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Some Applications of Plurisubharmonic Functions to Orbits of Real Reductive Groups

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INTRODUCTION

The principal aim of this paper is to prove the following results. Section 1 explains the terminology involved; applications are given in Section 3.

THEOREM 1. *Let $G \subset GL(n,\mathbb{R})$ be a real reductive group with Cartan decomposition*

$$\mathsf{g} = \mathsf{k} \oplus \mathsf{p}, \quad \mathsf{g} \ \textit{being the Lie algebra of } G.$$

Let $G^{\mathbb{C}}$ be the subgroup of $GL(n,\mathbb{C})$ with Lie algebra $\mathsf{g} \oplus i\mathsf{g}$ and $\tilde{K}$ the subgroup of $G^{\mathbb{C}}$ whose Lie algebra is $\mathsf{k} \oplus i\mathsf{p}$.

(i) If G operates on a real vector space V and $G^{\mathbb{C}}$ on $V^{\mathbb{C}} = V \bigotimes_{\mathbb{R}} \mathbb{C}$ and N is the norm square function on $V^{\mathbb{C}}$ obtained from a $\tilde{K}$-invariant hermitian inner product on $V^{\mathbb{C}}$, then for all $v \in V$, one has

$$N(gv) = N(\overline{g}v), \quad \forall\, g \in G^{\mathbb{C}},$$

where $\overline{g}$ denotes the complex conjugate of g.

(ii) If φ is a regular function on $G^{\mathbb{C}}$ such that $\varphi(g) = \varphi(\overline{g})$, then $\varphi|G$ has the identity e as a critical point $\Leftrightarrow$ φ has e as a critical point.

(iii) If $v \in V$ and $\Omega = G \cdot v$, $\Omega^{\mathbb{C}} = G^{\mathbb{C}} \cdot v$, then v is a critical point of $N|\Omega \Leftrightarrow v$ is a critical point of $N|\Omega^{\mathbb{C}}$ (N as in (i))

THEOREM 2. *Using the notations of Theorem 1, let $\Omega^{\mathbb{C}}$ be a complex homogeneous space for $G^{\mathbb{C}}$ and φ a $\tilde{K}$-invariant strictly plurisubharmonic function on $\Omega^{\mathbb{C}}$.*

If Ω is a G-orbit in $\Omega^{\mathbb{C}}$ and $f = \varphi|\Omega$ has a critical point then f is proper, Ω is closed in $\Omega^{\mathbb{C}}$ and the critical set of f is a single K-orbit, K being the subgroup of G

whose Lie algebra is k. *Moreover, the function* f *achieves its minimum value on its critical set.*

The following theorem reduces the study of orbits of real reductive groups to that of their complexifications (see (3.2) infra).

THEOREM 3. *Let* G *be a Lie group,* σ *an automorphism of finite order of* G *and* H *a* σ*-invariant closed subgroup of* G. *Let* $(G/H)_\sigma$ *be the set of fixed points of* σ *in* G/H. *If* $\xi \in (G/H)_\sigma$ *then the* G_σ°*-orbit of* ξ *is open in the connected component of* $(G/H)_\sigma$ *which contains* ξ. *Hence, the components are single* G_σ°*-orbits.*

As in [1] and [2], the proofs use fundamental results of G.D. Mostow on fibrations of homogeneous spaces [11, 12] – in fact, one needs only Theorem 3 of [11] – and elementary convexity properties of plurisubharmonic functions. The applications of these results to orbit closures of real reductive groups are given in Section 3. Further results in this direction are given in D. Luna [10]. Related results are in Richardson–Slodowy [13].

Of late, the group-theoretic aspects of plurisubharmonic functions have been applied by U. Helmke to problems of systems and control engineering in a series of papers (see e.g. [5, 6]) and there is interest in the analogue of the results of [1, 2] for real reductive groups.

1. PRELIMINARIES

Standard references for reductive groups are Borel-Harish Chandra [4] and Springer [15]. For this paper, we will take the following as a working definition. All groups and subgroups will henceforth be Lie. G° will denote the connected component of G, G' its commutator and $Z(G)$ its center.

