King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 228

April 1998

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PP-RINGS OF GENERALIZED POWER SERIES

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1. Introduction and Preliminaries

This paper is motivated by one of Fraser and Nicholson [3] in which it was proved that $R[[x]]$ is a reduced PP-ring if and only if R is a reduced PP-ring and any countable family of idempotents of R has a least upper bound in $B(R)$, the set of all central idempotents, and by a series of papers on rings of generalized power series (cf. [2, 5–9]). We will show that if R is a commutative ring, and $(S, \leq)$ is a strictly totally ordered monoid, then the ring $[[R^{S,\leq}]]$ of generalized power series is a PP-ring if and only if R is a PP-ring and every S-indexed subset C of $B(R)$ has a least upper bound in $B(R)$ (Theorem 2.3). A ring R is called a *weakly PP-ring* if every principal left ideal ${}_RRer$, considered as a left R-module, is projective for each $r \in R$ and each primitive idempotent $e \in R$. In [1], it was proved that if R is a normal ring then R is a weakly PP-ring if and only if $R[x]$ is a weakly PP-ring, and also an example of weakly PP-ring but not PP-ring was given. In section 3 we will show that if R is a commutative ring, and $(S, \leq)$ is a strictly totally ordered monoid which satisfies the condition that $0 \leq s$ for each $s \in S$, then the ring $[[R^{S,\leq}]]$ is weakly PP if and only if R is weakly PP. As a corollary, we obtain that a commutative ring R is weakly PP if and only if $R[[x]]$ is weakly PP, and an example of *commutative* weakly PP-ring but not PP-ring is given.

All rings considered here are associative with identity. Any concept and notation not defined here can be found in [6–9]. The right and left annihilators of a subset X of R will be denoted $r(X)$ and $\ell(X)$ respectively.

Let $(S \leq)$ be an ordered set. Recall that $(S, \leq)$ is *artinian* if every strictly decreasing sequence of elements of S is finite, and that $(S, \leq)$ is *narrow* if every subset of pairwise order-incomparable elements of S is finite. Let S be a commutative monoid. Unless stated otherwise, the operation of S will be written additively, and the neutral element will be denoted by 0. The following definition is due to [5–8].

Let $(S, \leq)$ be a strictly ordered monoid (that is, $(S, \leq)$ is an ordered monoid satisfying the condition that if $s, s', t \in S$ and $s < s'$, then $s + t < s' + t$), and R a commutative ring. Let $A = [[R^{S,\leq}]]$ be the set of all maps $\phi : S \to R$ such that $\text{supp}(\phi) = \{s \in S | \phi(s) \neq 0\}$ is artinian and narrow. With pointwise addition, A is an additive abelian group. For every $s \in S$ and $\phi, \psi \in A$, let $X_s(\phi, \psi) = \{(u, v) \in S \times S | s = u + v, \phi(u) \neq 0, \psi(v) \neq 0\}$. It follows from [8, 1.16] that $X_s(\phi, \psi)$ is finite. This fact allows one to define the operation of convolution

$$(\phi\psi)(s) = \sum_{(u,v) \in X_s(\phi,\psi)} \phi(u)\psi(v).$$

With this operation, and pointwise addition, A becomes a commutative ring, which is called the *ring of generalized power series.* The elements of A are called *generalized power series* with coefficients in R and exponents in S.

For example, if $S = N$ and $\leq$ is the usual order, then $[[R^{N,\leq}]] \cong R[[x]]$, the usual ring of power series. If S is a commutative monoid and $\leq$ is the trivial order, then $[[R^{S,\leq}]] = R[S]$, the monoid-ring of S over R. Further examples are given in [7]. The following results appeared in [8]. Recall that a monoid S is torsion-free if the following property holds: if $s, t \in S$, k is an integer, $k \geq 1$ and $ks = kt$, then $s = t$.

