King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 233

July 1998

Construction of Even Order Magic Squares

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CONSTRUCTION OF EVEN ORDER MAGIC SQUARES

Abstract

The aim of this note is to introduce a general method for the construction of single and double even order magic squares. The method for double even order magic squares is straightforward but some adjustment is necessary for single even order magic squares.

Introductory facts

A *primitive magic square* (referred to as a magic square in what follows) of *order* n is a square consisting of the n^2 distinct numbers $1, 2, \ldots, n^2$ in n^2 subsquares such that the sum of each row, column and main diagonals adds up to the same total, $n(n^2+1)/2$.

A *single even order* magic square is one whose order is divisible by 2 but not divisible by 4.

A *double even order* magic square is one whose order is divisible by 4.

A pair of natural numbers (a, b) is $(\pm k)$-*complementary* if $a + b = n^2 + 1 \pm k$, where n is the order of the magic square. If $k = 0$, the pair is simply called complementary. There are $n^2/2$ distinct complementary pairs in an even order magic square.

A magic square is *parallel* if all its complementary pairs are placed parallel to one another.

An even order magic square is *associated* if all its complementary pairs are equidistant from the centre of symmetry of the square, i.e., the point where all the lines of symmetry meet.

A magic square is *mixed* if it is neither parallel nor associated.

Introduction

The sum of the first n^2 natural numbers is $n^2(n^2+1)/2$ and the row sum, column sum and main diagonal sum is $n(n^2+1)/2$.

There are several methods for constructing magic squares of any order[1,2,3,4]. The fastest and general method for constructing odd order magic squares by consecutive numbering was found by the Middle Easterners. This method appears fully developed in Persian manuscripts of the 13th century[2]. However, even order magic squares are constructed differently.

The construction of double even order magic squares

Since a magic square is a sort of symmetrical or balanced square we may therefore impose a simple symmetry on the even-odd number distribution, and as a result a simple even-odd symmetrical picture is given by Figure 1.

Odd Numbers	Even Numbers
Even Numbers	Odd Numbers

Fig. 1. The general form of a double even square.

Beginning with the pair of outermost columns, place the complementary pairs horizontally from top to bottom in the following order:

$$(1,n^2),(2,n^2-1),(3,n^2-2),\ldots,(n,n^2-n+1),$$

observing the even-odd distribution symmetry in Figure 1. Next consider the pair of next outermost columns and place the complementary pairs horizontally from top to bottom in the following order:

$$(n+1, n^2-n), (n+2, n^2-n+1), \ldots, (2n, n^2-2n+1),$$

again, observing the even-odd distribution symmetry in Figure 1. Continue the process until you reach the pair of innermost columns always observing the even-odd distribution symmetry in Figure 1. Interchanging the lower diagonal and bottom complementary pairs of the pair of innermost columns yields a parallel magic square.

The construction of the 8 by 8 square is shown in Figure 2a with the parallel magic square shown in Figure 2b after the two interchanges have taken place as shown by * and ∧. 36 was in the position of 32 and vice-versa before the interchange and similarly for 33 and 29.

1	9	17	25	40	48	56	64
2	10	18	26	39	47	55	63
3	11	19	27	38	46	54	62
4	12	20	28	37	45	53	61
5	13	21	29	36	44	52	60
6	14	22	30	35	43	51	59
7	15	23	31	34	42	50	58
8	16	24	32	33	41	49	57

1	9	17	25	40	48	56	64
63	55	47	39	26	18	10	2
3	11	19	27	38	46	54	62
61	53	45	37	28	20	12	4
60	52	44	32*	33∧	21	13	5
6	14	22	30	35	43	51	59
58	50	42	34	31	23	15	7
8	16	24	36*	29∧	41	49	57

Fig.2 (a) Constructing an 8 by 8 magic square, (b) A complete 8 by 8 magic square

The numbers $1, 2, \ldots, n$ (here $n = 8$) are placed consecutively down the first column, the numbers $n+1, n+2, \ldots, 2n$ are placed consecutively, down the second column and continue the process until the numbers $(n/2-1)n+1, (n/2-1)n+2, \ldots, n^2/2$ are placed in the $(n/2)$-th column. Then starting at the last column place the numbers in descending order in a similar fashion. Interchanging the odd and even numbers symmetrically within each row to adhere to Figure 1 and carrying out the interchanges shown by * and ∧ gives the required square.

The general result for an n by n magic square is shown in Figure 3 where $p = n^2$.

