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THE STRICT TOPOLOGY ON TOPOLOGICAL ALGEBRAS

L.A. Khan*, N. Mohammad**, and A.B. Thaheem***

Abstract

In this paper we introduce strict, uniform, and compact-open topologies on topological algebras and investigate their properties. In particular, we extend several results of R.C. Busby [Trans. Amer. Math. Soc. 132(1968), 77–79] and F.D. Sentilles and D.C. Taylor [ibid 142(1969), 141–152] on Banach algebras and the general strict topology to this general setting.

1. Introduction

Let $C_b(S)$ be the Banach algebra of all bounded continuous scalar-valued functions on a locally compact space S and $C_0(S)$ the subalgebra of all functions in $C_b(S)$ which vanish at infinity. The strict topology β on $C_b(S)$ was first introduced by R.C. Buck [3] in 1958 as the locally convex topology generated by the seminorms $f \longrightarrow \|\varphi f\|$ for $\varphi \in C_0(S)$. Wang [24] defined in an analogous way the strict topology on the multiplier (or centralizer) algebra $M(A)$ of a commutative Banach algebra A and showed in particular that (1) A is a norm-closed ideal in $M(A)$, (2) A is strictly dense in $M(A)$ if A has an approximate identity, and (3) $M(A) = C_b(S)$ if $A = C_0(S)$. In the case of A a non-commutative B^*-algebra, Busby [4] introduced the strict topology on the double multiplier algebra $M_d(A)$ as the one given by the seminorms $x \longrightarrow \max\{\|ax\|, \|xa\|\}$ for $a \in A$. The concept of double multiplier algebra was initially developed by Johnson

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[7] and it later became an extensively studied object; see e.g. [1, 4, 6, 8, 11, 13, 20, 22, 23, 24].

Busby in [4] also defined in the above way a general strict topology on any B^*-algebra X which contains A as a closed two-sided ideal. Independently, this topology was studied in greater detail and in a more general setting by Sentilles and Taylor in an interesting paper [18] where A is a Banach algebra having an approximate identity and X is a left Banach A-module. There have been several applications of this paper in the literature; see e.g. [17, 19, 20, 21]. As pointed out in [18], the essential ingredients of arguments used by Buck [3] and later authors is the presence of an approximate identity in $C_0(S)$ or A.

The aim of the present paper is to extend several results of [4, 18] to a class of topological algebras which are not necessarily locally convex. For simplicity, we take X to be a topological algebra which contains A as a closed two-sided ideal. Our main results are given in section 3 while section 2 is devoted to some technical preliminaries required for the development of our results. The authors had initially established these results assuming X a locally m-convex algebra but later realized that "local convexity" can be dispensed with. In view of the increasing applications of (non-normed) topological algebras in other fields such as quantum mechanics and quantum statistics (see e.g. Lassner [9, 10]), these results would provide a framework for further applications.

2. Preliminaries

An algebra X (over $\mathbb{R}$ or $\mathbb{C}$) with a topology τ is called a *topological algebra* (resp. *semitopological algebra*) if it is a topological vector space in which multiplication is jointly (resp. separately) continuous. If (X, τ) is a topological algebra, there exists a base $\mathcal{W}$ of τ-neighborhoods of 0 consisting of closed, balanced, and absorbing sets such that for each $G \in \mathcal{W}$, there exists a $H \in \mathcal{W}$ with $H^2 \subseteq G$. (Here $H^2 = HH = \{ab : a, b \in H\}$). A subset H of X is called *idempotent* if $H^2 \subseteq H$. Following Żelako ([25], p. 31), X is said to be *locally idempotent* if it has a base of neighborhoods of 0 consisting of idempotent sets. X is called *locally bounded* if there is a bounded neighborhood of 0. Ever normed algebra and, more generally, every locally m-convex algebra is locally idempotent; every Hausdorff locally bounded algebra is metrizable. If a semitopological algebra is complete and metrizable, then it is a topological algebra. A complete metrizable topological algebra is called an *F-algebra.* A net $\{e_\lambda : \lambda \in I\}$ in a topological algebra X is called a two-sided *approximate identity* if, for all $x \in X, \lim_\lambda e_\lambda x = \lim_\lambda x e_\lambda = x$. A topological vector space (TVS) E is called *ultrabornological* if every linear map from E into any TVS which takes bounded sets into bounded sets is continuous. Every metrizable TVS is ultrabornological. For the general theory of TVS, see e.g. [15, 16] and of topological algebras, see [12] and the comments after each chapter in [26].

