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Abstract

In this paper, we discuss a property of large systems of equations over universal algebras which does not appear to be generally known. It is shown, for example, that if α is a cardinal number with uncountable cofinality, then every finitely solvable system of α equations over any countable algebra has a solvable subsystem consisting also of α equations. Some pathological (and some not so pathological) aspects of this notion when compared with equational compactness are presented. As an application, this property is used to generalize some model-theoretic results of Jensen and Lenzing on the non-compactness of ultrapowers of modules.

In [12], McKenzie called a universal algebra $\mathfrak{A}$ (equationally) α-incompact, where α is an infinite cardinal, if there exists a system of α equations with constants in $\mathfrak{A}$ and with no solution in $\mathfrak{A}$, but such that each subsystem of less than α equations is solvable in $\mathfrak{A}$ (in the terminology of [8], an algebra is α-incompact if and only if it is not (α, α^+)-compact). The abelian group $\mathbb{Z}$ is known to be α-incompact when $\alpha = \aleph_0$ or $\aleph_1$ (see Mycielski [13] and McKenzie [12]), and Łoś [11], proved that for any non-measurable regular cardinal β, there exists a cardinal α with $\beta \leq \alpha \leq 2^\beta$ such that $\mathbb{Z}$ is α-incompact. However, as we shall prove in this paper, $\mathbb{Z}$ (as well as all countable algebras and all modules over any countable ring) has the following interesting property: every finitely solvable system consisting of α equations over $\mathbb{Z}$ (where α is any cardinal with uncountable cofinality) has a solvable subsystem consisting also of α equations. This property is motivated by Mycielski's seminal paper [13], where he mentions the possibility of infinite systems of equations which are finitely solvable, but such that no infinite subsystem of which is solvable: the system of equations

$$x = y_n^2 + n \qquad (n \in \mathbb{N})$$

over the ring $\mathbb{R}$ of real numbers is finitely solvable but no infinite subsystem of it is solvable. Our aim is to discuss such a possibility for general algebras with emphasis on modules, and as an application, to derive a sufficient condition for the non-compactness of certain reduced powers (Lemma 5). This will yield a generalization of some results of Jensen and Lenzing [5]. Specifically, let us say that an infinite system of equations over an algebra $\mathfrak{A}$ is *α-subsolvable*, where α is an infinite cardinal, if it has a solvable subsystem in $\mathfrak{A}$ consisting of α equations. An algebra $\mathfrak{A}$ will be called *α-subcompact*, if every finitely solvable system of α equations over $\mathfrak{A}$ is α-subsolvable. We recall that an algebra $\mathfrak{A}$ is *(equationally) α-compact* if every finitely solvable system of α equations with constants in $\mathfrak{A}$ is solvable. If $\mathfrak{A}$ is α-compact for all α, it is said to be *equationally compact.* (By a result of Mycielski and Ryll-Nardzewski [14], $\mathfrak{A}$ is equationally compact if it is $|A|$-compact.) It is clear that every α-compact algebra is α-subcompact, and, in fact, as we shall see (Proposition 3), these two notions are equivalent for abelian groups when $\alpha = \aleph_0$; but, in general α-subcompact modules need not be α-compact. We should point out that equational compactness of general algebras was first introduced by Mycielski in [13], and is equivalent, for modules, to algebraic compactness (or pure-injectivity) in the sense of Fuchs [2] or Warfield [15].

The paper is divided into two parts. In the first one, we give some general properties of α-subcompactness and obtain such results as:

(i) Countable general algebras and modules over countable rings are α-subcompact for each cardinal α with uncountable cofinality.

(ii) If a commutative noetherian domain is $\aleph_0$-subcompact as a module over itself, then it is local.

The second part is devoted to the construction of systems of equations $S_n (n \geq 1)$ that are $\aleph_{n-1}$-subsolvable but not $\aleph_n$-subsolvable, and modules $M_n (n \geq 1)$ that are $\aleph_{n-1}$-compact but not $\aleph_n$-subcompact (and therefore not $\aleph_n$-compact). It is interesting to note that these S_n and M_n will be determined once we construct a finitely solvable system of equations S_0 which is not countably solvable, and a module M_0 which is not $\aleph_0$-compact.