R/C	1	2	3		$n/2$	$n/2+1$		$n-2$	$n-1$	n
1	1	$n+1$	$2n+1$	$\cdots$	$p/2-n+1$	$p/2+n$	$\cdots$	$p-2n$	$p-n$	p
2	$p-1$	$p-n-1$	$p-2n-1$	$\cdots$	$p/2+n-1$	$p/2-n+2$	$\cdots$	$2n+2$	$n+2$	2
3	3	$n+3$	$2n+3$	$\cdots$	$p/2-n+3$	$p/2+n-2$	$\cdots$	$p-2n-2$	$p-n-2$	$p-2$
4	$p-3$	$p-n-3$	$p-2n-3$	$\cdots$	$p/2+n-3$	$p/2-n+4$	$\cdots$	$2n+4$	$n+4$	4
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
$n/2$	$p-n/2+1$	$p-3n/2+1$	$p-5n/2+1$	$\cdots$	$p/2+n/2+1$	$p/2-n/2$	$\cdots$	$5n/2$	$3n/2$	$n/2$
$n/2+1$	$p-n/2$	$p-3n/2$	$p-5n/2$	$\cdots$	$p/2*$	$p/2+1\wedge$	$\cdots$	$5n/2+1$	$3n/2+1$	$n/2+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
$n-3$	$p-n+4$	$p-2n+4$	$p-3n+4$	$\cdots$	$p/2+4$	$p/2-3$	$\cdots$	$3n-3$	$2n-3$	$n-3$
$n-2$	$n-2$	$2n-2$	$3n-2$	$\cdots$	$p/2-2$	$p/2+3$	$\cdots$	$p-3n+3$	$p-2n+3$	$p-n+3$
$n-1$	$p-n+2$	$p-2n+2$	$p-3n+2$	$\cdots$	$p/2+2$	$p/2-1$	$\cdots$	$3n-1$	$2n-1$	$n-1$
n	n	$2n$	$3n$	$\cdots$	$p/2+n/2*$	$p/2-n/2+1\wedge$	$\cdots$	$p-3n+1$	$p-2n+1$	$p-n+1$

Fig. 3. A complete double even order n by n magic square after the interchange

Each row of the n by n square consists of $n/2$ complementary pairs thus making a total of $n(n^2+1)/2$, the required sum. Each column consists of $n/4(+1)$-complementary pairs and $n/4(-1)$-complementary pairs thus making a total of $(n/4)(n^2+2)+(n/4)\, n^2 = n(n^2+1)/4$, the required sum. The sums of the entries in the two main diagonals are easily seen to be $(n/2)n^2$ and $(n/2)n^2+n$. The interchange of the lower diagonal and bottom complementary pairs of the pair of innermost columns can be shown to increase the leading diagonal by $n/2$ and simultaneously decrease the other main diagonal by the same amount thus giving the required sum. Hence the square obtained is a magic square. Moreover, by construction the magic square is parallel.

The construction of single even order magic squares

Since the row sum $n(n^2+1)/2$ is odd ($n/2$ is odd and n^2+1 is odd), Figure 1 must be modified as shown in Figure 4. The two innermost rows are omitted and the centre $2n$ numbers of the numbers $1, 2, \ldots, n^2$ are also omitted and these are the n complementary pairs

$$(n^2-2n)/2+1, (n^2+2n)/2; (n^2-2n)/2+2, (n^2+2n)/2-1; n^2/2, n^2/2+1.$$

The process is similar to that of the double even case.

<table>
<tr><td>Odd Numbers</td><td>Even Numbers</td></tr>
<tr><td colspan="2">$(n/2)$-th row</td></tr>
<tr><td colspan="2">$(n/2+1)$-th row</td></tr>
<tr><td>Even Numbers</td><td>Odd Numbers</td></tr>
</table>

Fig. 4. The general form of a single even square

The construction of the 10 by 10 square is shown in Figure 5a and 5b with the two centre rows omitted.

1	9	17	25	33	68	76	84	92	100
2	10	18	26	34	67	75	83	91	99
3	11	19	27	35	66	74	82	90	98
4	12	20	28	36	65	73	81	89	97
5	13	21	29	37	64	72	80	88	96
6	14	22	30	38	63	71	79	87	95
7	15	23	31	39	62	70	78	86	94
8	16	24	32	40	61	69	77	85	93

Fig. 5. (a) Constructing a 10 by 10 magic square omitting the centre two rows

1	9	17	25	33	68	76	84	92	100
99	91	83	75	67	34	26	18	10	2
3	11	19	27	35	66	74	82	90	98
97	89	81	73	65	36	28	20	12	4
96	88	80	72	64	37	29	21	13	5
6	14	22	30	38	63	71	79	87	95
94	86	78	70	62	39	31	23	15	7
8	16	24	32	40	61	69	77	85	93

Fig. 5. (b) A 10 by 10 magic square with the centre two rows omitted

To complete the magic square, first we define some sequences involving the unused complementary pairs. Let $(a_n), (b_n), (f_n)$ and (g_n) be finite sequences, all with common difference $d = 2$ whose first and last terms are given by

$$\begin{aligned} a_n &: p/2 - n + 3, \ldots, p/2 - n/2 - 2; \\ b_n &: p/2 - n/2 + 1, \ldots, p/2 - 2; \\ f_n &: p/2 + 4, \ldots, p/2 + n/2 - 1; \\ g_n &: p/2 + n/2 + 2, \ldots, p/2 + n - 3; \end{aligned}$$

where $p = n^2$ and $n \geq 6$ for (b_n) and $n \geq 10$ for the other sequences. If $n = 6$ all the sequences vanish except (b_n).