LEMMA 1.1. (1) *If S has a compatible strict total order $\leq$, then S is cancellative and torsion-free.*

(2) *If S is a cancellative and torsion-free monoid, and $\leq$ is any compatible order on S (for example, the trivial order), then there exists a compatible total order on S, which is finer than $\leq$.*

LEMMA 1.2. *Let S be a cancellative and torsion-free monoid and R a commutative ring. If $\phi, \psi \in [[R^{S,\leq}]]$ and $\phi\psi = 0$ then $\phi(s)\psi(t)$ is a nilpotent element of R for every $s, t \in S$.*

2. PP-rings

A ring R is called a *left PP-ring* if every principal left ideal of R is projective. These rings have been studied since Hattori [4] introduced them. According to Fraser and Nicholson [3], a ring R is called a *reduced left PP-ring* if it is a left PP-ring with no nonzero nilpotent elements. An element a in R will be called *entire* if $\ell(a) = 0 = r(a)$. The following result appeared in [3].

LEMMA 2.1. *The following conditions are equivalent for a ring R:*

(1) *R is a reduced left PP-ring.*

(2) *If $a \in R$ then $a = eb = be$ where $e^2 = e \in R$ and $b \in R$ is entire.*

(3) *R is a left PP-ring with every idempotent central.*

We will use the following notations.

Let $r \in R$. Define a mapping $c_r \in A$ as follows:

$$c_r(0) = r, \ \ c_r(s) = 0, \quad 0 \neq s \in S.$$

LEMMA 2.2. *Let R be a reduced commutative ring and S a cancellative and torsion-free monoid. If $\phi^2 = \phi \in [[R^{S,\leq}]]$ then there exists an idempotent $e \in R$ such that $\phi = c_e$.*

PROOF. If $\phi^2 = \phi$, then $\phi(\phi - c_1) = 0$. Thus, by Lemma 1.2, $\phi(s)\phi(t) = 0$ for any $s \in S$ and $0 \neq t \in S$, which implies that $\phi(t) = 0$ for any $0 \neq t \in S$. Set $\phi(0) = e \in R$. Then $\phi = c_e$. Now form $c_e^2 = c_e$, it follows that $e^2 = e$.

In any commutative ring R the set $B(R)$ of all idempotents is a Boolean algebra where $e \leq f$ means $ef = e$, and where the join, meet and complement are given by $e \vee f = e + f - ef, e \wedge f = ef$ and $e' = 1 - e$, respectively. Let C be a subset of $B(R)$. We will say C is *S-indexed* if there exists an artinian and narrow subset I of S such that C is indexed by I. In [3], it was proved that $R[[x]]$ is a reduced left PP-ring if and only if R is a reduced left PP-ring and any countable family of idempotents of R has a least upper bound in $B(R)$. We now have:

THEOREM 2.3. *Let R be a commutative ring and $(S, \leq)$ a strictly totally ordered monoid. Set $A = [[R^{S,\leq}]]$, the ring of generalized power series. Then A is a PP-ring if and only if R is a PP-ring and every S-indexed subset C of $B(R)$ has a least upper bound in $B(R)$.*

PROOF. Note that since $(S, \leq)$ is a strictly totally ordered monoid, S is cancellative and torsion-free by Lemma 1.1.

"if": Suppose that ϕ belongs to A. For every $s \in \text{supp}(\phi)$, write $\phi(s) = e_s b_s$ where $e_s^2 = e_s \in R$ and $b_s \in R$ is entire. Set $I = \text{supp}(\phi)$, then $\{e_s | s \in I\}$ is an S-indexed subset of $B(R)$. Let e be a least upper bound of $\{e_s | s \in I\}$ in $B(R)$. Then $(1-e)e_s = 0$ for all $s \in I$. Thus

$$\begin{aligned} ((c_1 - c_e)\phi)(s) &= \sum_{(u,v) \in X_s(c_1 - c_e, \phi)} (c_1 - c_e)(u)\phi(v) \\ &= (1-e)\phi(s) = \begin{cases} (1-e)e_s b_s = 0, & s \in I \\ 0, & s \notin I \end{cases} \end{aligned}$$

and so $(c_1 - c_e)\phi = 0$. This means that $A(c_1 - c_e) \leq \ell_A(\phi)$. Now it remains to show that $\ell_A(\phi) \leq A(c_1 - c_e)$.