Let (X, τ) be a topological algebra, A a fixed closed two-sided ideal in X, and $\mathcal{W}$ a base of τ-neighborhoods of 0 in X. For any bounded set $D \subseteq A$ and $G \in \mathcal{W}$, we set

$$N(D, G) = \{x \in X : Dx \subseteq G, \quad xD \subseteq G\}.$$

If $\{a\} \subseteq A$ is a singleton, we write $N(a, G)$ instead of $N(\{a\}, G)$. The *uniform topology* $u = u_A$ (resp. *strict topology* $\beta = \beta_A$) on X is defined as the linear topology which has a base of neighborhoods of 0 consisting of all sets of the form $N(D, G)$, where D is a bounded (resp. finite) subset of A and $G \in \mathcal{W}$. If A has a two-sided approximate identity $\{e_\lambda : \lambda \in I\}$, the *compact-open topology* k on X is defined just as above with

D being a finite subset of $\{e_\lambda : \lambda \in I\}$ (cf. [18], §.3).

As mentioned earlier, β was introduced independently by Busby ([4], p. 83) and Sentilles and Taylor ([18], p. 145) in the case of X a Banach algebra. As a classical example, let $X = C_b(S)$ and $A = C_0(S)$ with S a locally compact space and X endowed with the sup-norm topology. Then A is a closed two-sided ideal in X and has an approximate identity. In this example, β is the original strict topology introduced by Buck [3], $u = \tau$ is the sup-norm topology, and k is the topology of uniform convergence on compact subsets of S. Another instance of the above, due to Wang [24] and Johnson [7], is that A is a topological algebra and $X = M_d(A)$. This is discussed in the later part of section 3.

3. Main Results

Throughout this section we shall assume, unless stated otherwise, that (X, τ) is a locally idempotent F-algebra, A a closed two-sided ideal in X having a bounded two-sided approximate identity $\{e_\lambda : \lambda \in I\}$, and $\mathcal{W}$ a base of τ-neighborhoods of 0 in X consisting of balanced idempotent sets. We shall also assume that A is *faithful* in X in the sense that, for each $x \neq 0$ in X, there exists $a \in A$ with $ax \neq 0$; that is, if $\{x \in X : Ax = \{0\}\} = \{0\}$ (cf [7, 18]).

Theorem 3.1. *(a)* $k \leq \beta \leq u \leq \tau$.

(b) k, β, *and* u *are Hausdorff.*

(c) (X, k), (X, β), *and* (X, u) *are semi-topological algebras.*

(d) For each $x \in X, e_\lambda x \xrightarrow{\beta} x$ *and* $x e_\lambda \xrightarrow{\beta} x$.

(e) $\bigcup_{\lambda \in I} e_\lambda X$ *and* $\bigcup_{\lambda \in I} X e_\lambda$ *are* β*-dense in* X*; in particular,* A *is* β*-dense in* X.

Proof. (a) It is clear that $k \leq \beta \leq u$. Let D be a bounded set in A and $G \in \mathcal{W}$.

Choose $H \in \mathcal{W}$ with $H^2 \subseteq G$ and $r > 0$ with $D \subseteq rH$. Then $V = \frac{1}{r}H \in \mathcal{W}$ and it is easy to see that $V \subseteq N(D, G)$. This proves that $u \leq \tau$.

