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Abstract

The n-type indecomposable Crump-Mode-Jagers branching process with immigration is considered. Using the martingale approach a limit theorem proved for such processes, when the totality of immigrating individuals at a given time depends on evolution of the processes generating by immigrated before individuals. The main assumption is that a limit theorem for corresponding process without immigration holds. Corollaries of the limit theorem are given for the cases of finite and infinite variances of offspring distribution in critical processes. As examples of applications of these results some infinitely divisible limit distributions are obtained.

Key words: MULTITYPE, INDECOMPOSABLE, GENERAL PROCESS, DEPENDENT IMMIGRATION, STOPPING TIME.
AMS 1991 SUBJECT CLASSIFICATION: PRIMARY 60J80, SECONDARY 60G70.

1 Introduction

Let for any $u \in [0, \infty)$ the piar $(\theta_k^u, \mathbf{Y}_k^u)$ be some n-variate point process, where θ_k^u are random variables with values from the set $R_+ = [0, \infty)$ and $\mathbf{Y}_k^u = (Y_{k1}^u, ..., Y_{kn}^u)$ be random vectors taking "values" from $N^n, N = \{1, 2, ...\}$, $u \in R_+$. Here θ_k^u and $\mathbf{Y}_k^u, k \geq 1$, are the time and "size" of kth jump of the point process respectively.

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We consider $(\theta_k^u, \mathbf{Y}_k^u)$ for any fixed u as an immigration process, namely suppose that $\mathbf{Y}_k^u$ is a random vector of immigrating at time θ_k^u individuals of types $T_1, T_2, ..., T_n$. These immigrating individuals generate independent and identically distributed n-type general Crump-Mode-Jagers branching processes.

If we enumerate simultaneously immigrating individuals of the type T_j by $i = 1, 2, ...$, then the triple (k, i, j) corresponds to the ith individual of the type T_j, $\quad j = 1, 2, ..., n$, immigrating at time θ_k^u. We shall call the multi-type general branching process generated by individual (k, i, j) as "(k, i, j)-process".

We denote

$$(1)\ \mathbf{X} = \{\mathbf{X}_{ki}^j(t) = (X_{ki}^{j1}(t), X_{ki}^{j2}(t), ..., X_{ki}^{jn}(t)), k \geq 1, i \geq 1, j = 1, 2, ..., n\}$$

the family of all possible (k, i, j)-processes. Here $X_{ki}^{jm}(t)$ is the number of individuals of type T_m in the (k, i, j)-process at time t. Then the branching process with n-types and immigration is defined by $\mathbf{Z}^u(t) = (Z_1^u(t), Z_2^u(t), ..., Z_n^u(t))$, $t \geq 0, \mathbf{Z}^u(0) = \mathbf{0}$, where

$$Z_m^u(t) = \sum_{j=1}^{n} \sum_{k=1}^{N^u(t)} \sum_{i=1}^{Y_{kj}^u} X_{ki}^{jm}(t - \theta_k^u) \tag{2}$$

is the number of individuals of type T_m at time t and $N^u(t)$ is the number of jumps of the point process $(\theta_k^u, \mathbf{Y}_k^u)$ up to time t, that is

$$N^u(t) = \sum_{k:\theta_k^u \leq t} 1.$$

It should be noted that we are dealing here with a double array of immigration processes depending on $u \in [0, \infty)$. However (k, i, j)-processes do not depend on u except for the starting time. For convenience we also use the process $\mathbf{Y}^u(t) = (Y_1^u(t), ..., Y_n^u(t))$, where

$$Y_j^u(t) = \sum_{k \geq 1} \chi(\theta_k^u \leq t) Y_{kj}^u, j = 1, ..., n,$$

is the number of the T_j type individuals immigrating up to time t. For instance in the process $Z(t) = Z^t(t)$ the immigration process $(\theta_k^t, \mathbf{Y}_k^t)$ may depend on the observation time. In this case we use $N(t) = N^t(t)$ for the number of immigration occurances up to time t. Considering of the double array of immigration processes makes our theorem below applicable, for example, in cases when immigration may have different intensity during the observation time.

Multitype branching processes with immigration have been studied widely in the literature (see Quine(1970), Kaplan(1974), Shurenkov(1976)). One can find a sufficiently full bibliography of such papers in the books Mode(1971), Sevastyanov(1971), Jagers(1975), Badalbaev and Rahimov(1993) and in the review by Vatutin and Zubkov(1993). However, the independence of processes of reproduction and immigration was assumed in previous publications. In the described above model this assumption means that the family $\mathbf{X}$ of independent and identically distributed branching processes and point processes $(\theta_k^u, \mathbf{Y}_k^u)$ are independent. Under this assumption the study of $\mathbf{Z}^u(t)$ can be reduced to the analyses of a relation for its generating function. If $\mathbf{X}$ and the point processes are not independent, it is not possible to get an explicit expression for the generating function of the process $\mathbf{Z}^u(t)$ in terms of the generating functions of corresponding processes without immigration and the immigration generating function.

On the other hand in applications of branching processes often the immigration process depends on reproduction. For instance, if we consider the process of urban population growth, the number of immigrants at present depends on the lives of past immigrants and their descendants. Another example is the neutron chain reaction in a nuclear reactor with an external neutron source. Actually in any branching model, regulated by immigration, the immigration process should depend on the reproduction.

Here the process $\mathbf{Z}^u(t)$ will be considered without the assumption of independence of processes of reproduction and immigration. We prove a limit theorem for $\mathbf{Z}^u(t)$ under the assumption that the limit theorem for corresponding process without immigration holds. Such theorems we shall call "-transfer theorems" for branching processes (see Rahimov (1995)). Corollaries of the transfer theorem will be obtained for cases of finite and infinite second moments of offspring distribution in critical processes and examples of applications of these results will be discussed. Single-type branching processes with reproduction-dependent immigration were studied in Rahimov(1992, 1995).