Assume $\psi\phi = 0$ where $\psi \in A$. Then $\psi(t)\phi(s) = 0$ for all $s, t \in S$ by Lemmas 1.2 and 2.1. Thus $\psi(t)e_s = 0$ for all $s \in I$ and $t \in J = \mathrm{supp}(\psi)$ since b_s is entire. For every $t \in J$, write $\psi(t) = f_t a_t$ where $f_t^2 = f_t \in R$ and $a_t \in R$ is entire. Then $f_t e_s = 0$ and so $e_s \leq 1 - f_t$ for all $s \in I$ and $t \in J$. Thus $e \leq 1 - f_t$ for all $t \in J$. It follows that $\psi(t)e = 0$ and so

$$(c_e\psi)(t) = \sum_{(u,v)\in X_t(c_e,\psi)} c_e(u)\psi(v) = e\psi(t) = 0$$

for all $t \in J$. Thus $c_e\psi = 0$ which implies that $\psi = \psi(c_1 - c_e) \in A(c_1 - c_e)$.

Hence $\ell_A(\phi) = A(c_1 - c_e)$ and so A is a PP-ring.

"only if": Let A be a PP-ring, and C an S-indexed subset of $B(R)$. Then there exists an artinian and narrow subset I of S such tat C is indexed by I. Define $\phi : S \to R$ via

$$\phi(s) = \begin{cases} e_s \in C, & s \in I \\ 0, & s \notin I \end{cases}.$$

Then $\mathrm{supp}(\phi) = I$ is artinian and narrow, and so $\phi \in A$. By Lemma 2.1, it follows that $\phi = \psi\alpha$ where $\alpha^2 = \alpha \in A$ and ψ is entire in A. Since A is a PP-ring, it follows that A is reduced, and so is R. By Lemma 2.2, there exists an idempotent $e^2 = e \in R$ such that $\alpha = c_e$. We claim that e is a least upper bound of C. First $(c_1 - c_e)\phi = 0$ implies that for every $s \in I$, $(1-e)e_s = (1-e)\phi(s) = 0$, and thus $e_s \leq e$ for all $s \in I$. On the other hand, suppose that $e_s \leq f$ for all $s \in I$ where $f^2 = f \in R$. Then $(1-f)e_s = 0$, which implies $(c_1 - c_f)\phi = 0$. Thus $(c_1 - c_f)\psi c_e = 0$, which implies that $(c_1 - c_f)c_e = 0$ since ψ is entire. Thus $(1-f)e = 0$ and so $e \leq f$.

Let $r \in R$. Then $\ell_A(c_r) = A\phi$ where $\phi^2 = \phi \in A$. By Lemma 2.2, $\phi = c_e$ for some $e^2 = e \in R$. We will show that $\ell_R(r) = Re$. From $c_e c_r = 0$ it follows that $er = (c_e c_r)(0) = 0$, and so $e \in \ell_R(r)$. Suppose that $p \in \ell_R(r)$. Then it is easy to see that $c_p c_r = 0$. Thus $c_p \in \ell_A(c_r) = Ac_e$. Let $c_p = \alpha c_e$ where $\alpha \in A$. Then $p = c_p(0) = \alpha(0)e \in Re$. Thus $\ell_R(r) = Re$ and so R is a PP-ring.

The following corollaries will give other examples of PP-rings.