DEFINITION. A connected subgroup G of $GL(n,\mathbb{R})$ is reductive if its Lie algebra g has a decomposition

$$\mathsf{g} = \mathsf{k} \oplus \mathsf{p}$$

where

(i) $[\mathsf{k},\mathsf{k}] \subset \mathsf{k},\ [\mathsf{k},\mathsf{p}] \subset \mathsf{p},\ [\mathsf{p},\mathsf{p}] \subset \mathsf{k}$

and

(ii) the Lie group $\tilde{K}$ of $GL(n,\mathbb{C})$ whose Lie algebra is $\tilde{\mathsf{k}} = \mathsf{k} \oplus i\mathsf{p}$ is compact.

This definition has the following notable consequences. Since $\mathsf{g} \oplus i\mathsf{g} = \tilde{\mathsf{k}} \oplus i\tilde{\mathsf{k}}$ and the group $\tilde{K}$ is compact, it follows that $G^{\mathbb{C}}$ is the Zariski closure of $\tilde{K}$ [16, § 8], so $G^{\mathbb{C}}$ is closed in $GL(n,\mathbb{C})$. Therefore the group G is also closed, as it is the connected component of $G^{\mathbb{C}}_\sigma$, σ being complex conjugation in $GL(n,\mathbb{C})$.

Similarly, $K = \langle \exp(X) : X \in \mathsf{k} \rangle = \tilde{K}_\sigma$ is also closed. Moreover, as $\tilde{K}^{\mathbb{C}} = G^{\mathbb{C}} = \tilde{K}\exp(i\tilde{\mathsf{k}})$ and $\tilde{K}\exp(i\tilde{\mathsf{k}})$ is homeomorphic to $\tilde{K} \times \exp(i\tilde{\mathsf{k}})$ [16, § 8], we also have $G = KP$, where $P = \exp(\mathsf{p})$.

Now $\tilde{K} = Z(\tilde{K})\tilde{K}'$ with $Z(\tilde{K}) \cap K'$ finite, so $\tilde{K} = (Z(\tilde{K}))^\circ \tilde{K}'$. Therefore $G^{\mathbb{C}} = (Z(G^{\mathbb{C}}))^\circ (G^{\mathbb{C}})'$ and $G = (Z(G))^\circ G'$; also, $Z(G)$ consists of semisimple elements. In particular, if $Z(G)$ is finite, then in any linear representation ρ of G in $GL(V)$, the image $\rho(G)$ is also reductive. On the other hand, if $Z(G)$ is infinite, its image in a linear representation may no longer be reductive. We shall therefore consider only those representations in which the connected component of the center of G is represented as a reductive group.

In this paper, we will have no occasion to use non-differentiable functions. So, in this paper, a plurisubharmonic (briefly psh) function on an n-dimensional complex manifold is a function f whose complex hessian matrix $[(\partial^2 f/\partial z_i \partial \overline{z}_j)]$, in a system of local holomorphic coordinates $z_1, \dots, z_n$, is positive semi-definite. And f is strictly plurisubharmonic (briefly spsh), if its complex hessian is strictly positive definite. We refer the reader to [8, 14] for further details on psh functions.

We conclude this section by proving the following fundamental lemma, a version of which already occurs in [1, 9]. We make no assumptions about reductivity or compactness of the groups involved.

LEMMA. *Let $\tilde{G}$ be a complex Lie group, $\tilde{K}$ a subgroup of $\tilde{G}$ and $\tilde{P} = \{\exp(X) : X \in i\ Lie(\tilde{K})\}$ with $\tilde{G} = \tilde{K}\tilde{P}$. Let G be a subgroup of $\tilde{G}$ with $G = KP$, where K is a subgroup of $\tilde{K}$ and $P = \{\exp(Y) :\ Y \in$ a subspace $\mathfrak{m}$ of i $\mathrm{Lie}(\tilde{K})\}$.*

Let φ be a $\tilde{K}$-invariant strictly plurisubharmonic function on a complex homogeneous space $\Omega^{\mathbb{C}}$ of $\tilde{G}$ and Ω a G-orbit in $\Omega^{\mathbb{C}}$. If $f = \varphi|\Omega$ has a critical point, then the critical set of f is a single K-orbit and f achieves its absolute minimum there. Moreover, if ξ is a critical point of f then the stabilizer G_ξ of ξ in G factorizes as $G_\xi = K_\xi P_\xi$, where K_ξ is the stabilizer of ξ in K and

$$\begin{aligned} P_\xi &= \{(\exp Y) : Y \in \mathfrak{p}, (\exp Y)\xi = \xi\} \\ &= \{\exp Y) : Y \in \mathfrak{p}, (\exp tY)\xi = \xi\ \ \forall t \in \mathbb{R}\}. \end{aligned}$$

Proof. Let ξ be a critical point of f and η another critical point of f. By K-invariance, we may assume that $\eta = \exp(X)\xi$ for some $X \in i$ Lie $(\tilde{K})$.