Let $\Re_{k,i}^j(t)$ be the σ-algebra generated by the evolution of the (k, i, j)-

process up to time t. One can imagine $\Re^j_{k,i}(t)$ as the direct product of life σ-algebras of individuals born before time t in the (k,i,j)-process, where, in particular, the life-length and the reproduction process of the individual are defined. We have not spoken on the independence of (k,i,j)-processes and the immigration process until now. Presuppose now that the following basic assumptions are fulfilled:
(a) the family $\{\theta^t_k, k \in N, t \in R_+\}$ is measurable with respect to some σ-algebra $\Re_0$, which is independent of the family $\Re^j_{ki}(t)$.
(b) for any $r = 0, 1, ...$and $p = 1, 2, ..., n$,

$$\{Y^t_{kp} \leq r\} \in \Im_{kr}(t), \tag{3}$$

where

$$\Im_{kr}(t) = \prod_{l=1}^{k-1} \prod_{q=1}^{n} \prod_{i=1}^{Y^t_{lq}} \Re^q_{li}(t) \times \prod_{q=1}^{n} \prod_{i=1}^{r} \Re^q_{ki}(t) \times \Re_0.$$

The direct products of the random number of σ-algebras we shall understand as

$$\prod_{i=1}^{Y} \Re_i = \{A : A \bigcap \{Y = j\} \in \prod_{i=1}^{j} \Re_i\}, \quad \prod_{i=1}^{0} \Re_i = \{, \Omega\}.$$

Under the condition (3) the totality of immigrating individuals at time θ^t_k may depend on evolution of the processes generated by the individuals which immigrated up to time θ^t_k. Note also that for processes with reproduction-independent immigration the condition (3)is fulfilled with $\mathbf{Y}^t_k$ are $\Re_0$-measurable and in this case σ-algebras $\Re^j_{ki}(t)$ and $\Re_0$are independent.

It is clear that $\Re^j_{ki}(\infty)$is the σ-algebra generating by lives of individuals who was ever born in (k,i,j)-process. It also can be seen from the further consideration that all our arguments and results remain true, if we replace $\Re^j_{ki}(t)$ by σ-algebras $\Re^j_{ki}(\infty)$. Thus considering here model may include the situation when the immigration process depends on the predicted evolution of daughter processes of past immigrants.

2 The main theorem and corollaries

For n-dimesional vectors $\mathbf{x} = (x_1, x_2, ..., x_n), \mathbf{y} = (y_1, y_2, ..., y_n)$we denote $\mathbf{x} \oplus \mathbf{y} = (x_1 y_1, ..., x_n y_n), \mathbf{x}^{\mathbf{y}} = (x_1^{y_1}, ..., x_n^{y_n}), (\mathbf{x}, \mathbf{y}) = x_1 y_1 + ... + x_n y_n, \mathbf{1} =$

$(1,1,...,1), \mathbf{0} = (0,0,...,0), \mathbf{e} = (e,e,...,e)$and $\mathbf{x} \geq \mathbf{y}$or $\mathbf{x} > \mathbf{y}$if $x_i \geq y_i$or $x_i > y_i, i = 1,2,...,n$,respectively.

We also denote for $j = 1,2,...,n$,

$$F_j(t,\mathbf{S}) = ES_1^{X_{ki}^{j1}(t)} S_2^{X_{ki}^{j2}(t)} \ldots S_n^{X_{ki}^{jn}(t)}, \quad \mathbf{S} = (S_1, S_2, ..., S_n)$$

the generating functions of the (k,i,j)-processes.

Assume that

$$\sup_t EX_{ki}^{jl}(t) \leq C_0 < \infty,$$

and for any fixed $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_n) > \mathbf{0}$ and for some non-increasing functions $\mathbf{Q}(t) = (Q_1(t), ..., Q_n(t))$

$$\lim_{t\to\infty} \frac{1 - F_j(t, \mathbf{e}^{-\lambda \oplus \mathbf{Q}(t)})}{Q_j(t)} = 1 - \varphi(\boldsymbol{\lambda}), \tag{4}$$

where $\varphi(\boldsymbol{\lambda}) = \varphi(\lambda_1, ..., \lambda_n)$ is the Laplace transform of a random vector having finite expectation, $\mathbf{Q}(t) \to \mathbf{0}$ and for any $x \in [0,1]$,

$$\lim_{t\to\infty} \frac{Q_j(t)}{Q_j(tx)} = \pi(x) \tag{5}$$

and this convergence is uniform in each interval of the form $[\varepsilon, 1)$ for any $\varepsilon > 0$ and $\pi(x)$ is a continuous function for $x \in (0,1]$.

Conditions (4)and (5)can be satisfied for critical or close to critical (in the case of transition phenomena) discrete or continuous time multitype Markov branching processes. For the critical multitype Bellman-Harris processes these conditions are fulfilled in the case of finite or infinite second moments of the offspring distribution and under some conditions on the life-time distributions of the individuals. In the case of general branching processes conditions ensuring fulfilment of (4) and (5) are known for the offspring distribution with finite variance. Examples concerning to these cases will be considered below. The limit function $\pi(x)$ in (5) necessarily has the form $x^\alpha$for some $\alpha \in [0,\infty)$(see S.I Resnick(1987), p.14).

For the immigration process we assume that

$$\sum_{j=1}^{n} Q_j(t) Y_j^t(tx) \xrightarrow{P} T(x), \tag{6}$$

as $t \to \infty$for $0 \le x \le 1$, where $T(x)$ is some $\Re_0$- measurable, stochastically continuous for $x = 1$ stochastic process with non-decreasing trajectories, $T(0) = 0$ and $T(1) < \infty$ almost everywhere.

Theorem. *If conditions (3) - (6) are satisfied, then*

$$\mathbf{W}(t) = \mathbf{Q}(t) \oplus \mathbf{Z}(t) \xrightarrow{\mathcal{D}} \mathbf{W} = (W_1, W_2, ..., W_n),$$

where

$$Ee^{-(\boldsymbol{\lambda},\mathbf{W})} = E\exp\left\{-\int_0^1 \frac{1-\varphi(\lambda_1\pi(1-x), ..., \lambda_n\pi(1-x))}{\pi(1-x)} dT(x)\right\}$$

(the value of the integrand at $x = 1$ is defined by continuity).