COROLLARY 2.4. *Let* $Q^+ = \{a \in Q | a \geq 0\}, R^+ = \{a \in R | a \geq 0\}$. *Then the rings* $[[Z^{N,\leq}]], [[Z^{Q^+,\leq}]], [[Z^{R^+,\leq}]], [[Z^{Z,\leq}]], [[Z^{Q,\leq}]]$ *and* $[[Z^{R,\leq}]]$ *are PP-rings, where* $\leq$ *is the usual order.*

COROLLARY 2.5. *Let* $(S_1, \leq_1), \ldots, (S_n, \leq_n)$ *be strictly totally ordered monoids. Denote by* (lex $\leq$) *and* (revlex $\leq$) *the lexicographic order, the reverse lexicographic order, respectively, on the monoid* $S_1 \times \cdots \times S_n$. *If* R *is a commutative PP-ring such that every subset of* $B(R)$ *has a least upper bound in* $B(R)$, *then the rings*

$$[[R^{S_1 \times \cdots \times S_n, (\mathrm{lex} \leq)}]] \quad \textit{and} \quad [[R^{S_1 \times \cdots \times S_n, (\mathrm{revlex} \leq)}]]$$

are PP-rings.

PROOF. It is easy to see that $(S_1 \times \cdots \times S_n, (\mathrm{lex} \leq))$ is a strictly totally ordered monoid. Thus, by Theorem 2.3, $[[R^{S_1 \times \cdots \times S_n, (\mathrm{lex} \leq)}]]$ is a PP-ring. The proof of the second assertion is similar.

Let R be a commutative ring, and consider the multiplicative monoid $N_{\geq 1}$, endowed with the usual order $\leq$. Then $A = [[R^{N_{\geq 1}, \leq}]]$ is the ring of arithmetical functions with values in R, endowed with the Dirichlet convolution:

$$(\phi\psi)(n) = \sum_{d|n} \phi(d)\psi(n/d), \quad \text{for each} \quad n \geq 1.$$

COROLLARY 2.6. *If* R *is a commutative ring, then* $A = [[R^{N_{\geq 1}, \leq}]]$ *is a PP-ring if and only if* R *is a PP-ring and any countable family of idempotents in* R *has a least upper bound in* $B(R)$.

COROLLARY 2.7. *Let* R *be a commutative PP-ring and* $(S, \leq)$ *a strictly ordered monoid with* S *cancellative and torsion-free. If every subset of* $B(R)$ *has a least upper bound in* $B(R)$ *and* $(S, \leq)$ *is narrow, then* $A = [[R^{S,\leq}]]$ *is a PP-ring.*

PROOF. Note that since S is cancellative and torsion-free, by Lemma 1.1, there exists a compatible strict total order $\leq'$ on S, which is finer than $\leq$. Let $A' = [[R^{S,\leq'}]]$, $A = [[R^{S,\leq}]]$. Then A is a subring of A' by [9, 1.12]. Since $(S, \leq)$ is narrow, it follows that $A = A'$ by [9, 1.12]. Now the result follows from Theorem 2.3.

3. Weakly PP-rings

A ring R is called a *weakly PP-ring* if every principal left ideal ${}_RRer$ is projective for each $r \in R$ and each primitive idempotent $e \in R$.

LEMMA 3.1. *The following conditions are equivalent for a commutative ring R:*

(1) *R is a weakly PP-ring.*

(2) *For every $r \in R$ and every primitive idempotent $e \in R$, if $er \neq 0$ then $er = ea$ where $a \in R$ is entire.*

PROOF. (2) $\Rightarrow$ (1). For every $r \in R$ and every primitive idempotent $e \in R$, if $er = 0$ then ${}_RRer$ is projective. Suppose that $er \neq 0$. Define a homomorphism $h : Rer \longrightarrow Re$ via $h(wer) = we$ for $w \in R$. It is easy to see that h is well-defined and so $Rer \simeq Re$ is projective. Thus R is a weakly PP-ring.