(b) We need only show that k is Hausdorff. Let $x \in \bigcap_{\lambda \in I} \bigcap_{G \in \mathcal{W}} N(e_\lambda, G)$. Since τ is Hausdorff, $\bigcap_{G \in \mathcal{W}} G = \{0\}$ and so $e_\lambda x = 0 = xe_\lambda$ for all $\lambda \in I$. Then, for any $a \in A$, $ax = \lim_\lambda (ae_\lambda)x = 0$. Since A is faithful in X, we have $x = 0$.

(c) We prove the result only for (X, u) as the proofs for (X, β) and (X, k) are similar to it. Fix $z \neq 0$ in X, and let D be a bounded set in A and $G \in \mathcal{W}$. Choose a balanced idempotent $H \in \mathcal{W}$ with $H \subseteq G$ and $r > 1$ with $z \in rH$. Put $D_1 = D \cup Dz \cup zD$ and $H_1 = \frac{1}{r}H$. Then it is easy to verify that $zN(D_1, H_1) \subseteq N(D, G)$ and $N(D_1, H_1)z \subseteq N(D, G)$. This implies that the multiplication in (X, u) is separately continuous.

(d) Let $x \in X$, and let D be a finite set in A and $G \in \mathcal{W}$. Choose a balanced idempotent $H \in \mathcal{W}$ with $H \subseteq G$ and $r > 1$ with $x \in rH$. There exists $\lambda_0 \in I$ such that $ae_\lambda - a \in \frac{1}{r}H$ and $e_\lambda xa - xa \in \frac{1}{r}H$ for all $a \in D$ and $\lambda \geq \lambda_0$. Then, for any $a \in D$ and $\lambda \geq \lambda_0$, we have $a(e_\lambda x - x) \in G$ and $(e_\lambda x - x)a \in G$. Hence $e_\lambda x \xrightarrow{\beta} x$. In a similar way, we have $xe_\lambda \xrightarrow{\beta} x$.

(e) This follows immediately from (d).

The following result is an extension of ([5], Lemma 4).

Lemma 3.2. *Let S be a β-bounded set in X. Then, for any bounded set $D \subseteq A$, $DS \cup SD$ is τ-bounded in X.*

Proof. For any $x \in X$, define $L_x, R_x : A \to A$ by $L_x(a) = xa$ and $R_x(a) = ax, a \in A$. Since S is β-bounded, for each $a \in A$, $aS \cup Sa$ is τ-bounded in X. Hence $B = \{L_x, R_x : x \in S\}$ is a pointwise bounded subset of $CL(A, A)$, the space of all continuous linear maps from A into A. By the principle of uniform boundedness ([15], Theorem 2.6), B is equicontinuous and hence uniformly bounded in $CL(A, A)$. This implies that, for any bounded set $D \subseteq A$, $DS \cup SD$ is τ-bounded.

Theorem 3.3. *(a) β and u have the same bounded sets.*

(b) If (X, β) is ultrabornological, then $\beta = u$ on X.

Proof. (a) This follows immediately from Lemma 3.2.

(b) By (a), the identity map $i = (X, \beta) \to (X, u)$ takes bounded sets into bounded sets. Hence, by hypothesis, i is continuous; that is, $u \leq \beta$.

Theorem 3.4 *(a) $k = \beta$ on τ-bounded sets in X.*

(b) If a sequence $\{x_n\} \subseteq X$ is β-convergent, it is u-bounded and k-convergent; conversely, if $\{x_n\}$ is τ-bounded and k-convergent, it is β-convergent.

Proof. (a) The proof is omitted since it is similar to and simpler than the one given for part (a) of the next theorem.

(b) Let $\{x_n\} \subseteq X$ with $x_n \xrightarrow{\beta} x \in x$. Then it is β-bounded and hence u-bounded by Theorem 3.3(a). Further, since $k \leq \beta$, $x_n \xrightarrow{k} x$. Conversely, suppose $\{x_n\}$ is τ-bounded and $x_n \xrightarrow{k} x$. By (a), $k = \beta$ on the τ-bounded set $\{x, x_n = n \geq 1\}$; hence $x_n \xrightarrow{\beta} x$.