Let us mention here that modules (over principal ideal domains) which are $\aleph_n$-compact but not equationally compact have been discussed by Fuchs in [2], via a topological argument. However, the construction suggested there used a result which has been shown in [6] to be true if and only if α is not weakly inaccessible. (See also [9] for a construction over perfect rings of modules which are α-compact but not α^+-compact.) In addition, the construction we give here is purely algebraic.

Along another direction, Jensen and Lenzing proved in [5, Proposition 8.40] that if R is the polynomial ring $\mathbb{C}[X]$, then there is no ultrafilter $\mathcal{F}$ on $\mathbb{N}$ for which the R-module $R^{\mathbb{N}}/\mathcal{F}$ is equationally compact. We end this paper with a generalization of this and another result in [5], using subcompactness. In particular, our Proposition 11 yields that if K is any uncountable field with R its polynomial ring $K[X]$ and if $\mathcal{F}$ is any non-trivial filter on the integers $\mathbb{N}$, then $R^{\mathbb{N}}/\mathcal{F}$ is not an $\aleph_1$-compact R-module.

Throughout this paper, R denotes an associative ring with 1 and all modules are unitary left R-modules. A theory of ordinals is assumed where each ordinal $x = \{y : y < x\}$, and where cardinals are initial ordinals. For any infinite cardinal α, α^+ denotes the successor cardinal of α and $\mathcal{F}_\alpha$ denotes the Fréchet filter $\{J \subseteq \alpha : |\alpha \backslash J| < \alpha\}$. Given a set I and a module M, $|I|$, M^I and $M^{(I)}$ denote respectively the cardinality of I, the direct product and the direct sum of $|I|$ copies of M.

1. Results

Proposition 1. Let R be $\aleph_0$-subcompact as a left module over itself, let $r \in R$ and let J be the Jacobson radical of R. If $\bigcap_{n\in\mathbb{N}} (1-r)^n R \subseteq J$, then r has a right inverse.

Proof. Let $r \in R$ and assume that $\bigcap_{n\in\mathbb{N}} (1-r)^n R \subseteq J$, it is easy to see that the system

$$rx_0 + (1-r)^n x_n = 1 \qquad (n \in \mathbb{N})$$

is finitely solvable in R. It has therefore a solvable subsystem consisting of $\aleph_0$ equations. So there exist a strictly increasing sequence $\{n_i : i \in \mathbb{N}\}$ of natural numbers and elements a_0, a_i $(i \in \mathbb{N})$ in R such that

$$1 - ra_0 = (1-r)^{n_1} a_1 = (1-r)^{n_2} a_2 = \cdots .$$

Hence $1 - ra_0 \in \bigcap_{n\in\mathbb{N}} (1-r)^n R \subseteq J$. This implies that ra_0 is a unit and so r has a right inverse.

Corollary 2. (i) Let R be a commutative noetherian domain, and suppose it is $\aleph_0$-subcompact as a module over itself, then R is local.

(ii) For any ring R, $R[x]$ is not $\aleph_0$-subcompact.

Proof. (i) Suppose that R is a non-local commutative noetherian domain and assume it is $\aleph_0$-subcompact. Since it is not local, R contains a non-unit r for which $1-r$ is also a non-unit. By Krull's Intersection Theorem, $\bigcap_{n\in\mathbb{N}} (1-r)^n R = 0$, and by Proposition 1, r has an inverse, which is a contradiction.

(ii) Let J be the Jacobson radical of the ring $R[x]$. Then $\bigcap_{n\in\mathbb{N}} x^n R[x] = 0 \subseteq J$, but $1 - x$ does not have a right inverse. By Proposition 1, $R[x]$ is not $\aleph_0$-subcompact.

Remark. If, in Corollary 2 (i), $\aleph_0$-subcompactness of R as an R-module is replaced by the stronger equational compactness of R as a ring (so that we allow for systems of equations to contain polynomial equations instead of restricting them to the linear ones), then R is not only local, but has finite residue field and, if R is infinite, it has the power of the continuum as well (see [3] and [4]). The existence of fields of arbitrary infinite cardinality, and the fact that fields are always compact when considered as modules over themselves, show that the conclusion of Corollary 2 (i) cannot be strengthened in this direction.