Now we consider some applications of the theorem. Let $\tau = (\tau_1, \tau_2, ..., \tau_n)$ be the vector of the life-times and $\xi(t) = (\xi^1(t), \xi^2(t), ..., \xi^n(t))$ be the vector of (multivariate) point processes of reproduction , where τ_i is the lifespan of an individual of type T_i and $\xi^i(t) = (\xi^i_1(t), ..., \xi^i_n(t)), \xi^i_j(t)$ is the number of the offspring of type T_j born by an individual of type T_i during age interval $[0, t)$. Note that $\xi(t)$ and τ are different for different (k, i, j)-processes and for different individuals, although the pairs $(\xi(t), \tau)$ are independent and identically distributed. For simplicity of notations we do not indicate these dependences.

Let $m_{ij}(t) = E[\xi^i_j(t)]$ be the mean number of type T_j that born to a type T_i parent in the age interval $[0, t)$ and let $m(t) = [m_{ij}(t)]$ be the $n \times n$ matrix of measures. We assume that each $m_{ij}(t)$ is a nonlattice measure; $m_{ij}(0) < m_{ij}(\infty)$, for some i,j; $m(\infty)$ is indecomposable and has largest eigenvalue 1 and $m^p(\infty) > 0$ for some $p \in N$.

Under the above assumptions there are the right and the left eigenvectors $\mathbf{U} = (u_1, u_2, ..., u_n)$ and $\mathbf{V} = (v_1, v_2, ..., v_n)$ of the matrix $m(\infty)$ corresponding to the eigenvalue 1 such that

$$m(\infty)\mathbf{U} = \mathbf{U}, \quad \mathbf{V}m(\infty) = \mathbf{V}, \qquad \sum_{i=1}^n u_i v_i = 1, \quad \sum_{i=1}^n u_i = 1.$$

We also assume, (see Holte(1982)), that

C1.

$$\left(E\,\mathbf{S}^{\xi^i(\infty)}, i = 1, ..., n\right) \neq m(\infty)\mathbf{S},$$

which excludes the case where each parent gives birth to exactly one offspring;
C2.

$$q^i_{jk} = E\left[\xi^i_k(\infty)(\xi^i_j(\infty) - \delta_{jk})\right] < \infty$$

for any $i, j, k = 1, ..., n$;
C3.

$$m(\infty) - m(t) = o(t^2), \quad 1 - \mathbf{G}(t) = o(t^2),$$

where $\mathbf{G}(t) = (G_1(t), ..., G_n(t)), \quad G_i(t) = P\{\tau_i \le t\}$. Let

$$B = (\mathbf{V}, \mathbf{q}[\mathbf{U}]), \qquad q^i[\mathbf{U}] = \sum_{j=1}^{n}\sum_{k=1}^{n} q^i_{jk} u_j u_k,$$

$$\beta = \sum_{i=1}^{n}\sum_{j=1}^{n} \mu_{ij} v_i u_j, \quad \mu_{ij} = \int_0^\infty y m_{ij}(dy)$$

It is known, (see Holte(1982)), that under the above assumptions and conditions C1-C3 the limit theorem for critical multitype general branching processes holds. More precisely for any $j = 1, 2, ..., n$

$$\lim_{t\to\infty} E\left[e^{-(\lambda, \mathbf{X}^j_{ki}(t)\oplus\mathbf{Q}(t))} \mid \mathbf{X}^j_{ki}(t) \neq \mathbf{0}\right] = [1 + (\mathbf{C}, \boldsymbol{\lambda})]^{-1},$$

where

$$Q_j(t) = P\{\mathbf{X}^j_{ki}(t) \neq \mathbf{0}\} \sim 2u_j\beta(Bt)^{-1}, t \to \infty$$

$$\mathbf{C} = (C_1, ..., C_n), \mathbf{R} = (R_1, ..., R_n), C_i = \beta^{-1} u_i v_i R_i, R_i = E(\tau_i).$$

Thus in this case conditions (4) and (5) are satisfied with

$$\varphi(\boldsymbol{\lambda}) = (1 + (\mathbf{C}, \lambda))^{-1}, \quad Q_j(t) \sim \frac{2u_j\beta}{Bt}, t \to \infty, \quad \pi(x) = x,$$

and we have the following result.

Corollary 1. *If conditions (3), (6) and C1-C3 are satisfied, then*

$$\lim_{t\to\infty} P\left\{\frac{2u_j\beta Z_j(t)}{Bt} \le y_j, j = 1, 2, ..., n\right\} = A(\mathbf{y}),$$

where the distribution $A(\mathbf{y}), \quad \mathbf{y} = (y_1, y_2, \ldots, y_n)$, has the Laplace transform

$$\int_{R^n_+} e^{-(\mathbf{y},\boldsymbol{\lambda})} dA(\mathbf{y}) = E \exp\left\{-(\mathbf{C}, \lambda)\int_0^1 (1 + (1 - x)(\mathbf{C}, \lambda))^{-1} dT(x)\right\}.$$

Now we consider the multitype Bellman-Harris process under the broader assumptions on the offspring distribution. For the Bellman-Harris processes

$$\xi_j^i(t) = \begin{cases} 0, & \text{for } t < \tau_i \\ \xi_j^i(\infty) & \text{for } t \geq \tau_i, \end{cases}$$

and, therefore $\mu_{ij}(t) = G_i(t)E\xi_j^i(\infty)$. Let

$$F^j(\mathbf{S}) = E\left[\mathbf{S}^{\xi^j(\infty)}\right], \quad \mathbf{F}(\mathbf{S}) = (F^1(\mathbf{S}), ..., F^n(\mathbf{S})), \quad |\mathbf{S}| \leq \mathbf{1},$$

and let $\mathbf{F}_k(\mathbf{S})$ be the kth iteration of $\mathbf{F}(\mathbf{S})$. We assume that

$$x - (\mathbf{V}, \mathbf{1} - \mathbf{F}(\mathbf{1} - \mathbf{U}x)) = x^{1+\alpha}L(x), \tag{7}$$

where $0 < x \leq 1$, $\alpha \in (0,1]$ and $L(x)$ varies slowly as $x \to +0$.