If, in addition, X is locally bounded, we obtain a stronger version of Theorem 3.4 which generalizes the corresponding results of Buck [3] and Sentilles and Taylor [18].

Theorem 3.5. *Suppose (X, τ) is locally bounded. Then*

(a) $k = \beta$ on u-bounded sets.

(b) A sequence $\{x_n\} \subseteq X$ is β-convergent iff it is u-bounded and k-convergent.

Proof. (a) Let S be a u-bounded set in X, and let $\{x_\alpha : \alpha \in J\} \subseteq S$ with $x_\alpha \xrightarrow{k} x \in S$. Let D be a finite set in A and $G \in \mathcal{W}$. Choose a balanced idempotent $H \in \mathcal{W}$ with

$H+H+H \subseteq G$. Let V be a closed bounded τ-neighborhood of 0 in X. Choose $r > 1$ with $V \subseteq rH$. Since S is u-bounded, it follows that $(V \cap A)S$ is τ-bounded. Choose $t > 1$ with $\{x\} \cup D \cup (V \cap A)S \subseteq tV$. There exists $\lambda_0 \in I$ such that

$$ae_{\lambda_0} - a \in \frac{1}{r^2t}V, \quad ae_{\lambda_0} - a \in \frac{1}{r^2t}V$$

for all $a \in D$. Choose $\alpha_0 \in J$ such that

$$e_{\lambda_0}(x_\alpha - x) \in \frac{1}{r^2t}V, \quad (x_\alpha - x)e_{\lambda_0} \in \frac{1}{r^2t}V$$

for all $\alpha \geq \alpha_0$. Now, let $a \in D$ and $\alpha \geq \alpha_0$. Then

$$a(x_\alpha - x) = (a - ae_{\lambda_0})x_\alpha + ae_{\lambda_0}(x_\alpha - x) + (ae_{\lambda_0} - a)x.$$

If $p = p_V$ is the Minkowski functional of V, then

$$\frac{(a - ae_{\lambda_0})x_\alpha}{p(a - ae_{\lambda_0})} \in (V \cap A)S \subseteq tV \quad \text{and so} \quad (a - ae_{\lambda_0})x_\alpha \in \frac{1}{r^2}V \subseteq H.$$

Hence $a(x_\alpha - x) \in H + H^2 + H^2 \subseteq G$. Similarly, we have $(x_\alpha - x)a \in G$. So $x_\alpha \xrightarrow{\beta} x$ and hence $\beta \leq k$.

(b) The proof is the same as that of Theorem 3.4(b).

The next two results characterize the equivalence of two of the topologies β, u, and τ on X.

Theorem 3.6. *Suppose (X, τ) is locally bounded. Then the following are equivalent:*

(a) u and τ have the same bounded sets.

(b) β and τ have the same bounded sets.

(c) The β-closure of a τ-bounded set is τ-bounded.

(d) $u = \tau$.

Proof. (a) $\Rightarrow$ (b). This follows immediately from Theorem 3.3(a).

(b) $\Rightarrow$ (c). Let S be a τ-bounded set in X. Since $\{e_\lambda : \lambda \in I\}$ is τ-bounded, $P = \bigcup_{\lambda \in I} e_\lambda S$ is also τ-bounded. Since $\beta \leq \tau$, P and hence $\overline{P}^\beta$ is β-bounded. By (b), $\overline{P}^\beta$ is τ-bounded. By Theorem 3.1(d), for any $x \in S, e_\lambda x \xrightarrow{\beta} x$ and so $S \subseteq \overline{P}^\beta$. Hence $\overline{S}^\beta$ is τ-bounded.