Proposition 3. An abelian group is $\aleph_0$-subcompact if, and only if, it is (equationally) compact.

Proof. We need only prove that if G is an abelian group which is $\aleph_0$-subcompact, then it is compact. The proof is based on a result of Balcerzyk (see [1] and [16, Problem 11]). Let

$$x_0 - n!x_n = a_n \qquad (n \in \mathbb{N})$$

be a finitely solvable system of equations over G. It is enough to prove that this system is solvable in G. Since G is $\aleph_0$-subcompact, the system has a countable subsystem $x_0 - n_i!x_{n_i} = a_{n_i}$ $(i \in \mathbb{N},\ n_1 < n_2 < \cdots)$ solvable in G by b_0, b_{n_i} $(i \in \mathbb{N})$, say. Assume that the whole system is not solvable. We can clearly suppose that there exist $i, j \in \mathbb{N}$ such that

(a) $j < n_i$

(b) $b_0 - k!x_k = a_k$ is solvable whenever $j < k \leq n_i$

(c) $b_0 - j!x_j = a_k$ is not solvable.

Now $b_0 - (j+1)!b_{j+1} = a_{j+1}$ for some $b_{j+1} \in G$ by (b), and $x_0 - j!x_j = a_j$, $x_0 - (j+1)!x_{j+1} = a_{j+1}$ are simultaneously solvable by c_0, c_j, c_{j+1} say. Put $b_j = (j+1)(b_{j+1} - c_{j+1}) + c_j$, then $b_0 - j!b_j = a_j$, which contradicts (c). This completes the proof.

Although $\aleph_0$-subcompactness and $\aleph_0$-compactness coincide for abelian groups, it is in general not true that α-subcompact modules are α-compact. To see this, let us first note the following

Proposition 4. Let $\{M_i\}_{i \in I}$ be a family of α-subcompact R-modules such that $cf(\alpha) > |I|$. Then $\bigoplus_{i \in I} M_i$ is α-subcompact.

Proof. Let $\sum_{k \in K} r_{jk}x_k = a_j \quad (a_j \in \bigoplus_{i \in I} M_i,\ j < \alpha)$ be a finitely solvable system of equations over $\bigoplus_{i \in I} M_i$. Assume first that I is finite, say $I = \{1, 2, \ldots, n\}$. Since $\sum_{k \in K} r_{jk}x_k = a_j(1)$ is finitely solvable in M_1, there exist $\{m_k(1)\}_{k \in K}$ in M_1 and $J_1 \subseteq \alpha$ with $|J_1| = \alpha$ such that $\sum_{k \in K} r_{jk}m_k(1) = a_j(1)$ for each $j \in J_1$. Similarly, there exist $\{m_k(2)\}_{k \in K}$ in M_2 and $J_2 \subseteq J_1$ with $|J_2| = \alpha$ such that $\sum_{k \in K} r_{jk}m_k(2) = a_j(2)$ for each $j \in J_2$. Continuing in this way, we obtain for each i in I with $i \geq 2$, a set $\{m_k(i)\}_{k \in K}$ in M_i and a subset $J_i \subseteq J_{i-1}$ such that $|J_i| = \alpha$ and $\sum_{k \in K} r_{jk}m_k(i) = a_j(i)$ for all $j \in J_i$. It is now easy to see that $\{(m_k(1), m_k(2), \ldots, m_k(n))\}_{k \in K}$ is a solution of $\sum_{k \in K} r_j x_k = a_j \quad (j \in J_n)$. Now suppose that I is infinite and let $f : \alpha \to \mathcal{P}_\omega(I)$, where $\mathcal{P}_\omega(I) = \{X \subseteq I : X \text{ is finite}\}$, be the function given by $f(j) = s(a_j)$, the support of a_j in $\bigoplus_{i \in I} M_i$. Since $cf(\alpha) > |I| = |I|^{\underset{\smile}{\omega}} = |\mathcal{P}_\omega(I)|$, it follows that for some $X \subseteq \mathcal{P}_\omega(I)$, $|f^{-1}(\{X\})| = \alpha$. By the first part of this proof, $\bigoplus_{i \in X} M_i$ is α-subcompact, and hence the finitely solvable subsystem $\sum_{k \in nK} r_{jk}x_k = a_j \quad (j \in f^{-1}(\{X\}))$ has a solvable subsystem consisting of α equations, as required.