Note that under the condition (7) the second moments of the offspring distribution can be not finite. Thus it is weaker than C2. On the other hand, if $q_{jk}^i < \infty$, then condition (7) is satisfied with $\alpha = 1$ and $L(x)$ is a function having a finite limit as $x \to +0$. Now we replace the condition C3 by the following.

C4.

$$\lim_{k\to\infty} k(1 - G_i(k))/(\mathbf{V}, \mathbf{1} - \mathbf{F}_k(\mathbf{0})) = 0.$$

It is known that, under the assumption (7) the difference $\mathbf{1}$-$\mathbf{F}_k(\mathbf{0}) \sim \mathbf{U}k^{1/\alpha}L_1(k), k \to \infty$,where $L_1(k)$varies slowly as $k \to \infty$.

In this case the following limit theorem for the critical multitype Bellman-Harris process holds (see Vatutin (1978)). We denote

$$q(t) = \sum_{j=1}^{n} v_j(1 - F_j(t, \mathbf{0})), \qquad D = (\mathbf{U} \oplus \mathbf{V}, \mathbf{R}),$$

Proposition. *Let* $m(\infty)$is indecomposable and has the largest eigenvalue 1, $G_j(t)$are nonlattice and conditions (7) and C4 are satisfied. Then

(a)

$$P\left\{\mathbf{X}_{ki}^j(t) \neq \mathbf{0}\right\} \sim u_j(\frac{t}{D})^{-1/\alpha}L_1(t),$$

where $L_1(t)$the same as in the asymptotics of $\mathbf{1} - \mathbf{F}_k(\mathbf{0})$;

(b)

$$\lim_{t\to\infty} P\left\{\mathbf{X}_{ki}^j(t)q(t) \leq \mathbf{y}, | \, \mathbf{X}_{ki}^j(t) \neq \mathbf{0}\right\} = S_\alpha(D\psi(\mathbf{y})),$$

where $S_\alpha(\mathbf{y}) = S_\alpha(y_1, ..., y_n)$is the distribution having the Laplace transform

$$\int_{R_+^n} e^{-(\mathbf{x},\lambda)} dS_\alpha(\mathbf{x}) = 1 - (1 + \bar{\lambda}^{-\alpha})^{-1/\alpha}, \quad \bar{\lambda} = (\boldsymbol{\lambda}, \mathbf{1}),$$

$$\psi(\mathbf{y}) = min\left\{\frac{y_1}{v_1 R_1}, \frac{y_2}{v_2 R_2}, ..., \frac{y_n}{v_n R_n}\right\}.$$

It is not difficult to see that

$$(\lambda, \mathbf{X}_{ki}^j(t) \oplus \mathbf{Q}(t)) = (\lambda \oplus \mathbf{U}, \mathbf{X}_{ki}^j(t)) q^*(t)$$

where

$$q^*(t) \sim \left(\frac{t}{D}\right)^{-1/\alpha} L_1(t) \sim q(t), \quad t \to \infty.$$

Therefore it follows from the Proposition that in this case conditions (4) and (5) are satisfied with

$$\varphi(\boldsymbol{\lambda}) = 1 - (1 + (\lambda, \gamma)^{-\alpha})^{-1/\alpha}, Q_j(t) \sim u_j \left(\frac{t}{D}\right)^{-1/\alpha} L_1(t), \pi(x) = x^{1/\alpha},$$

where $\gamma = (\gamma_1, ..., \gamma_n), \gamma_j = D^{-1} u_j v_j R_j$.Thus the folowing corollary holds.

Corollary 2.*If conditions (3), (6), (7) and C4 are satisfied, then*

$$\lim_{t\to\infty} P\{Q_j(t) Z_j(t) \le y_j, j = 1, 2, ..., n\} = B(y_1, y_2, ..., y_n),$$

where the distribution $B(\mathbf{y}), \mathbf{y} = (y_1, y_2, ..., y_n)$has the Laplace transform

$$\int_{R_+^n} e^{-(\mathbf{y},\boldsymbol{\lambda})} dB(\mathbf{y}) = E \exp\left\{- \int_0^1 [1 - x + (\lambda, \gamma)^{-\alpha}]^{-1/\alpha} dT(x)\right\}$$

Corollaries 1 and 2 give examples of branching processes for which conditions (4) and (5) of the theorem are fulfilled. Now we consider some examples of immigration processes satisfying condition (6).

Example 1. Let $\mathbf{Y}_k^t \equiv \mathbf{Y}_k$and for $r = 0, 1, ..., p = 1, 2, ..., n$

$$\{Y_{kp} \le r\} \in \mathcal{B}_r(k, t) = \prod_{i=1}^{r} \prod_{q=1}^{n} \Re_{ki}^q(t).$$

Since $(k,i,j)-$processes are independent, the vector $\mathbf{Y}_k$and processes $\{\mathbf{X}^j_{li}(t),$ $l,i \geq 1,\ l \neq k, j = 1,...,p\}$are also independent. If $\mathbf{Y}_k, k = 1,2,...$have the same distribution, and $N(t)$is a Pisson process with the intensity ν, then $\mathbf{Y}(t) = (Y_1(t),...,Y_n(t))$is the multivariate compound Poisson process. Therefore the Laplace transform of $Y_j(t)$has the following form

$$Ee^{-uY_j(t)} = \exp\{\nu t(f_j(u) - 1)\},$$

where $f_j(u)$the Laplace transform of Y_{kj}. Using this representation it is not difficult to show that, if $E\mathbf{Y}_k = \mathbf{a} = (a_1, a_2, ..., a_n)$is finite, then

$$\frac{Y_j(t)}{t} \xrightarrow{P} \nu a_j$$

In fact, since

$$\lim_{t\to\infty} t(1 - f_j(\frac{u}{t})) = ua_j,$$

we have that $E\exp\{-ut^{-1}Y_j(t)\}$tends to $\exp\{-\nu a_j u\}$as $t \to \infty$for any $u > 0$. So $t^{-1}Y_j(t)$converges to $a_j\nu$in distribution.