Consider the function $g(z) = \varphi(\exp(zX) \cdot \xi)$, $z \in \mathbb{C}$. As φ is $\tilde{K}$-invariant, we have $g(x + iy) = g(x)$. Since g is subharmonic, $\Delta g \geq 0$ implies $g''(x) \geq 0$. So g is convex and it achieves its absolute minimum at any critical point. Since $x = 0$ and $x = 1$ are critical points of $g(x)$, we see that $g(x)$ is constant for $0 \leq x \leq 1$. Now $g(z) = g(\mathrm{Re}(z))$, so the function g is constant on the strip $0 \leq$ Re $z \leq 1$. Therefore $\varphi(\gamma(z)) = g(z) =$ constant on $0 \leq$ Re $z \leq 1$. Hence

$$(i\partial\overline{\partial}\varphi)(\gamma(z))(\gamma'(z), \overline{\gamma'(z)}) = 0, \quad 0 \leq \mathrm{Re}\ z \leq 1$$

and since $i\partial\overline{\partial}\varphi$ is positive definite, we must have $\gamma'(z) \equiv 0$. Hence $\gamma(z)$ is constant, so $\gamma(0) = p = \gamma(1) = q$.

By the same argument, if $g = k\exp(Y) \in G$ with $Y \in \mathsf{m} \subset i$ Lie $(\tilde{K})$ and $g\xi = \xi$, then $\exp(tY) \cdot \xi = \xi, \quad 0 \leq t \leq 1$, so $k \cdot \xi = \xi$. But if $\exp(tY) \cdot \xi = \xi$, then $\exp(-tY) \cdot \xi = \xi$, so the entire 1-parameter subgroup $\{\exp(tY)\}_{t \in \mathbb{R}}$ stabilizes ξ. Hence G_ξ has the factorization claimed above.

2. PROOFS

We shall use the notation set up in Section 1 without further comment.

Proof of Theorem 1. (i) The group G operates on V. Let $\pi : G \to GL(V)$ be the corresponding representation. Let $X_1, \dots, X_r$ be a basis of the Lie algebra of G. The complexification $\pi(G)$ of G in $GL(V^{\mathbb{C}})$ is generated by the complex 1-parameter subgroups $\exp(z\pi(X_k))$, $z \in \mathbb{C}$, $1 \leq k \leq r$, and clearly

$$[\exp(z\pi(X_k))]^{-} = \exp(\overline{z}\pi(X_k)).$$

Hence complex conjugation leaves $(\pi(G))^{\mathbb{C}}$ stable, and one has $[\pi(g) \cdot v]^{-} = \overline{\pi(g)}\, \overline{v}$, for $g \in G$, $v \in V^{\mathbb{C}}$. Take a hermitian inner product on $V^{\mathbb{C}}$invariant under the compact group $\tilde{K}$. Now any orthonormal basis of V over $\mathbb{R}$ remains an orthonormal basis of $V^{\mathbb{C}}$ over $\mathbb{C}$. Taking matrices relative to such a basis we have, for $g \in G^{\mathbb{C}}$ and $v \in V$

$$\begin{aligned} N(gv) &= \left[(gv)^{-}\right]^{t} gv \\ &= \overline{v}^{t}\overline{g}^{t} gv \end{aligned}$$

and

$$N(\overline{g}v) = \overline{v}^{t} g^{t} \overline{g} v,$$

where t denotes the transpose and bar the complex conjugate. Since N is real, we have

$$N(gv) = N(\overline{g}v) \Leftrightarrow \overline{v}^{t}\overline{g}^{t} gv = v^{t}\overline{g}^{t} g\overline{v},$$

which holds for $v = \overline{v}$, i.e. for $v \in V$. This proves part (i).

