then G and K are interpolated by G_N and $K_N \in T_N$ where

$$G_N(x) = \frac{1}{N} \sum_{q=0}^{N-1} \hat{G}_{N,q} \exp(i\omega_q x) \tag{3.1}$$

$$\hat{G}_{N,q} = \sum_{n=0}^{N-1} \exp(-i\omega_q x_n) \tag{3.2}$$

$$\text{and } G(x_n) = G_n = G_N(x_n),\ \ \omega_q = \frac{2\pi q}{T}. \tag{3.3}$$

Similar expressions as (3.1) and (3.2) can be obtained for K_N.

Consider (2.3). The Fredholm integral equation of the first kind of convolution-type

$$(KF)(x) = \int_{-\infty}^{\infty} K(x-y)F(y)dy = G(x), \qquad -\infty < x < \infty \tag{3.4}$$

where G and K are known functions in $L_2(R)$ and $F \in H^p(R)$ is to be found. Then from the convolution theorem we have

$$\hat{K}(\omega)\hat{F}(\omega) = \hat{G}(\omega) \tag{3.5}$$

whence

$$F(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\hat{G}(\omega)}{\hat{K}(\omega)} \exp(i\omega y)d\omega. \tag{3.6}$$

The ill–posedness of (3.4) is reflected by the fact that any small perturbation ϵ in G, whose transform $\hat{\epsilon}(\omega)$ does not decay faster than $\hat{K}(\omega)$ as $|\omega| \to \infty$ will result in a perturbation in $\frac{\hat{G}(\omega)}{\hat{K}(\omega)}$, which will grow without bound, when G is inexact. Therefore, we may seek a stable or filtered approximation to F given by

$$F_\lambda(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} z(\omega;\lambda)\frac{\hat{G}(\omega)}{\hat{K}(\omega)} \exp(i\omega y)d\omega \tag{3.7}$$

where $z(\omega;\lambda)$ is a stabilizing or filter function dependent on a parameter λ. In this paper we restrict our attention to filters generated from regularization theory.

The smoothing functional

$$C(F;\lambda) = \|KF - G\|_2^2 + \lambda\Omega(F) \tag{3.8}$$

is minimized in an appropriate subspace of L_2, where $\Omega[F]$ is a stabilizing functional in the form of a smoothing norm

$$\Omega[F] = \|LF\|^2 \tag{3.9}$$

and L is a linear operator. The regularization parameter λ controls the trade-off between smoothness, as imposed by Ω and the extent to which (3.4) is satisfied.

Our method is a simple extension of the ideas of (Anderssen and Bloomfield [2, 3]) who consider the problem of numerically differentiating noisy data. We restrict our attention to regularization of order p ($p = 2$, in our case), where L in (3.9) is the p-th order differential operator, $LF = F^{(p)}$ and the norm in (3.9) is L_2. The minimizer of (3.8) in H^p is then given by (3.7) where

$$z(\omega; \lambda) = \frac{|\hat{K}(\omega)|^2}{|\hat{K}(\omega)|^2 + \lambda\omega^{2p}}. \tag{3.10}$$

If (3.4) is now replaced by

$$(K_N F_N)(x) = \int_0^T K_N(x-y) F_N(y) dy = G_N(x) \tag{3.11}$$

where K_N is periodically continued outside $(0, T)$. Then we may prove (a) above.

Lemma 3.1. *Let $F \in T_N$ and $\underline{F} = (F(x_0), \ldots, F(x_{N-1}))^T \in R^N$. Then the $N \times N$ matrix*

$$K = \psi \text{diag}(\hat{K}_{N,q}) \psi^H \tag{3.12}$$

where ψ is the unitary matrix with elements

$$\psi_{rs} = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi}{N} irs\right), \qquad r, s = 0, \ldots, N-1 \tag{3.13}$$

has the property

$$(K\underline{F})_n = (K_N F)(x_n). \tag{3.14}$$

Thus from the infinite-support hypothesis and (3.3), it follows that at $\{x_n\}$, (2.3) is identical to the discrete system