Hence, under the assumptions C1-C3 the condition (6) holds with $T(x) = 2\beta x\nu B^{-1}\sum_{j=1}^n a_j u_j$.It is not difficult to see that in this case the Laplace transform in Corollary 1 is

$$(1 + (\lambda, \mathbf{C}))^{-\alpha}, \quad \alpha = 2\beta\nu B^{-1}\sum_{j=1}^{n} a_j u_j.$$

Therefore we have the following result which is a generalization of the well-known theorem on convergence to the gamma distribution.

Corollary 3. *If conditions C1-C3 are satisfied, the coordinates of the vector* $\mathbf{Y}_{kp}, p = 1,2,...,n$ are stopping times with respect to the family $\mathcal{B}_r(k,t), r = 0,1,...,$ it has the same distribution for different k, $\mathbf{a} < \infty$ and $N(t)$ is a Poisson process with the density ν, then

$$\mathbf{W}(t) = (\frac{2u_j\beta Z_j(t)}{Bt}, j = 1,2,...,n) \xrightarrow{\mathcal{D}} \frac{1}{\beta}\mathbf{U} \oplus \mathbf{V} \oplus \mathbf{R} \oplus \mathbf{W},$$

where $\mathbf{W} = (W_1,...,W_n), W_i = W_j$ with probability 1 and W_i has the gamma distridution of the parameter α.

Remark. It seems that Corollary 3 is new even in the case of reproduction-independent immigration. We can compare it with the theorem on the convergance to the gamma distribution for multitype Sevastyanov (Shurenkov (1976)) or singletype general (Jagers (1975)) branching processes. The corollary shows that the gamma limit theorem holds, without additional conditions, when the totality of the number of immigrating individuals is the stopping time with respect to the family of σ-algebras generated by reproduction processes.

Example 2. Let again coordinates of the vector $\mathbf{Y}_k$ are stopping times with respect to $\mathcal{B}_r(k,t)$. If conditions C1-C3 are satisfied and

$$\frac{1}{k}\sum_{j=0}^{k}\mathbf{Y}_j \xrightarrow{P} \boldsymbol{\eta} = (\eta_1, ..., \eta_n),$$

as $k \to \infty$, where η is some random vector, then the condition (6) holds with $T(x) = 2\beta x B^{-1}(\mathbf{U}, \boldsymbol{\eta})$ and the Laplace transform in Corollary 1 is

$$E[(1 + (\lambda, \mathbf{C}))^{-2\beta T/B}], \qquad T = (\mathbf{U}, \boldsymbol{\eta}).$$

In addition, if $\mathbf{Y}_k, k = 1, 2, ...$ have the same distribution, then $W_k = (\mathbf{U}, \mathbf{Y}_k), k = 1, 2, ...$ are independent and identically distributed. Therefore, if $N(t)$ is a Poisson process, then the process

$$\sum_{j=1}^{n} u_j Y_j(t) = \sum_{j=1}^{n} u_j \sum_{k\geq 1} \chi(\theta_k \leq t) Y_{kj} = \sum_{k=1}^{N(t)} W_k$$

is compound Poisson. Then the Laplace transform of T is infintely divisible as a limit of infinitely divisible Laplace transforms. Therefore it can be written in the following form (see Feller (1967), Sec. 7, Chap. XII)

$$Ee^{-uT} = \exp\left\{-\int_0^\infty \frac{1 - e^{-ux}}{x} P(dx)\right\},$$

where $P(x)$ is a measure such that $\int_0^\infty x^{-1} P(dx) < \infty$. Since

$$E[(1 + (\lambda, \mathbf{C}))^{-2\beta T/B}] = E\exp\{-\frac{2\beta T}{B}\log(1 + (\lambda, \mathbf{C}))\},$$

using this fact we obtain the following result.

Corollary 4. *If coordinates of* $\mathbf{Y}(k)$ *are stopping times with respect to the family* $\mathcal{B}_r(k,t), r = 0,1,\ldots,$ *then, under the assumptions mentioned above* $\mathbf{W}(t) \xrightarrow{\mathcal{D}} \mathbf{W} \oplus \mathbf{C}, \quad t \to \infty,$ *where* $\mathbf{W} = (W_1, \ldots, W_n), W_i = W_j$ *with probability 1 and* W_1 *has the Laplace transform*

$$Ee^{-yW_1} = \exp\left\{-\int_0^\infty \frac{(1+y)^{2\beta x/B} - 1}{x(1+y)^{2\beta x/B}} P(dx)\right\}.$$

and $\mathbf{C} = (C_1, \ldots, C_n)$ *is as in the Corollary 1.*

Now we proceed to prove our main theorem.

3 The proof of the theorem

First we consider the function

$$H(t,\boldsymbol{\lambda}) = \prod_{k=1}^{N(t)} \prod_{j=1}^{n} [F_j(t-\theta_k, \mathrm{e}^{-\boldsymbol{\lambda}\oplus\mathbf{Q}(t)})]^{Y_{kj}^t}$$

and we shall prove the following lemma.

Lemma 1. *If conditions (4), (5) and (6) are satisfied, then*

$$H(t,\boldsymbol{\lambda}) \xrightarrow{P} H(\boldsymbol{\lambda}) = \exp\left\{-\int_0^1 \frac{1-\varphi(\pi(1-x)\boldsymbol{\lambda})}{\pi(1-x)} dT(x)\right\}. \tag{8}$$

Proof. Since $Y_{kj}^t = \Delta Y_j^t(\theta_k^t) = Y_j^t(\theta_k^t) - Y_j^t(\theta_k^t - 0)$ and

$$\log H(t,\boldsymbol{\lambda}) = \sum_{j=1}^{n} \sum_{k=1}^{N(t)} Y_{kj}^t \log F_j(t-\theta_k^t, \mathrm{e}^{-\boldsymbol{\lambda}\oplus\mathbf{Q}(t)}),$$

we get

$$H(t,\lambda) = \exp\left\{\sum_{j=1}^{n} \int_0^t \log F_j(t-u, \mathrm{e}^{-\boldsymbol{\lambda}\oplus\mathbf{Q}(t)}) dY_j^t(u)\right\}.$$

We choose $\varepsilon \in (0,1)$, put $a = 1-\varepsilon$ and consider

$$A_1 = \sum_{j=1}^{n} \int_0^{ta} \log F_j(t-u, \mathrm{e}^{-\boldsymbol{\lambda}\oplus \mathbf{Q}(t)}) dY_j^t(u). \tag{9}$$