(c) $\Rightarrow$ (a). Let S be a u-bounded set in X. Then $P = \bigcup_{\lambda \in I} e_\lambda S$ is τ-bounded. Indeed, if $G \in \mathcal{W}$, then taking $D = \{e_\lambda : \lambda \in I\}$ as a bounded set in A, there exists $r > 0$ such that $S \subseteq r(D, G)$ and so $P \subseteq rG$. By (c), $\overline{P}^\beta$ is τ-bounded. But, as before, $S \subseteq \overline{P}^\beta$; hence S is τ-bounded.

(a) $\Rightarrow$ (d). Let $G \in \mathcal{W}$, and let V be a bounded τ-neighborhood of 0 in X. Then $D = V \cap A$ is a bounded τ-neighborhood of 0 in A, and so it easily follows that $N(D, V)$ is a bounded u-neighborhood of 0 in X. By (a), $N(D, V)$ is τ-bounded. There exists $r > 0$ such that $N(D, V) \subseteq rG$ or equivalently $N\left(D, \frac{1}{r}V\right) \subseteq G$. This shows that $\tau \leq u$.

(d) $\Rightarrow$ (a). This is obvious.

Theorem 3.7. *Consider the following conditions:*

(a) There exists $a \in A$ such that the map $x \to ax$ is an isomorphism.

(b) $\beta = \tau$ on X.

(c) (X, β) is metrizable.

(d) (X, β) is ultrabornological.

(e) $\beta = u$ on X.

(f) For each $x \in X, e_\lambda x \xrightarrow{u} x$ and $x e_\lambda \xrightarrow{u} x$.

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e). If (X,τ) is a Banach algebra, then (f) $\Rightarrow$ (a); that is, (a) – (f) are equivalent.

Proof. (a) $\Rightarrow$ (b). Suppose (a) holds, and let $x_\alpha \xrightarrow{\beta} x$ in X. Now $\varphi : x \to ax$ is a continuous, linear, and one-one map from (X,τ) onto (X,τ). As a consequence of the open mapping theorem [15], the inverse map $\varphi^{-1} : ax \to x$ is continuous. Since $ax_\alpha \xrightarrow{\tau} ax$, we have $\varphi^{-1}(ax_\alpha) \xrightarrow{\tau} \varphi^{-1}(ax)$; that is, $x_\alpha \xrightarrow{\tau} x$. So $\tau \leq \beta$.

(b) $\Rightarrow$ (c). This is obvious since (X,τ) is metrizable.

(c) $\Rightarrow$ (d). This follows from ([15], Theorem 1.32).

(d) $\Rightarrow$ (e). This follows from Theorem 3.3(b).

(e) $\Rightarrow$ (f). This follows from Theorem 3.1(d).

Now suppose that (X,τ) is a Banach algebra. Then (f) $\Rightarrow$ (a) follows from ([18], Theorem 2.4).

Following Rieffel [14], the *essential part* of X_e is defined as the u-closed linear subalgebra of X spanned by $AX \cup XA$; that is $X_e = \overline{AX \cup XA}^u$. X is called *essential* if $X = X_e$.

Theorem 3.8. $X_e = \{x \in X : e_\lambda x \xrightarrow{u} x, xe_\lambda \xrightarrow{u} x\}$.

Proof. If $x \in X$ with $e_\lambda x \xrightarrow{u} x$ and $xe_\lambda \xrightarrow{u} x$, then clearly $x \in X_e$. Now let $x \in X_e$, and consider any bounded set $D \subseteq A$ and $G \in \mathcal{W}$. Choose a balanced idempotent $H \in \mathcal{W}$ with $H + H + H \subseteq G$. Put $D_1 = \{a, e_\lambda, ae_\lambda, e_\lambda a : a \in D, \lambda \in I\}$, and choose $r > 1$ with $D_1 \subseteq rH$. We may assume that $x \in \overline{AX}^u$ and so there exist $a_1, \ldots, a_n \in A$ and $x_1, \ldots, x_n \in X$ such that $\sum_{i=1}^{n} a_i x_i - x \in N\left(D_1, \frac{1}{r}H\right)$. Choose $\lambda_0 \in I$ such that