$$(K\underline{F})_n = G_n \tag{3.15}$$

where K is given in (3.12). In T_N it is easily shown that F_λ in (3.7) is approximated by

$$\hat{F}_{N,\lambda}(x) = \sum_{q=0}^{N-1} z_{q;\lambda} \frac{\hat{G}_{N,q}}{\hat{K}_{N,q}} \exp(i\omega_q x) \tag{3.16}$$

where the discrete p–th order filter is

$$z_{q;\lambda} = \frac{|\hat{K}_{N,q}|^2}{|\hat{K}_{N,q}|^2 + N^2 \lambda \tilde{\omega}_q^{2p}} \tag{3.17}$$

and

$$\tilde{\omega}_q = \begin{cases} \omega_q, & 0 \le q < \frac{1}{2}N \\ \omega_{N-q}, & \frac{1}{2}N \le q < N-1 \end{cases} \tag{3.18}$$

To show (b) above, we note that $\sqrt{N}\psi^H$ is the discrete FT matrix representing (3.2) and so (3.15) is equivalent to the diagonal system

$$\hat{K}_{N,q}\hat{F}_{N,q} - \hat{G}_{N,q}. \tag{3.19}$$

After regularization, (3.19) is replaced by

$$\hat{K}_{N,q}\hat{F}_{N;q;\lambda} = z_{q;\lambda}\hat{G}_{N,q} \tag{3.20}$$

so that $F_{N;q;\lambda}(x)$ may be found by multiplying the FFT of $\{G_N\}$ by the filter, dividing by the FFT of $\{K_N\}$, and then taking the inverse FFT (fast Fourier transform).

4. THE FILTER IN A STOCHASTIC SETTING.

In this section we relate the p–th order convolution filter (3.17) to certain spectral densities which play a role in the maximum likelihood (ML) optimization of λ in the next section.

Assume that the data $\{G_n\}$ are noisy and that there is an underlying function $U_N \in T_N$ such that

$$G_n = U_n(x_n) + \epsilon_n \equiv U_n + \epsilon_n. \tag{4.1}$$

In the limit $N \to \infty$, $h \to 0$, for any discrete process X_n, we may write (see, for example, [21])

$$X_n = \int_0^T \exp(2\pi i \omega n) ds_x(\omega) \tag{4.2}$$

where $s_x(\omega)$ is a stochastic process defined on $[0, T]$.

Lemma 4.1. *The variance of any integral* $\int \theta(\omega) ds_x(\omega)$ *is given by* $\int |\theta(\omega)|^2 dG_x(\omega)$ *where* $dG_x(\omega) = E(|ds_x(\omega)|^2)$. $G_x(\omega)$ *may be interpolated as a spectral distribution function, and accordingly we shall write* $dG_x(\omega) = p_x(\omega) d\omega$ *where* $p_x(\omega)$ *is a spectral density.*

Now consider $F_N \in T_N$ with $\underline{F} = (F_n) \equiv F_N(x_n)$ defined by $(K\underline{F})_n = U_n, n = 0, 1, 2, \ldots, N-1$ with K given by (3.12).

From (4.2) we have

$$F_n = \int_{m=0}^{N-1} \left\{ (K^{-1})_{mn} \int_0^T \exp(2\pi i \omega m) ds_u(\omega) \right\}$$

$$= \int_0^T \left[\hat{K}_N(\omega)\right]^{-1} \exp(2\pi i\omega n) ds_u(\omega) \tag{4.3}$$

where

$$\hat{K}_N(\omega) = \frac{1}{N} \sum_{n=0}^{N-1} K_n \exp(-2\pi i\omega n). \tag{4.4}$$