If we denote

$$\pi_j(t,x) = \frac{Q_j(t)}{Q_j(tx)}, \quad \boldsymbol{\pi}(t,x) = (\pi_1(t,x), ..., \pi_n(t,x)), \delta = \delta_u(t) = 1 - \frac{u}{t},$$

then for sufficiently large t and $0 \leq u \leq ta$

$$0 < \pi(\varepsilon) - \varepsilon_1 \leq \pi_j(t,\delta) \leq 1 \tag{10}$$

for some $\varepsilon_1 > 0$ and $j = 1, 2, ..., n$. Therefore it follows from (10) and the condition (4) that

$$\sup_{1 \leq k \leq [at]} \left| \frac{1 - F_j(t-u, \mathrm{e}^{-\boldsymbol{\lambda}\oplus \mathbf{Q}(t)})}{Q_j(t-u)} - 1 + \varphi((\boldsymbol{\lambda}, \boldsymbol{\pi}(t,\delta)) \right| \to 0 \tag{11}$$

as $t \to \infty$. On the other hand, since $\varepsilon \leq \delta_u(t) < 1$ for $0 \leq u \leq ta$, it follows from (5) that

$$\sup_{0 \leq u \leq ta} |\pi_j(t,\delta) - \pi(\delta)| \to 0 \tag{12}$$

as $t \to \infty$ for $j = 1, 2, ..., n$. We obtain from (11) and (12) the following representation

$$-\frac{\log F_j(t-u, \mathrm{e}^{-\boldsymbol{\lambda}\oplus \mathbf{Q}(t)})}{Q_j(t)} = \frac{1 - \varphi(\pi(\delta)\boldsymbol{\lambda})}{\pi(\delta)}(1 + \alpha_j(u,t)), \tag{13}$$

where $\alpha_j(u,t) \to 0$ for $j = 1, 2, ..., n$, as $t \to \infty$ uniformly with respect to $o \leq u \leq ta$.

It is not difficult to see that the integral

$$\int_0^{1-\varepsilon} \frac{1 - \varphi(\pi(1-x)\boldsymbol{\lambda})}{\pi(1-x)} d\xi_t(x), \tag{14}$$

where $\xi_t(x) = \sum_{j=1}^{n} Q_j(t) Y_j^t(tx)$, conveges in probability to

$$B(\boldsymbol{\lambda}, \varepsilon) = \int_0^{1-\varepsilon} \frac{1 - \varphi(\pi(1-x)\boldsymbol{\lambda})}{\pi(1-x)} dT(x)$$

due to condition (6). Since $T(x)$ is stochastically contiuous at $x = 1$ and $\varphi(\boldsymbol{\lambda})$ is the Laplace transform of a random vector having finite expectation, one can show by standard arguments that the last integral converges in probability to the $B(\boldsymbol{\lambda}, 0)$ as $\varepsilon \to 0$. Hence it follows from (13) that A_1 converges in probability to $-B(\boldsymbol{\lambda}, 0)$ as $t \to \infty$ and $\varepsilon \to 0$.

We now consider

$$A_2 = \sum_{j=1}^{n} \int_{ta}^{t} \log F_j(t-u, \mathbf{e}^{-\boldsymbol{\lambda} \oplus \mathbf{Q}(t)}) dY_j^t(u).$$

Using the simple inequality $\log(1-x) \geq -x - x^2/(1-x), 0 \leq x < 1$, we get

$$0 > A_2 \geq -\sum_{j=1}^{n} \int_{ta}^{t} (R_j(u) + \frac{R_j^2(u)}{1-R_j(u)}) dY_j^t(u),$$

where

$$R_j(u) = R_j(u, \boldsymbol{\lambda}) = 1 - F_j(t-u, \mathbf{e}^{-\boldsymbol{\lambda} \oplus \mathbf{Q}(t)}).$$

It is not difficult to see that

$$\sum_{j=1}^{n} \int_{ta}^{t} R_j(u) dY_j^t(u) \leq C_0 \sum_{j=1}^{n} \lambda_j [\xi_t(1) - \xi_t(1-\varepsilon)]$$

and the last difference in probability converges to $T(1) - T(1-\varepsilon)$ as $t \to \infty$. Using again the stochastic continuity of $T(x)$ at $x = 1$ we obtain that this difference in probability converges to 0 as $\varepsilon \to 0$, and, therefore, A_2 converges in probability to 0 as $t \to \infty$ and $\varepsilon \to 0$. The lemma is proved.

Let $\mathbf{X}_i(n) = (X_{i1}(n), X_{i2}(n), ..., X_{ip}(n)), i = 1, 2, ...$ be non-negative p-dimensional random vectors such that $\mathbf{X}_i(n)$ is $F_i(n)$-measurable, where $F_i(n)$ are some σ-algebras such that $F_i(n) \subseteq F_{i+1}(n), i = 0, 1, 2, ...$ for any n. We consider the following sum

$$S_n = \sum_{i=1}^{n} \nu_i(n) \mathbf{X}_i(n), \tag{15}$$

where $\nu_i(n)$ are random variables taking values 0 and 1 and $F_{i-1}(n)$-measurable. Denote

$$f_j^{(n)}(\boldsymbol{\lambda}) = E[e^{-(\boldsymbol{\lambda}, \mathbf{X}_j(n))} | F_{j-1}(n)], \boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_p) \in R_+^p.$$

The following lemma for $p=1$ was proved in Rahimov(1995, p.29) using a semimartingale technique. A similar result for simple random sums can be seen in Beska et. al.(1982).