$$e_\lambda a_i x_i - a_i x_i \in \frac{1}{nr}H, \quad a_i x_i e_\lambda - a_i x_i \in \frac{1}{nr}H$$

for all $\lambda \geq \lambda_0$ and $1 \leq i \leq n$. Then, for any $a \in D$ and $\lambda \geq \lambda_0$,

$$a(e_\lambda x - x) = ae_\lambda \left(x - \sum_{i=1}^{n} a_i x_i \right) + a \sum_{i=1}^{n} (e_\lambda a_i x_i - a_i x_i) + a \left(\sum_{i=1}^{n} a_i x_i - x \right) \in H + H^2 + H \subseteq G$$

and similarly $(e_\lambda x - x)a \in G$. Hence $e_\lambda x \xrightarrow{u} x$. In a similar way we can show that $xe_\lambda \xrightarrow{u} x$.

Under some additional hypotheses and using a factorization theorem we shall next show that $X_e = AX = XA$. But first we introduce some terminology. Following [2], a topological vector space E is called *fundamental* if there exists $r > 1$ such that for any sequence $\{x_n\}$ in E the convergence of $r^n\,(x_{n+1} - x_n)$ to 0 in E implies that $\{x_n\}$ is a Cauchy sequence. Every locally convex space and every locally bounded space is fundamental. An approximate identity $\{a_\lambda : \lambda \in I\}$ in a topological algebra E is said to be *uniformly bounded* if there exists $r > 0$ such that $\left\{ \left(\frac{a_\lambda}{r}\right)^n : \lambda \in I, n = 1, 2, \ldots \right\}$ is a bounded set in E.

We shall require the following generalization of the famous Cohen factorization theorem due to Ansari-Puri [2].

Theorem 3.9. *Suppose X is essential, and let A be fundamental with $\{e_\lambda : \lambda \in I\}$ uniformly bounded. Then*

(i) for each $x \in X$, there exists $a \in A$ and $y \in X$ such that $x = ay$;

(ii) for any sequence $\{b_n\}$ in A with $b_n \to 0$, there exist $a \in A$ and a sequence $\{c_n\}$ in A with $c_n \to 0$ such that $b_n = ac_n, n = 1, 2, \ldots$.

Note that this is a so-called left-hand version of the factorization theorem. Clearly its right-hand version also holds.

Theorem 3.10. *Let X and A be as in Theorem 3.9. Then $X_e = AX = XA$.*

Proof. This follows immediately from Theorem 3.9.

The following result is an analogue of ([18], Theorem 3.1) and ([20], Lemma 2.5).

Theorem 3.11. *Let X and A be as in Theorem 3.8, and suppose that A is also commutative. Then the collection of all sets $N(a, H)$ for $a \in A$ and $G \in \mathcal{W}$ is a base for β-neighborhoods of 0 in X.*

Proof. Let D be a finite set in X and $G \in \mathcal{W}$. Choose a balanced $H_1 \in \mathcal{W}$ with $H_1^2 \subseteq G$. Suppose $D = \{b_1, \ldots, b_m\}$. Taking $b_{m+1} = b_{m+2} = \cdots = 0$ in Theorem 3.9(ii), there exist $a, c_1, \ldots, c_m \in A$ such that $b_i = ac_i, i = 1, \ldots, m$. Choose $r > 1$ with $\{c_1, \ldots, c_m\} \subseteq rH_1$, and put $H = \frac{1}{r}H_1$. Then using the commutativity of A it is easy to see that $N(a, H) \subseteq N(D, G)$, as required.

Remark 3.12. If A is a B^*-algebra, then the above theorem holds without the assumption of commutativity of A. Indeed, in view of the factorization result of Taylor ([20], Lemma 2.4), we can choose in the above proof elements $a, c_1, \ldots, c_m, c_1', \ldots, c_m'$ in A such that $b_i = ac_i = c_i'a$. This theorem also holds without commutativity if we modify the definition of β by setting either $N(D, G) = \{x \in X : Dx \subseteq G\}$ or $N(D, G) = \{x \in X : xD \subseteq G\}$ (cf. [18], §3).