Assume that F_n is estimated by $\sum_{m=0}^{N-1} \ell_m G_{n-m}$, where $\{\ell_m\}$ is a filter which we shall relate to $z_{q;\lambda}$ and $\{G_n\}$ is periodically continued for $n \notin [0, N)$. Then the error

$$F_n - \sum_{m=0}^{N-1} \ell_m G_{n-m} \tag{4.5}$$

is given by

$$\int_0^T \exp(2\pi i\omega n) \left(\hat{K}_N(\omega)\right)^{-1} \hat{\ell}_N(\omega) ds_u(\omega) \\ - \int_0^T \exp(2\pi i\omega n) \hat{\ell}_n(\omega) ds_\epsilon(\omega) \tag{4.6}$$

where $\hat{\ell}_N(\omega)$ is defined as in (4.4).

From Lemma (4.1) the variance of this error is clearly

$$\int_0^T \left|\left[\hat{K}_N(\omega)\right]^{-1} - \hat{\ell}_N(\omega)\right|^2 P_u(\omega) d\omega + \int_0^T |\hat{\ell}_N(\omega)|^2 p_\epsilon(\omega) d\omega \tag{4.7}$$

which is minimized when

$$\hat{\ell}_N(\omega)\hat{K}_N(\omega) = \frac{P_u(\omega)}{P_u(\omega) + P_\epsilon(\omega)}. \tag{4.8}$$

Since the Fourier coefficients of the filtered solution must satisfy

$$\hat{F}_{N,q;\lambda} = \hat{\ell}_{N,q}\hat{G}_{N,q} = z_{q;\lambda}\hat{G}_{N,q}\left[\hat{K}_{N,q}\right]^{-1}.$$

We find from (4.8)

$$\begin{aligned} z_{q;\lambda} &= \hat{\ell}_{N,q}\hat{K}_{N,q} \\ &= \frac{P_u(qh)}{P_u(qh)+P_\epsilon(qh)} \end{aligned} \tag{4.9}$$

where $\hat{\ell}_{N,q} = \hat{\ell}_N(qh)$ and $\hat{K}_{N,q} = \hat{K}_N(qh)$.

5. OPTIMIZATION BY ML.

We now simply relate the filter (4.9) to the p–th order filter (3.17). Assuming that the errors are uncorrelated, $P_\epsilon(\omega)$ has the form

$$P_\epsilon(\omega) = \sigma^2 = \text{constant} \tag{5.1}$$

where σ^2 is the unknown variance of the noise in the data.

Choosing

$$P_u(\omega) = \frac{\sigma^2|\hat{K}_N(\omega)|^2}{\lambda\tilde{\omega}^{2p}} \tag{5.2}$$

where

$$\tilde{\omega} = \begin{cases} 2\pi N\omega, & 0 \le \omega < \frac{1}{2}T \\ 2\pi N(T-\omega), & \frac{1}{2}T \le \omega < T \end{cases}$$

Then yields (3.17) from (4.9). Moreover, the spectral density for $\{G_n\}$ is then

$$P_G(\omega) = P_u(\omega) + P_\epsilon(\omega) = \sigma^2\left[1 + \frac{|\hat{K}_N(\omega)|^2}{\lambda\tilde{\omega}^{2p}}\right]$$

whence

$$P_G(\omega) = \sigma^2\,(1 - z_{q;\lambda})^{-1}\,. \tag{5.3}$$

The statistical likelihood of any suggested values of σ^2 and λ may now be estimated from the data. Following (Whittle [30]), the logarithm of likelihood function of P_G is given approximately by

$$\text{Constant } -\frac{1}{2}\sum_{q=0}^{N-1}[\log P_G(qh) + I(qh)/P_G(qh)] \tag{5.4}$$

where

$$I(\omega) = \left|\sum_{n=0}^{N-1} G_n \exp(-2\pi i\omega n)\right|^2$$

is the periodogram of the data, with $I(qh) = |\hat{G}_{N,q}|^2$.