(ii) The map $(X, Y) \mapsto \exp(X + iY)$ $X, Y \in \mathsf{g}$ gives local coordinates at the identity e. Now if $\psi(X, Y) = \varphi(\exp(X + iY))$, then $\psi(X, Y) = \psi(X, -Y)$, which clearly implies (ii).

(iii) This is a direct consequence of parts (i) and (ii).

Proof of Theorem 2. For the proof we need a special case of a theorem of G.D. Mostow [11, 12] in the following formulation.

Using the notations of Section 1, the Lie algebra g has the Cartan decomposition $\mathsf{g} = \mathsf{k} \oplus \mathsf{p}$ and $G = K\exp(\mathsf{p})$. Take a hermitian norm on $\mathbb{C}^n$ invariant under the compact group $\tilde{K}$ and fix an orthonormal basis of $\mathbb{R}^n$ (over $\mathbb{R}$), which will remain an orthonormal basis of $\mathbb{C}^n$ over $\mathbb{C}$. Taking matrices relative to this basis, we see that k is represented by real skew-symmetric matrices and p by symmetric matrices. So the form $B(X,Y) = \mathrm{Tr}(XY)$ is nondegenerate on $\mathfrak{g}$. It is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$.

Let H be a closed subgroup of G such that $H = L\exp(\mathfrak{q})$, where L is a subgroup of K and $\mathfrak{q}$ is an L-invariant subspace of $\mathfrak{p}$ such that $[X,[X,Y] \in \mathfrak{q}\ \forall X, Y \in \mathfrak{q}$. Let $\mathfrak{q}'$ be the orthogonal complement of $\mathfrak{q}$ in $\mathfrak{p}$ relative to B. Then the argument in Mostow [11, Thm. 3, p. 40] is directly applicable to this situation and one has the decomposition

$$G = K\exp(\mathfrak{q}')\exp(\mathfrak{q})$$

with uniqueness of expressions. Since $H = L\exp(\mathfrak{q})$, this gives immediately $G/H \cong K \underset{L}{\times} \exp(\mathfrak{q}')$.

Now let $\Omega^{\mathbb{C}}$ be a complex homogeneous space for $G^{\mathbb{C}}$, φ a $\tilde{K}$-invariant spsh function on $\Omega^{\mathbb{C}}$ whose restriction f to a G-orbit Ω has a critical point ξ. By the lemma of Section 1, the stabilizer $G_\xi = K_\xi P_\xi$, where K_ξ is the stabilizer of ξ in K and

$$\begin{aligned} P_\xi &= \{\exp(Y) : Y \in \mathfrak{p}, \exp(Y)\cdot\xi = \xi\} \\ &= \{\exp(Y) : Y \in \mathfrak{p},\ (\exp tY)\cdot\xi = \xi \quad \forall t \in \mathbb{R}\}. \end{aligned}$$

Hence all conditions of Mostow's theorem hold for the pair (G, G_ξ). Now applying the lemma of Section 1 and repeating the argument of [2] word for word completes the proof of the lemma.

Proof of Theorem 3. First, notice that the connected components of $(G/H)_\sigma$ are submanifolds of G/H. Indeed, if a compact Lie group K operates on a manifold M, one can give to M a K-invariant Riemannian metric. Using the existence of geodesic strongly convex neighbourhoods one sees readily that the components of the fixed point set M_K of K in M are submanifolds of M. So, as σ has finite order, the same holds for $(G/H)_\sigma$

The component containing $\xi_0 = eH$ has the same dimension as $(T_{\xi_0})_\sigma$, T_{ξ_0} being the tangent space of G/H at ξ_0. Now $(T_{\xi_0})_\sigma = (\mathfrak{g}/\mathfrak{h})_\sigma \cong \mathfrak{g}_\sigma/\mathfrak{h}_\sigma$, $\mathfrak{g}, \mathfrak{h}$ being the Lie algebras of G and H, so the component containing ξ_0 has the same dimension as $\mathfrak{g}_\sigma/\mathfrak{h}_\sigma$. Therefore, the G_σ° orbit of ξ_0 is open in the component $C(\xi_0)$ containing ξ_0. Hence $C(\xi_0)$ must be a single orbit of G_σ°. This completes the proof of Theorem 3.

3. APPLICATIONS

In this section, $G, G^{\mathbb{C}}, N$ etc. have the same meaning as in the statement of Theorem 1.