Now consider (A, τ) itself a Hausdorff topological algebra having a two-sided bounded approximate identity. A pair (S, T) of mappings $S, T : A \to A$ is called a *double multiplier* (or a double centralizer) on A if $aS(b) = T(a)b$ for all $a, b \in A$. Let $M_d(A)$ denote the algebra of all double multipliers (S, T) on A with S and T continuous linear (see

Johnson [7] and Busby [4] for details). We define a topology σ (resp. s) on $M_d(A)$ as the linear topology which has a base of neighborhoods of 0 consisting of all sets of the form

$$U(D,G) = \{(S,T) \in M_d(A) : S(D) \subseteq G, \ \ T(D) \subseteq G\}$$

where D is a bounded (resp. finite) subset of A and G is a neighborhood of 0 in A. Clearly $s \leq \sigma$. If we define a map $\mu_0 : A \to M_d(A)$ by $\mu_0(a) = (L_a, R_a)$, where $L_a(x) = ax$ and $R_a(x) = xa$ for all $a \in A$, then μ_0 is an algebra isomorphism and continuous with $\mu_0(A)$ a two-sided ideal in $M_d(A)$. Further, if A is complete, then under the above embedding, A is σ-closed and s-dense in $M_d(A)$; if A is complete and metrizable, then $(M_d(A), \sigma), (M_d(A), s)$ are complete and σ, s have the same bounded sets in $M_d(A)$ (see [4, 7, 8] for details).

Finally we consider the problem of completeness of (X, β) via an embedding of (X, β) into $(M_d(A), s)$. Define $\mu : X \to M_d(A)$ by $\mu(y) = (L_y, R_y), \ \ y \in X$. Then, as in ([4], Proposition 3.7), μ is a unique isomorphism satisfying $\mu(a) = \mu_0(a)$ for all $a \in A$.

The following result is given in [4] for X a Banach algebra (see also [18], Theorem 3.5) and in [8] for X a topological algebra . For completeness, we include it here in a slightly improved form.

Theorem 3.13. *Let μ and s be as above. Then*

(a) $\mu : (X, \beta) \to (M_d(A), s)$ is continuous and open onto $\mu(X)$.

(b) If A is complete and metrizable, then (X, β) is complete iff $\mu(X) = M_d(A)$.

Proof. (a) Let $\{y_\alpha\}$ be a net in X with $y_\alpha \xrightarrow{\beta} y \in X$. Then, for each $a \in A$, $y_\alpha a \xrightarrow{\tau} ya$ and $a y_\alpha \xrightarrow{\tau} ay$ in A; that is, $(L_{y_\alpha}, R_{y_\alpha}) \xrightarrow{s} (L_y, R_y)$ in $M_d(A)$. So μ is continuous. For any finite set $D \subseteq A$ and open $G \in \mathcal{W}$, we have

$$\begin{aligned} \mu(N(D,G)) &= \{(L_x, R_x) : x \in X \text{ with } L_x(D) \subseteq G, R_x(D) \subseteq G\} \\ &= \{\mu(x) : x \in X \text{ with } \mu(x) \in U(D,G)\} \end{aligned}$$

which is s-open in $\mu(X)$. Hence μ is open onto $\mu(X)$.

(b) If (X, β) is complete, then $\mu(X)$ is complete and hence a closed subset of $(M_d(A)s)$. Since $A \subseteq X$ and $\mu(A)$ is s-dense in $M_d(A)$, ([8], Theorem 3.2(b)), $\mu(X)$ is also s-dense in $M_d(A)$. By ([8], Theorem 3.2(a)), $(M_d(A), s)$ is complete. Thus (X, β) is complete iff $\mu(X) = M_d(A)$.

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