(1) $\Rightarrow$ (2). For every $r \in R$ and every primitive idempotent $e \in R$, it is clear that $\ell(er) = Rf$ where $f^2 = f \in R$. Suppose that $er \neq 0$. Put $g = 1 - f$ and $a = f + er$. Then by analogy with the proof of [3, Proposition 2], it follows that $er = ga$ and a is entire. Now $ega = er = ga$, and so $eg = g$ since a is entire. This means that $g \leq e$. If $g = 0$ then $er = 0$, a contradiction. Thus $g \neq 0$ which implies that $e = g$ since e is primitive. Now the result follows.

LEMMA 3.2. *Let R be a commutative ring and $(S, \leq)$ a strictly totally ordered monoid which satisfies the condition that $0 \leq s$ for every $s \in S$. If $\alpha \in A = [[R^{S,\leq}]]$ is an idempotent, then there exists an idempotent $e \in R$ such that $\alpha = c_e$.*

PROOF. Since $0 \leq s$ for every $s \in S$, it is easy to see that $\alpha(0) = \alpha^2(0) = (\alpha(0))^2$, and we write $e = \alpha(0)$ for convenience. We claim that $\alpha(s) = 0$ for any $0 < s \in S$.

Suppose that $\alpha(u) = 0$ for any $0 < u < s$. We will show that $\alpha(s) = 0$. By hypothesis, it is easy to see that

$$\begin{aligned} \alpha(s) &= \alpha^2(s) = \sum_{(u,v)\in X_s(\alpha,\alpha)} \alpha(u)\alpha(v) \\ &= 2\alpha(0)\alpha(s) + \sum_{(u,v)\in X} \alpha(u)\alpha(v) \\ &= 2\alpha(0)\alpha(s) \end{aligned}$$

where $X = \{(u,v) \in X_s(\alpha,\alpha) | 0 < u < s, 0 < v < s\}$. Thus $\alpha(s) = 0$. This shows that our claim holds.

Now it is clear that $\alpha = c_e$.

LEMMA 3.3. *Let R be a commutative ring and $(S, \leq)$ a strictly totally ordered monoid which satisfies the condition that $0 \leq s$ for every $s \in S$. If $\alpha \in A = [[R^{S,\leq}]]$ is a primitive idempotent, then there exists a primitive idempotent $e \in R$ such that $\alpha = c_e$.*

PROOF. There exists an idempotent $e \in R$ such that $\alpha = c_e$ by Lemma 3.2. If $0 \neq f \leq e$, then $(1-e)f = 0$. Thus $(c_1 - c_e)c_f = 0$, which implies that $c_f \leq c_e$. Hence $c_e = c_f$, and so $e = f$. This means that $e \in R$ is primitive.

THEOREM 3.4. *Let R be a commutative ring and $(S, \leq)$ a strictly totally ordered monoid which satisfies the condition that $0 \leq s$ for every $s \in S$. Then $A = [[R^{S,\leq}]]$ is a weakly PP-ring if and only if R is a weakly PP-ring.*

PROOF. Note that since $(S, \leq)$ is a strictly totally ordered monoid, S is cancellative and torsion-free by Lemma 1.1.

Suppose that R is a weakly PP-ring. If $\phi \in A$ and $\alpha \in A$ is a primitive idempotent, then there exists a primitive idempotent e of R such that $\alpha = c_e$ by Lemma 3.3. If $\alpha\phi = 0$ then $A\alpha\phi$ is projective. Suppose that $\alpha\phi \neq 0$. Then there exists $s_0 \in S$ such that $e\phi(s_0) = (c_e\phi)(s_0) \neq 0$. Thus, by Lemma 3.1, there exists an entire element $a \in R$ such that $e\phi(s_0) = ea$. Clearly $(c_1 - c_e)c_e\phi = 0$, which implies that $A(c_1 - c_e) \leq \ell_A(c_e\phi)$.