We now maximize (5.4) with respect to σ^2 and λ. The optimal maximum with respect to σ^2 may be found exactly (in terms sof λ) with the maximizing value of σ^2 given by

$$\sigma^2 = \frac{1}{N}\sum_{q=1}^{N-1}|\hat{G}_{N,q}|^2(1 - z_{q;\lambda}). \tag{5.5}$$

The maximum with respect to λ may then be found by minimizing

$$V_{ML}(\lambda) = \frac{1}{2}N\log\left[\sum_{q=1}^{N-1}|\hat{G}_{N,q}|^2(1 - z_{q;\lambda})\right] - \frac{1}{2}\sum_{q=1}^{N-1}\log(1 - z_{q;\lambda}). \tag{5.6}$$

Thus the optimal regularization parameter is given by the minimizer of the simple function of λ, depending upon the known Fourier coefficients $\hat{G}_{N,q}$ and $\hat{K}_{N,q}$. No prior knowledge of σ^2 is assumed but an *a posteriori*, estimate is given by (5.5). Now (5.6) can be minimized with resepct to λ. In order to minimize $V_{ML}(\lambda)$ in (5.6), we have used a subroutine which uses quadratic interpolation technique to obtain a minimum. In some problems, we have experienced more than one minimum.

6. MOMENTS METHOD.

We consider the classical moment problem where a positive density $F(x)$ is sought from knowledge of its power moments

$$\int_a^b x^n F(x)dx = \mu_n, \quad n = 0,1,2,\ldots, \qquad a \leq x \leq b \tag{6.1}$$

The extent to which the density $F(x)$ may be determined from its moments has been extensively discussed in the literature. In practice, only a finite number of moments, say $N+1$, is usually available. Clearly, then there exist an infinite variety of functions whose first $N+1$ moments coincide and a unique construction of $F(x)$ is not possible (Mead [18]). Nevertheless, various approximation procedures exist which aim at constructing specific sequences of functions $F_N(x)$ such that

$$\int_a^b x^n F_N(x)dx = \mu_n, \quad n = 0,1,2,\ldots,N, \tag{6.2}$$

which eventually converge to the true distribution $F(x)$ as N approaches infinity. A simple possible method is to expand $F(x)$ in some set of orthogonal polynomials. The resulting series is truncated after $N+1$ terms and the expansion coefficients are determined by requiring that the first $N+1$ moments be correct. This entails the solution of a $(N+1) \times (N+1)$ system of linear equations. Judicious choices of weighted orthogonal polynomials could lead to rapidly convergent sequences. In particular, the choice of a suitable weight is usually difficult; so the resulting sequences often produce notoriously oscillating approximation to $F(x)$ which are further impaired by lack of positivity at each finite stage of iteration.

The moment method is equivalent to the inversion of Laplace transformation, in the case when the Laplace transform is sampled at a sequence of equidistant points from the real axis only. Indeed, thanks to the Paley–Wiener theorem on Fourier–Laplace transform, this inversion problem is nothing but the analytic continuation problem. Suppose one gives a sequence $\{\mu_n\}_{n=0,1,2,\ldots}$ of numbers and asks for a real or complex–valued function F such that the domain f is $(0,\infty)$ and

$$\int_0^\infty e^{-st} F(t)dt, \tag{6.3}$$

the Laplace transform of F, equals μ_n as s equals $n + \frac{1}{2}$ and n is a non-negative integer.

The moment problem is an ill–posed one in the sense of Hadamard [13] and it can be shown by exhibiting functions F such that

$$\sum_{n=0}^{\infty} \left| \int_a^b x^n F(x) dx \right|^2, \qquad \text{is arbitrarily small}$$

and

$$\int_a^b |F(x)|^2 \, dx, \text{ is arbitrarily large.}$$

Lots of relevant examples are available in the literature for $[a, b] = [0, 1]$. A satisfactory algorithm for an ill–posed problem should involve the full set of data, including information on noise and *a priori* information on solutions. The moment method fails to meet such a requirement. We have not quoted the results of the method.