Remark. Let M be an equationally compact R-module such that $M^{(\mathbb{N})}$ is not $\aleph_0$-compact (such modules exist whenever the underlying ring R is not representation-finite). Then, for each cardinal α with uncountable cofinality, $M^{(\mathbb{N})}$ is α-subcompact by Proposition 4, but is not α-compact.

A further peculiarity of subcompactness is that α^+-subcompact modules need not be α-subcompact. To show this, we shall use reduced products. We first need

Lemma 5. Let $\mathfrak{A}$ be an algebra and let α, β be cardinals such that $cf(\alpha) > \beta$. If $\mathfrak{A}^\beta/\mathcal{F}$ is α-subcompact for some non-trivial filter $\mathcal{F}$ on β, then so also is $\mathfrak{A}$.

Proof. Let $\{R_j\}_{j<\alpha}$ be a finitely solvable system of equations in $\mathfrak{A}$ with a set of unknowns $\{x_k\}_{k\in K}$ and a set of constants $\mathcal{C}$. The system of equation $\{\overline{R}_j\}_{j<\alpha}$ obtained from $\{R_j\}_{j<\alpha}$ by replacing each $c \in \mathcal{C}$ by its image under the diagonal map $d : \mathfrak{A} \to \mathfrak{A}^\beta/\mathcal{F}$ is also finitely solvable in $\mathfrak{A}^\beta/\mathcal{F}$. Since this is α-subcompact, there exist $\{\overline{a}_k\}_{k\in K}$ in $\mathfrak{A}^\beta/\mathcal{F}$ satisfying $\{\overline{R}_j\}_{j\in J}$, where $J \subseteq \alpha$ and $|J| = \alpha$. This means that for each $j \in J$, the set $X_j = \{\sigma < \beta : \{a_k(\sigma)\}_{k\in K}$ satisfies $R_j\} \in \mathcal{F}$. For each $j \in J$ pick a σ_j in X_j, and consider the map $f : J \longrightarrow \beta$ given by $f(j) = \sigma_j$. Since $cf(\alpha) > \beta$,it follows that there exists $\sigma_0 \in \beta$ such that $|f^{-1}(\{\sigma_0\})| = \alpha$. It is now clear that $\{a_k(\sigma_0)\}_{k\in K}$ satisfies $\{R_j\}_{j\in f^{-1}(\{\sigma_0\})}$.

Remark. The foregoing lemma shows that, under rather mild conditions and in contrast to equational compactness, subcompactness of a reduced power of an algebra $\mathfrak{A}$ is inherited by $\mathfrak{A}$.

An immediate consequence of Lemma 5 is

Corollary 6. Let R be a ring and let α be a cardinal with $cf(\alpha) > \aleph_0 + |R|$. Then every R-module is α-subcompact. In particular, every abelian group is α-subcompact for each cardinal α with uncountable cofinality.

Proof. Using for example [5, Theorem 7.50], there exists an ultrafilter $\mathcal{F}$ on $\beta = \aleph_0 + |R|$ such that $M^\beta/\mathcal{F}$ is equationally compact for each R-module M. By Lemma 5, M is α-subcompact.

For algebras we infer the following

Corollary 7. Suppose that $\mathfrak{A}$ is an algebra such that $|A| = \aleph_m$ for some non-negative integer m, and let α be a cardinal such that $cf(\alpha) > \aleph_m$. Then $\mathfrak{A}$ is α-subcompact

Proof. Put $\mathfrak{A}_0 = \mathfrak{A}$ and for each $n \geq 0$ let $\mathfrak{A}_{n+1} = \mathfrak{A}_n^{\omega_n}/\mathcal{F}_{\omega_n}$. By [6], $\mathfrak{A}_{n+1}$ is $\aleph_n$-compact. Hence $\mathfrak{A}_{m+1}$ is $\aleph_m$-compact and therefore equationally compact (by [14]). This means $\mathfrak{A}_{m+1}$ is α-subcompact and so, using Lemma 5 repeatedly, we obtain that $\mathfrak{A}$ is α-subcompact.