3.1. (Birkes [3]) *If G operates on a real vector space V and $G^{\mathbb{C}}$ on $V^{\mathbb{C}}$ and a G-orbit of $p \in V$ is closed, then the $G^{\mathbb{C}}$-orbit of p in $V^{\mathbb{C}}$ is also closed.*

Proof. Let Ω be a closed G-orbit in V. Since the function N is proper, $N|\Omega$ achieves its minimum, say at q. By Theorem 1, the function $\varphi(g) = N(gq)$ $(q \in G^{\mathbb{C}})$

has a critical point at e, so by the Kempf-Ness theorem [7] or by [2], the orbit $\Omega^{\mathbb{C}} = G^{\mathbb{C}} \cdot q = G^{\mathbb{C}} \cdot p$ is closed.

3.2. (Borel-Harish Chandra [4]). *If $v \in V$ and $\Omega^C = G^{\mathbb{C}} \cdot v$, then $\Omega^{\mathbb{C}} \cap V$ is a union of equidimensional orbits of G, each of which is a component of $\Omega^{\mathbb{C}} \cap V$.*
In particular, if $\Omega^{\mathbb{C}}$ is closed, then all G-orbits in $\Omega^{\mathbb{C}} \cap V$ are closed.

Proof. For this result, G need not be reductive. It suffices, by Theorem 3, to show that all G-orbits in $\Omega^{\mathbb{C}} \cap V$ have the same dimension. Now if $v \in V$ and $X, Y \in \mathrm{Lie}(G)$ with $(X + iY)v = 0$, then $Xv = 0,\ Yv = 0$, so all G-orbits in $G^{\mathbb{C}}v \cap V$ have the same dimension.

REMARK. In [4] it is shown, using an argument from real algebraic geometry, that $G^{\mathbb{C}}v \cap V \quad (v \in V)$ is a finite union of G-orbits.

3.3. (Richardson–Slodowy [13]). *If N restricted to a G-orbit Ω has a critical point, then Ω is closed.*

Proof. This follows immediately from Theorem 1 and (3.2).

3.4. *If φ is a $\tilde{K}$-invariant spsh function on $\Omega^{\mathbb{C}} = G^{\mathbb{C}} \cdot v$ $(v \in V)$ and $\varphi|(G \cdot v)$ has a critical point, then G_v is reductive and $G \cdot v$ is closed in $G^{\mathbb{C}} \cdot v$.*

Proof. This is a consequence of Theorem 2 and the lemma of Section 1.

3.5. *If φ is a $\tilde{K}$-invariant spsh function defined in a neighbourhood of a G-orbit of $v \in V$ and $\varphi|G \cdot v$ has a critical point, then the G and the $G^{\mathbb{C}}$ orbits of v are closed.*

Proof. If $\varphi|G \cdot v$ has a critical point then $f = \varphi|\Omega$ is proper (Thm. 2) and since φ is defined in a neighbourhood of $G \cdot v$, the orbit $G \cdot v$ is closed in V; therefore by Theorem 1, $G^{\mathbb{C}} \cdot v$ is also closed in $V^{\mathbb{C}}$.

REMARK. If we take a compact group K and V a representation of K, then all $K^{\mathbb{C}}$-orbits of real points are closed in $V^{\mathbb{C}}$. The simplest example of a K-invariant spsh function which is critical along K-orbits but not along $K^{\mathbb{C}}$-orbits is given by $f(z) = (\log|z|) + |z|^2$ with $K = S^1, K^{\mathbb{C}} = \mathbb{C}^*$.

3.6. *If φ is a $\tilde{K}$-invariant spsh proper function on $V^{\mathbb{C}}$ such that $\varphi|G^{\mathbb{C}}v$ has a critical point $(v \in V)$, then $\varphi|Gv$ also has a critical point.*

Proof. The assumptions imply that $G^{\mathbb{C}} \cdot v$ is closed in $V^{\mathbb{C}}$; hence by (3.2), $G \cdot v$ is closed in V so $\varphi|G \cdot v$ achieves its minimum value on $G \cdot v$.

3.7. *If $v \in V$, then $\overline{G \cdot v}$ contains a closed G-orbit.*

Proof. The function N is proper, so $N|\overline{G \cdot v}$ achieves its minimum value, say at p. Hence the G-orbit of p is closed.

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