7. NUMERICAL RESULTS.

In this section we tabulate the results of the above method applied to the test problems taken from the literature. All data functions have the property $g(s) = O(s^{-1})$ and no noise is added apart from machine-rounding error; only optimal results have been quoted in the table and demonstrated in the diagrams. In each of the test examples, 256 sample points are used to calculate the discrete Fourier coefficients of G and K in (3.2).

Problem 1. This problem has been taken from Cristina [6].

$$g(s) = \frac{1}{(s+1.5)^2}$$
$$f(t) = t \cdot e^{-1.5\,t}.$$

The optimal results are shown in diag (1) and Table 1.

Problem 2. This problem has been taken from Gabuti [10].

$$g(s) = \frac{\beta}{(s+\alpha)^2+\beta^2}$$
$$f(t) = e^{-\alpha t}\sin\beta t.$$

The optimal results are shown in diag (2) and Table 1. In this example we have discussed a class of functions for various values of α and β and quoted the result for $\alpha = 5.0$ and $\beta = 2.2$ in diag (2) and Table 1.

Problem 3. This problem has been taken from Cristina [6].

$$g(s) = \tan^{-1}(1/s)$$

$$f(t) = \frac{\sin t}{t}.$$

The optimal results are shown in diag (3) and Table 1.

Problem 4. This problem has been taken from Gelfat [11].

$$g(s) = \frac{10}{s+0.002} + \frac{45}{s+0.009} + \frac{100}{s+0.02} + \frac{225}{s+0.045} + \frac{1000}{s+0.2} + \frac{2250}{s+0.45}$$

$$f(t) = 10\,e^{-0.002\,t} + 45\,e^{-0.009\,t} + 100\,e^{-0.02\,t} + 225\,e^{-0.045} + 1000\,e^{-0.2\,t} + 2250\,e^{-0.45\,t}$$

The optimal results are shown in diag (4) and Table 1.

Table 1

Problem	a	T	h	λ	$v(\lambda)$	$\lVert f - f_\lambda \rVert_\infty$	**Diags.**
1	10.0	12.50	0.04883	0.3499×10^{-12}	0.5223×10^{-2}	0.0003	1
2	5.0	11.50	0.0457	0.362×10^{-9}	0.1362×10^{4}	0.003	2
3	8.0	9.50	0.0371	0.11×10^{-7}	0.7939×10^{3}	0.07	3
4	10.0	12.10	0.4727	0.99×10^{-13}	0.2875×10^{4}	0.07	4

CONCLUDING REMARKS.

The maximum likelihood (ML) method worked very well over all the four test problems which are severely ill–posed. The optimal results are shown in diags (1–4) and Table 1. As regards the moment method, some authors (Mead [18] and others) have experienced that the procedure becomes unstable as n, the number of moments, increases. Small number of moments have been used and for large number of moments, it is not clear whether the numerical accuracy will improve or not and it is not possible to estimate the error accurately. Our method is simpler and less expensive and yields very good results.