It follows from Corollary 6 that, although $\mathbb{Z}$ is not $\aleph_0$-subcompact, it is α-subcompact for all α with uncountable cofinality. It is however not $\aleph_\omega$-subcompact as the following result shows.

Proposition 8. Let α, β be cardinals such that $cf(\alpha) = \beta$ and let $\mathfrak{A}$ be an α-subcompact algebra. Then $\mathfrak{A}$ is β-subcompact.

Proof. By definition, β is the least cardinal such that α is the sum of β cardinals each of which is less than α. So there are cardinals $c_t < \alpha$ $(t < \beta)$ such that $\alpha = \sum_{t<\beta} c_t$. Let

$$A_0 = \{x < \alpha : x < c_0\}, \ \ A_t = \{x < \alpha : x < c_t\} \setminus \bigcup_{s<t} A_s \qquad (t < \beta).$$

Define a function $f : \alpha \to \beta$ by writing $f(x) = t$, where $x \in A_t$ $(f$ is well-defined since $\alpha = \bigcup_{t<\beta} A_t$

and the A_t are mutually disjoint). Let $Q \subseteq \alpha$ with $|Q| = \alpha$. Then

$$Q \subseteq \bigcup_{x \in Q} A_{f(x)} \subseteq \bigcup_{y \in f(Q)} A_y.$$

Hence $\alpha = |Q| \leq |\bigcup_{y\in f(Q)} A_y| = \sum_{y \in f(Q)} |A_y| \leq \sum_{t<\beta} |A_t| = \alpha$. Thus $\alpha = \sum\limits_{y \in f(Q)} |A_y|$ and hence $|f(Q)| = \beta$. Now let $\{R_j\}_{j<\beta}$ be a finitely solvable system of equations over $\mathfrak{A}$ and consider the system $\{R_{f(i)}\}_{i<\alpha}$. This is clearly a finitely solvable system of α equations in $\mathfrak{A}$ and therefore has a solvable subsystem of α equations. Thus there exists $Q \subseteq \alpha$ with $|Q| = \alpha$ such that $\{R_{f(i)}\}_{i\in Q}$, i.e. $\{R_j\}_{j \in f(Q)}$, is solvable in $\mathfrak{A}$. Since $|f(Q)| = \beta$, by the first part, it follows that $\mathfrak{A}$ is β-subcompact.

2. Examples

Our objective in this section is to construct a ring R and for each $\aleph_n$ $(n \geq 0)$, an R-module which is $\aleph_n$-compact but not $\aleph_{n+1}$-subcompact. The following result is needed.

Proposition 9. Let M be an R-module, let α, β be cardinals such that $\alpha > \beta$ and α is regular and suppose that the system

$$S_\beta : \sum_{k \in K} r_{jk} x_k = a_j \qquad (a_j \in M,\ j < \alpha)$$

is γ-solvable for all $\gamma < \beta$ but is not β-subsolvable. Then the system

$$S_{\beta^+} : \sum_{k\in K} r_{jk} x_k = \overline{(\cdots, a_j, a_j, \ldots)} \qquad (j < \beta)$$

over $M^\beta / \mathcal{F}_\beta$ is β-solvable but is not β^+-subsolvable.

Proof. We first prove that S_{β^+} is β-solvable. It is enough to show that the subsystem S'_{β^+} : $\sum\limits_{k\in K} r_{jk}x_k = \overline{(\cdots, a_j, a_j, \ldots)}$ $\quad (j < \beta)$ is solvable. Since $\sum\limits_{k\in K} r_{jk}x_k = a_j$ $\quad (j < \alpha)$ is γ-solvable in M for all $\gamma < \beta$, it follows that for each $t < \beta$, there exist m_k^t $(k \in K)$ in M such that $\sum_{k\in K} r_{jk}m_k^t = a_j$ $(j \leq t)$. Let $m_k \in M^\beta$ be defined by $m_k(t) = m_k^t$ $(t < \beta)$. Then for each $j < \beta$,

$$|\beta \setminus z(\sum_{k\in K} r_{jk}m_k - (\cdots, a_j, a_j, \ldots))| \leq |j| < \beta.$$