It can be shown that $(a_n), (f_n)$ and (g_n) have $(n-6)/4$ terms and (b_n) has $(n-2)/4$ terms; these four sequences give $3(n-6)/4 + (n-2)/4 = n - 5$ terms. These four sequences are used for the $(n/2)$-th row. 5 independent terms are needed to complete the n terms for the $(n/2)$-th row.

The $(n/2)$-th row is filled out as shown in Figure 6. There are n positions to be filled and these are filled from the left numbered 1 to n. If a term is independent (i) or odd (o) or even (e) this is indicated.

Position	1	2	$\cdots$	$(n-2)/4$	$(n+2)/4$	$\cdots$	$n/2-1$
Number	$p/2-n+2$	$p/2-n+3$	$\cdots$	$p/2-n/2-2$	$p/2-n/2+1$	$\cdots$	$p/2-2$
Sequence/independent	i	a_1	$\cdots$	$a_{\frac{n-6}{4}}$	b_1	$\cdots$	$b_{\frac{n-2}{4}}$
Odd/Even	(e)	o	$\cdots$	o	e	$\cdots$	e

$n/2$	$n/2+1$	$n/2+2$	$n/2+3$	$n/2+4$	$\cdots$	$(3n+6)/4$	$(3n+10)/4$	$\cdots$	n
$p/2+n$	$p/2+n/2+1$	$p/2-1$	$p/2$	$p/2+4$	$\cdots$	$p/2+n/2-1$	$p/2+n/2+2$	$\cdots$	$p/2+n-3$
i	i	i	i	f_1	$\cdots$	$f_{\frac{n-6}{4}}$	g_1	$\cdots$	$g_{\frac{n-6}{4}}$
e	e	o	e	e	$\cdots$	e	o	$\cdots$	o

Fig. 6 The $(n/2)$-th row of a single even n by n magic square

The $(n/2+1)$-th row consists of numbers that make the numbers in each column (of the two middle rows) complementary.

It can be verified routinely by ordering all the even numbers in ascending order and similarly with the odd numbers that the $2n$ numbers of the two centre rows are distinct and are between $p/2-n+1$ and $p/2+n$ inclusive.

It can be shown that the two middle rows each sum to $n(n^2+1)/2$ and that the rest of the rows consist of $n/2$ complementary pairs thus giving the required sum. Each column consists of $(n-2)/4(+1)$-complementary pairs, $(n-2)/4(-1)$-complementary pairs and a single complementary pair (from the two middle rows) thus making a total of $[(n-2)/4](n^2+2)+[(n-2)/4]n^2+(n^2+1)=n(n^2+1)/2$, the required sum. The sum of the entries in the main diagonals can be verified to be $n(n^2+1)/2$. A completed example for the 10 by 10 magic square is shown in Figure 7.

1	9	17	25	33	68	76	84	92	100
99	91	83	75	67	34	26	18	10	2
3	11	19	27	35	66	74	82	90	98
97	89	81	73	65	36	28	20	12	4
42	43	46	48	60	56	49	50	54	57
59	58	55	53	41	45	52	51	47	44
96	88	80	72	64	37	29	21	13	5
6	14	22	30	38	63	71	79	87	95
94	86	78	70	62	39	31	23	15	7
8	16	24	32	40	61	69	77	85	93

Fig. 7. A 10 by 10 magic square

Concluding remarks

The interesting thing about this method of constructing a double even order magic square is that it can be regarded as a continuous or consecutive (i.e., by consecutive numbering) method by analogy with the fastest method for constructing odd order magic squares. The description for this method is as follows:

Place consecutively, the numbers $1, 2, \ldots, n$ from the first to the last row (one in each row) alternating from left to right between the pair of outermost columns, however $n/2$ and $n/2+1$ are placed in the same column after which you resume the alternate placement from left to right. Next place the numbers $n+1, n+2, \ldots, 2n$ in exactly the same fashion as above between the pair of next outermost columns, noting that $3n/2$ and $3n/2+1$ are placed in the same column. Continue the process until you have reached the innermost columns and placed $n^2/2$ at the bottom cell of the left hand column of the innermost pair. Now place $n^2/2+1$ adjacent to $n^2/2$ (i.e., in the bottom cell of the right hand column of the innermost pair), and continue the consecutive placement but reversing the whole process. That is, place the number $n^2/2+1, \ldots, n^2$ alternately from right to left, from the last to the first row and moving outwards from the innermost to the outermost pair of columns. Finally, interchanging the lower diagonal and the bottom pairs of the innermost columns completes the magic square.

Acknowledgment. I wish to thank the referee whose invaluable suggestions greatly improved the presentation of this paper.

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