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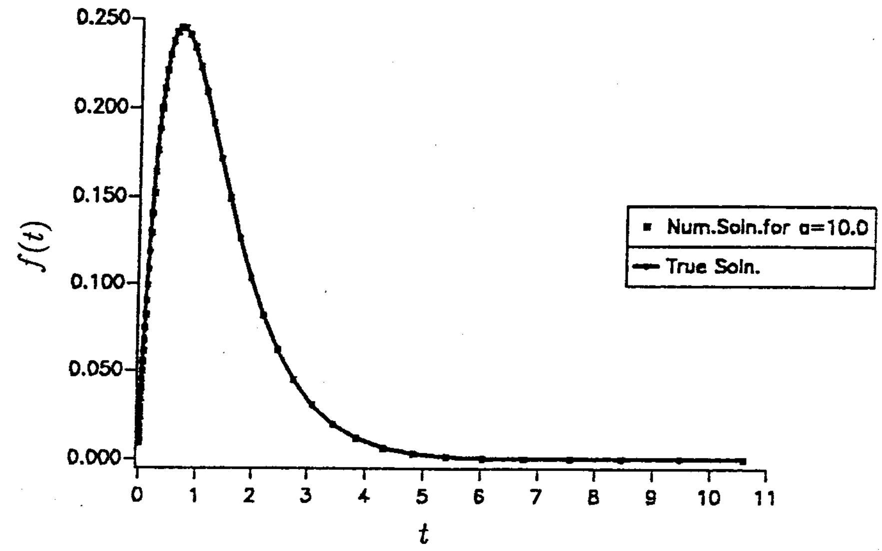
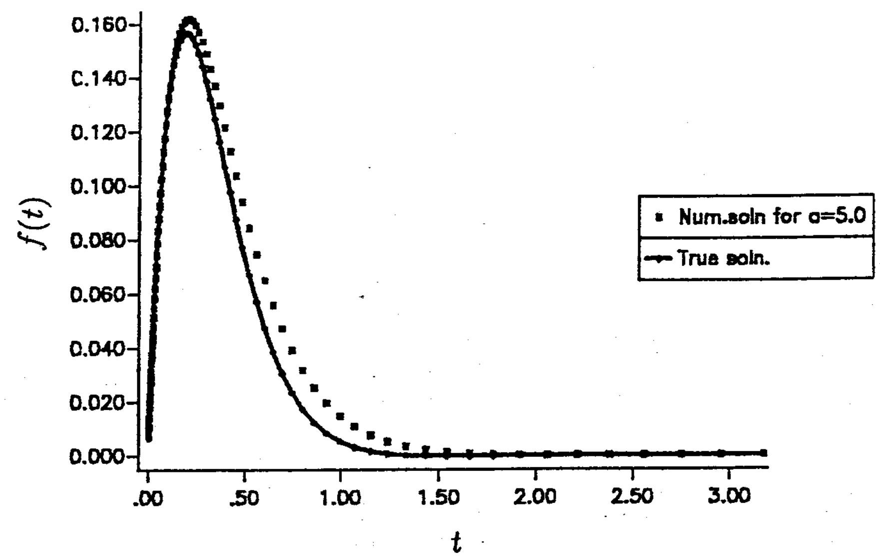
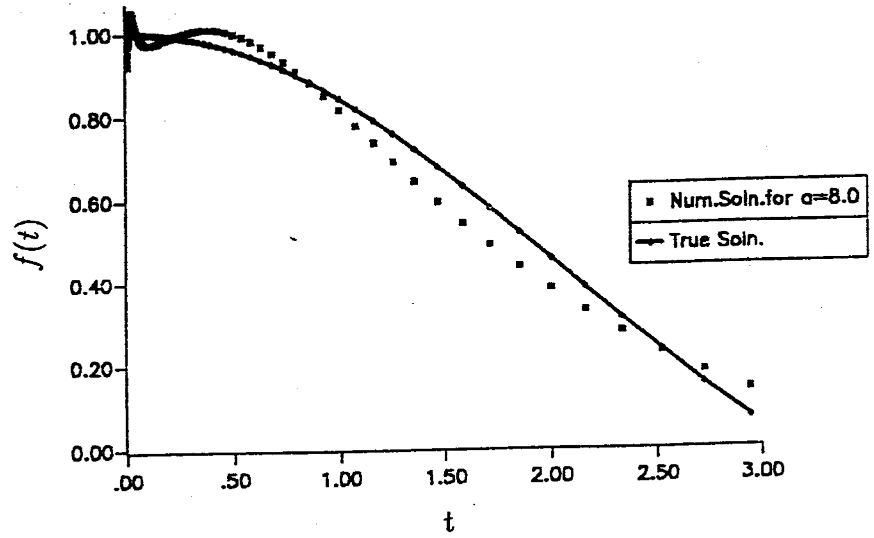
Diag(1) Maximum Likelihood Method



Diag(2) Maximum Likelihood Method



Diag(3) Maximum Likelihood Method



Diag(4) Maximum Likelihood Method