Therefore $\overline{m_k}$ $(k \in K)$ solves S'_{β^+} in $\mathcal{F}_\beta$. We next prove that no β-subsystem of S_{β^+} is solvable. Assume the contrary. Without loss of generality we can assume that

$$\sum_{k\in K} r_{jk}x_k = \overline{(\cdots, a_j, a_j, \ldots)} \quad (j < \beta^+)$$

has a solution $\overline{u}_k$ $(k \in K)$ in $M^\beta/\mathcal{F}_\beta$. Then, using the same argument as in the proof of Proposition 6, we see that there exists a subset Q of β^+ with $|Q| = \beta^+$ such that $\sum_{k\in K} r_{jk}u_k(i) = a_j$ $(j \in Q)$ for some $i < \beta$. This clearly contradicts the fact that no β-subsystem of S_β is solvable.

Corollary 10. Let M_0 be an R-module and let S_0 be a system of $\aleph_\omega$ equations over M_0 which is finitely solvable in M_0 but not $\aleph_0$-subsolvable in M_0. Then, for each positive integer n, there exist an R-module M_n and a system S_n of $\aleph_\omega$ equations over M_n which is $\aleph_{n-1}$-solvable in M_n but not $\aleph_n$-subsolvable in M_n.

Proof. Let S_0 be the system

$$\sum_{k\in K} r_{jk}x_k = a_j \qquad (a_j \in M_0,\ j < \omega_\omega).$$

For each $n \geq 1$, let $M_n = M_{n-1}^{\omega_{n-1}}/\mathcal{F}_{\omega_{n-1}}$, let $f_{n-1} : M_{n-1} \to M_n$ be the homomorphism given by $f_{n-1}(a) = (\ldots, a, a, a, \ldots) + \mathcal{F}_{\omega_{n-1}}$ (where $\mathcal{F}_{\omega_{n-1}}$ is identified with the submodule $\{m \in M_{n-1}^{\omega_{n-1}} : z(m) \in \mathcal{F}_{\omega_{n-1}}\}$), and let $g_n = f_{n-1} \circ f_{n-2} \circ \cdots \circ f_0$. Consider the systems $S_n : \sum_{k\in K} r_{jk}x_k = g_n(a_j)$ $(j <$

ω_ω). By Proposition 10, with $\alpha = \omega$, and using induction on n, we obtain that S_n is $\aleph_{n-1}$-solvable in M_n but no $\aleph_n$-subsystem of it is solvable in M_n.

Remark. By [2] or [6, Theorem 2], for each $n \geq 0$, M_n above is $\aleph_n$-compact but not $\aleph_{n+1}$-subcompact.

Construction (A.J. Douglas and A. Laradji). From Corollary 10, in order to construct the modules M_n and the systems S_n $(n \in \mathbb{N})$, we need only find a ring R, an R-module M_0 and a system S_0 of $\aleph_\omega$ equations over M_0 which is finitely solvable in M_0 but not $\aleph_0$-subsolvable.

Let A be a commutative ring and suppose that it contains a subset $T = \{a_i\}_{i \in I}$ of $\aleph_\omega$ elements that are not zero-divisors and such that $a_i - a_j$ is a unit whenever $i \neq j$. Let R be the polynomial ring $A[t]$ and let $M_0 = R$ considered as a module over itself. We introduce a partial order on the elements of M_0. We say that $p_1 \leq p_2$ $(p_1, p_2 \in M_0)$ if there exists $r \in R$ such that $tp_2 + 1 = r(tp_1 + 1)$. It is easy to check that $\leq$ is indeed a partial order. We show that $\{M_0, \leq\}$ is directed. Let $p_1, p_2 \in M_0$ and let $q = tp_2p_2 + p_1 + p_2$. Then $tq + 1 = (tp_1 + 1)(tp_2 + 1)$ and hence $q \leq p_1, p_2$. We further observe that if $a_{i_1}, \ldots, a_{i_n}$ are distinct elements of T, and if $q \geq a_{i_k}$ $(1 \leq k \leq n)$,where $q \in M_0$, then