Lemma 2. *Let for any* $j=1,2,...,n$ *random variable* $\nu_j(n)$ *is* $F_{j-1}(n)$ *-measurable and*

$$\prod_{j=1}^{n}\left\{f_j^{(n)}(\boldsymbol{\lambda})\right\}^{\nu_j(n)} \xrightarrow{P} \varphi(\boldsymbol{\lambda}), n\to\infty, \tag{16}$$

where $\varphi(\boldsymbol{\lambda})$ *is some* $\mathcal{F}_0$*-measurable random variable, such that* $\varphi(\boldsymbol{\lambda})>0$ *almost everywhere,* $\mathcal{F}_0\subset F_0(n)$*for all* $n=1,2,...$ *Then*

$$E\left[e^{-(\boldsymbol{\lambda},\mathbf{S}_n)}|\mathcal{F}_0\right] \xrightarrow{P} \varphi(\boldsymbol{\lambda}), n\to\infty. \tag{17}$$

Proof. *Let* $\tilde{\mathbf{X}}_i(n)=\mathbf{X}_i(n)\chi(\mathcal{A}_i(n))$, where

$$\mathcal{A}_i(n)=\left\{\prod_{j=1}^{i}\left\{f_j^{(n)}(\boldsymbol{\lambda})\right\}^{\nu_j(n)}\geq\frac{1}{2}\varphi(\boldsymbol{\lambda})\right\},$$

and $\chi(.)$is the indicator function. It is clear that $\mathcal{A}_i(n)\in F_{i-1}(n)$. First we shall prove that for any $n$and $m\leq n$

$$E\left[\prod_{k=1}^{m}Z_k^{\nu_k(n)}(n)|\mathcal{F}_0\right]=1, \tag{18}$$

where

$$Z_k(n)=\frac{e^{-(\boldsymbol{\lambda},\tilde{\mathbf{X}}_k(n))}}{\tilde{f}_k^{(n)}(\boldsymbol{\lambda})},\quad \tilde{f}_k^{(n)}(\boldsymbol{\lambda})=E[e^{-(\boldsymbol{\lambda},\tilde{\mathbf{X}}_k(n))}|F_{k-1}(n)].$$

If $m=1$, then

$$E\left[\prod_{k=1}^{1}Z_k^{\nu_k(n)}(n)|\mathcal{F}_0\right]=E[E[Z_1^{\nu_1(n)}(n)|F_0(n)]|\mathcal{F}_0].$$

Using the simple equality

$$(19)\qquad e^{-(\boldsymbol{\lambda},\tilde{\mathbf{X}}_k(n))\nu_k(n)} = \chi(\nu_k(n)=1)e^{-(\boldsymbol{\lambda},\tilde{\mathbf{X}}_k(n))} + \chi(\nu_k(n)=0)$$

we have

$$E\left[\prod_{k=1}^{1} Z_k^{\nu_k(n)}(n)|\mathcal{F}_0\right] = 1$$

Assuming that (18) holds for $m = i$, we can obtain by similar arguments that it is true for $m = i+1$. Thus (18) holds for any $m \geq 1$.

Using induction on $m$it can be shown that

$$(20)\qquad \prod_{i=1}^{m} \left\{\tilde{f}_i^{(n)}(\boldsymbol{\lambda})\right\}^{\nu_i(n)} \geq \frac{1}{2}\varphi(\boldsymbol{\lambda}).$$

almost everywhere, for all $n$and $m \leq n$.

Putting $\tilde{\mathbf{S}}_n = \sum_{k=1}^{n} \tilde{\mathbf{X}}_k(n)\nu_k(n)$and using (18) and (20) we have

$$(21)\qquad |E[e^{-(\boldsymbol{\lambda},\tilde{\mathbf{S}}_n)}|\mathcal{F}_0] - \varphi(\boldsymbol{\lambda})| \leq \frac{2}{\varphi(\boldsymbol{\lambda})}E[W_n(\boldsymbol{\lambda})|\mathcal{F}_0],$$

where

$$W_n(\boldsymbol{\lambda}) = \left|\prod_{k=1}^{n} \left\{\tilde{f}_k^{(n)}(\boldsymbol{\lambda})\right\}^{\nu_k(n)} - \varphi(\boldsymbol{\lambda})\right|.$$

The estimate (21) shows that in order to be

$$(22)\qquad E\left[e^{-(\boldsymbol{\lambda},\tilde{\mathbf{S}}_n)}|\mathcal{F}_0\right] \xrightarrow{P} \varphi(\boldsymbol{\lambda}), n \to \infty,$$

it is sufficient that $W_n(\boldsymbol{\lambda})$converges in probability to zero as $n \to \infty$. In fact, then we obtain from the dominated convergence theorem that $EW_n(\boldsymbol{\lambda})$tends to zero and, since

$$P\left\{E[W_n(\boldsymbol{\lambda})|\mathcal{F}_0] > \varepsilon\right\} \leq \frac{1}{\varepsilon}EW_n(\boldsymbol{\lambda}),$$

we have that $E[W_n(\boldsymbol{\lambda})|\mathcal{F}_0]$converges to zero in probability as $n \to \infty$.

For any $\varepsilon > 0$we have

$$(23)\quad P\{W_n(\boldsymbol{\lambda}) > \varepsilon\} \leq P\{V_n(\boldsymbol{\lambda}) > \varepsilon\} + P\{\bigcup_{k=1}^{n} R_k(n) \bigcap \{\nu_k(n) = 1\}\},$$

where

$$V_n(\boldsymbol{\lambda}) = \left| \prod_{k=1}^{n} f_k^{(n)}(\boldsymbol{\lambda})^{\nu_k(n)} - \varphi(\boldsymbol{\lambda}) \right|, R_k(n) = \left\{ f_k^{(n)}(\boldsymbol{\lambda}) \neq \tilde{f}_k^{(n)}(\boldsymbol{\lambda}) \right\}.$$

Since $R_k(n) = \bar{\mathcal{A}}_k(n)$, we have to show that

$$P\{T_n\} \to 0, n \to \infty, \tag{24}$$

where

$$T_n = \bigcup_{k=1}^{n} \bar{\mathcal{A}}_k(n) \bigcap \{\nu_k(n) = 1\}.$$

We have from the definition of $\mathcal{A}_k(n)$, that

$$\bar{\mathcal{A}}_k(n) \subseteq \left\{ \prod_{j=1}^{n} \left\{ f_j^{(n)}(\boldsymbol{\lambda}) \right\}^{\nu_j(n)} < \frac{1}{2}\varphi(\boldsymbol{\lambda}) \right\}, k \leq n.$$

Therefore we have

$$P\{T_n\} \leq P\left\{ \varphi(\boldsymbol{\lambda}) - \prod_{j=1}^{n} \{ f_j^{(n)}(\boldsymbol{\lambda})\}^{\nu_j(n)} > \frac{1}{2}\varphi(\boldsymbol{\lambda}) \right\}.$$

Choosing εsuch that $\varepsilon < \varphi(\boldsymbol{\lambda})$, we obtain due to condition (16)

$$P\{T_n\} \leq P\{V_n(\boldsymbol{\lambda}) > \frac{\varepsilon}{2}\} \to 0, n \to \infty. \tag{25}$$

Thus (24) holds. It follows from (23) and (24) that $W_n(\boldsymbol{\lambda})$converges to zero in probability.