Suppose that $\psi \in A$ such that $\psi c_e \phi = 0$. Then for all $s, t \in S, \psi(t)e\phi(s)$ is a nilpotent element of R by Lemma 1.2. If there exists $t \in S$ such that $e\psi(t) \neq 0$, then by Lemma 3.1, $e\psi(t) = eb$, where $b \in r$ is entire. Suppose that $(\psi(t)e\phi(s_0))^n = 0$. Then $ea^n b^n = 0$, and so $e = 0$, a contradiction. Thus $e\psi(t) = 0$ for all $t \in S$, which implies that $c_e\psi = 0$. Hence $\psi = \psi(c_1 - c_e) \in A(c_1 - c_e)$.

Thus $\ell_A(c_e\phi) = A(c_1 - c_e)$, and so $Ac_e\phi$ is projective. This shows that A is a weakly PP-ring.

Conversely suppose that A is a weakly PP-ring. Let r belong to R and $e \in R$ be a primitive idempotent. Then, by Lemma 3.2, it is easy to see that c_e is a primitive idempotent of A. Thus $\ell_A(c_e c_r) = A\phi$ where $\phi^2 = \phi \in A$. By Lemma 3.2, $\phi = c_f$ for some $f^2 = f \in R$. By analogy with the proof of Theorem 2.3, we can show that $\ell_R(er) = Rf$. Thus Rer is projective.

In [1], it was proved that a normal ring R is weakly PP if and only if $R[x]$ is weakly PP. Here we have

COROLLARY 3.5. *Let R be a commutative ring. Then $R[[x]]$ is a weakly PP-ring if and only if R is a weakly PP-ring.*

By analogy with the proof of Theorems 2.3 and 3.4, we have

COROLLARY 3.6. *Let R be a reduced commutative ring and $(S, \leq)$ a strictly totally ordered monoid. Then $A = [[R^{S,\leq}]]$ is a weakly PP-ring if and only if R is a weakly PP-ring.*

Note that by analogy with Corollaries 2.5–2.7, we can obtain more examples of weakly PP-rings, which we omit.

In [1], an example of a weakly PP-ring which is not PP-ring was given. Using

Theorems 2.3 and 3.4, we give an example of a commutative reduced weakly PP-ring which is not a PP-ring.

EXAMPLE 3.7. Let W be an infinite set and B the Boolean ring of all subsets of W. Let R be the subring of B consisting of all finite and cofinite subsets of W. Then R is a commutative reduced PP-ring. Thus $R[[x]]$ is weakly PP-ring by Theorem 3.4. But R fails to have the property that all countable families have a least upper bound in R ([3]). Thus, by Theorem 2.3, $R[[x]]$ is not a PP-ring.

REFERENCES

[1] L. Chen and W. Xue, Weakly PP-rings, Math. Japonica, 39(2) (1994), 361–365.

[2] G.A. Elliott and P. Ribenboim, Fields of generalized power series, Arch. Math., 54(1990), 365–371.

[3] J.A. Fraser and W.K. Nicholson, Reduced PP-rings, Math. Japonica, 34(5) (1989), 715–725.

[4] A. Hattori, A foundation of torsion theory for modules over general rings, Nagoya Math. J., 17(1960), 147–158.

[5] P. Ribenboim, Generalized power series, in "Lattices, Semigroups and Universal Algebra" (J. Almeida, G. Bordalo, and P. Dwinger, Eds.), Plenum, New York, 1990.

[6] P. Ribenboim, Rings of generalized power series: Nilpotent elements, Abh. Math. Sem. Univ. Hamburg, 61(1991), 15–33.

[7] P. Ribenboim, Noetherian rings of generalized power series, J. Pure and Appl. Algebra, 79(1992), 293–312.

[8] P. Ribenboim, Rings of generalized power series II: Units and zero-divisors, J. Algebra, 168(1994), 71–89.

[9] P. Ribenboim, Special properties of generalized power series, J. Algebra, 173(1995), 566-586.