$$tq + 1 \in \bigcap_{k=1}^{n} R(ta_{i_k} + 1) = \prod_{k=1}^{n} R(ta_{i_k} + 1),$$

since the ideals $R(ta_{i_k} + 1)$ $(1 \leq k \leq n)$ are mutually coprime.This implies that $degree(q) \geq n - 1$. Now consider the system over M_0

$$S_0 : x - (a_i t + 1)x_{a_i} = a_i \qquad (i \in I).$$

We claim that S_0 is finitely solvable in M_0. For, if $q \geq a_{i_k}$ $(1 \leq k \leq n)$ then there exist $p_{i_k} \in R$ such that $tq + 1 = p_{i_k}(a_{i_k}t + 1)$. Since $p_{i_k}(0) = 1$, it follows that $p_{i_k} = s_{i_k}t + 1$ for some $s_{i_k} \in R$. It is now easy to check that

$$q - (a_{i_k}t + 1)s_{i_k} = a_{i_k} \quad (1 \le k \le n),$$

which proves our claim. However, no countable (and hence no $\aleph_n$-)subsystem of S_0 is solvable in M_0. For, if

$$q - (a_{i_k}t + 1)s_{i_k} = a_{i_k} \qquad (k \in \mathbb{N})$$

for some q, s_{i_k} $(k \in \mathbb{N})$ in M_0, then $tq + 1 = (a_{i_k}t + 1)(s_{i_k}t + 1)$, and so $q \ge a_{i_k}$ $(k \in \mathbb{N})$ and this is impossible, by the above observation about the degree of q.

3. Application

The equational compactness of reduced products of modules has been studied for several types of rings. It is shown in many instances that the equational compactness of certain reduced products of modules, which, regardless of the structure of the underlying ring are always $\aleph_0$-compact, forces the ring to have certain properties (see for example [7] and [9] for more details and references). Lemma 5 provides a universal algebraic result in that direction. (Note that if there exist countably many members of a non-trivial filter $\mathcal{F}$ on β whose intersection is empty, e.g. if $\mathcal{F}$ is an ω-incomplete ultrafilter on β, then $\mathfrak{A}^\beta/\mathcal{F}$ is $\aleph_0$-compact for any algebra $\mathfrak{A}$.) In the case of modules this provides a supply of reduced powers which fail to be equationally compact, as the following generalization of [5, Proposition 8.40] shows

Proposition 11. Let n be a non-negative integer, and let A be a commutative ring containing an uncountable subset $\{a_i\}_{i<\omega_{n+1}}$ of non-zero divisors such that $a_i - a_j$ is a unit whenever $i \ne j$ (e.g. if A is a field with $|A| > \aleph_n$). If R is the polynomial ring $A[X]$, then for each non-trivial filter $\mathcal{F}$

on ω_n, the R-module $R^{\omega_n}/\mathcal{F}$ is not $\aleph_{n+1}$-subcompact, and therefore not equationally compact.

Proof. The construction above can be modified to show that R is not $\aleph_{n+1}$-subcompact, and so by Lemma 5, $R^{\omega_n}/\mathcal{F}$ is not $\aleph_{n+1}$-subcompact.

Proposition 11 also can also be used to extend [5, Theorem 8.45]:

Corollary 12. Let K be an uncountable field and let R be the polynomial ring in n indeterminates $K[X_1, X_2, ..., X_n]$. Then for each non-trivial filter $\mathcal{F}$ on $\mathbb{N}$, the R-module $R^{\mathbb{N}}/\mathcal{F}$ is not $\aleph_1$-subcompact, and therefore not equationally compact.

Let us note finally that subcompactness can be used to refine results appearing in [10]:

1. *If a non-zero free left R-module M is Σ-$\aleph_0$-subcompact (that is every direct sum of copies of M is $\aleph_0$-subcompact), then R is left perfect.*

2. *Let R be a commutative ring and assume that the residue fields of the local ring factors of R are infinite. If each R-module is $\aleph_0$-subcompact, then each R-module is equationally compact, and therefore R has finite representation type, that is the ring R is artinian and has only finitely many non-isomorphic indecomposable modules.*

The proofs follow from suitable modifications of those that appear in [10].

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