It remains to show that the variable $\tilde{\mathbf{S}}_n$in (22) can be replaced by S_n. It is not difficult to see that

$$E\left[e^{-(\boldsymbol{\lambda},\tilde{\mathbf{S}}_n)} | \mathcal{F}_0\right] = E\left[e^{-(\boldsymbol{\lambda},\mathbf{S}_n)} | \mathcal{F}_0\right] + E\left[\left(e^{-(\boldsymbol{\lambda},\tilde{\mathbf{S}}_n)} - e^{-(\boldsymbol{\lambda},\mathbf{S}_n)}\right) \chi(T_n) | \mathcal{F}_0\right]$$

and the variable $\chi(T_n)$converges to zero in probability due to (25).

The lemma is proved.

Proof of Theorem 1. It follows from (2) that $\mathbf{Z}(t)$can be written in the form

$$\mathbf{Z}(t) = \sum_{k=0}^{N(t)} \sum_{r=1}^{n} \sum_{i=1}^{\infty} \chi(i \leq Y_{kr}^t) \mathbf{X}_{ki}^r(t - \theta_k^t). \tag{26}$$

Let $\{M_t\}$and $\{N_t^r\}, t \geq 0$be a sequences of integers such that

$$(27) \qquad P\{N(t) > M_t\} \to 0, \quad P\{\bigvee_{k=1}^{N(t)} Y_{kr}^t > N_t^r\} \to 0$$

as $t \to \infty$for $r = 1, 2, ..., n$.Putting $N_t = \bigvee_{r=1}^n N_t^r$,we define process $\mathbf{W}^\star(t)$by the following relation

$$(28) \qquad \mathbf{W}^\star(t) = \sum_{l=1}^{k(t)} \nu_l(t)\mathbf{V}_l(t)$$

where $k(t) = nN_tM_t$,

$$\nu_l(t) = \chi(i \leq Y_{kr}^t)\chi(k \leq N(t)), \quad \mathbf{V}_l(t) = \mathbf{X}_{ki}^r(t-k)$$

for such $l$that

$$(29) \qquad l = N_t n(k-1) + j, \quad j = N_t(r-1) + i$$

and $1 \leq j \leq nN_t, 1 \leq i \leq N_t, 1 \leq k \leq M_t, 1 \leq r \leq n$.Then we have from (26) that

$$(30) \qquad T_t\mathbf{Z}(t) = T_t\mathbf{W}^\star(t),$$

where

$$T_t = \chi\{\bigcap_{r=1}^n\{\bigvee_{k=1}^t Y_{kr}^t \leq N_t\}\}\chi(N(t) \leq M_t).$$

It is not difficult to see that for any $l$satisfying (29) the vector $\mathbf{V}_l(t)$is $\mathcal{F}_{ki}(t)$-measurable. On the other hand it follows from condition (3) that $\nu_l(t)$is $\mathcal{F}_{ki-1}(t)$-measurable. Thus Lemma 2 is applicable to $\mathbf{W}^\star(t)$. Note that under our assumptions

$$E\left[e^{-(\lambda, X_{ki}^j(t-\theta_k^t))}|\mathcal{F}_{ki-1}(t)\right] = E\left[e^{-(\lambda, X_{ki}^j(t-\theta_k^t))}|\Re_0\right] = F_j(t-\theta_k^t, \mathbf{e}^{-\boldsymbol{\lambda}}).$$

According to that lemma, in order to be

$$(31) \qquad E\left[e^{-(\boldsymbol{\lambda}_t, \mathbf{W}^\star(t))}|\Re_0\right] \xrightarrow{P} H(\boldsymbol{\lambda})$$

with $\boldsymbol{\lambda}_t = \boldsymbol{\lambda} \oplus \mathbf{Q}(t)$for any $\boldsymbol{\lambda} \in R_+^n$, it is sufficient that

$$(32) \qquad D(t, \boldsymbol{\lambda}) \xrightarrow{P} H(\boldsymbol{\lambda}), t \to \infty,$$

where

$$D(t,\boldsymbol{\lambda}) = \prod_{k=1}^{M_t}\prod_{j=1}^{n}\prod_{i=1}^{N_t}\left\{F_j(t-\theta_k^t, \mathbf{e}^{-\lambda_t})\right\}^{\chi(i\le Y_{kj}^t)}.$$

Since

$$D(t,\boldsymbol{\lambda}) = T_t H(t,\boldsymbol{\lambda}) + (1-T_t)D(t,\boldsymbol{\lambda}),$$

we have from Lemma 1, (27) and the choice M_t, that (32) holds for any $\boldsymbol{\lambda} \in R_+^n$. Thus, due to the dominated convergence theorem, it follows from (31) that the Laplace transform of $\mathbf{Q}(t) \oplus \mathbf{W}^\star(t)$tends as $t \to \infty$to $EH(t,\boldsymbol{\lambda})$, that is

(33) $$\mathbf{Q}(t) \oplus \mathbf{W}^\star(t) \xrightarrow{\mathcal{D}} \mathbf{W}, t \to \infty.$$

It is not difficult to see that the inequality

(34) $$|P\{\chi\mathbf{X} \le \mathbf{x}\} - P\{\mathbf{X} \le \mathbf{x}\}| \le P\{\chi = 0\}$$

is true for any random vector $\mathbf{X}$, indicator χ and $\mathbf{x} \in R^n$.

The assertion of the theorem follows from (30), (33), (34) and the choice of M_t and N_t. The theorem is proved.

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