King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 235

November 1998

Estimation of the Scale Matrix of a Class of Elliptical Distributions

Anwar H. Joarder, S.E. Ahmed

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

Estimation of the Scale Matrix of a Class of Elliptical Distributions

Anwar H. Joarder
Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia
Internet: anwarj@kfupm.edu.sa

S.E. Ahmed
Department of Mathematics and Statistics
University of Regina, Saskatchewan, Canada S4S 0A2
Internet: ahmed@math.uregina.ca

Abstract

The problem of estimation of the scale matrix of a class of elliptical distributions is considered. We propose an improved class of estimators for scale matrix. The exact forms of the risk functions are derived as well. The relative merits of the class of improved estimators to the usual one are appraised in the light of a quadratic loss function. The conditions under which the class of proposed estimators outperform the class of usual estimators are obtained. Relative Risk is also computed for a special case. Some important characteristics of scale matrix are also considered for estimation.

Key Words and Phrases: Elliptical distribution; multivariate normal distribution; multivariate t-distribution; estimation of scale matrix; risk function.

AMS 1991 Subject Classification: Primary 62H05, Secondary:62H12

1 Introduction

In this paper we consider the estimation of the scale matrix of a class of elliptical distributions. The multivariate normal distribution and the multivariate t-distribution are the two important special cases of the class of elliptical distributions.

This class of distributions contains thin tailed as well as fat tailed distributions and hence is important in modeling many real data. It may be mentioned here that several authors have observed that the empirical distributions of rates of return of common stocks have somewhat fatter tails than that of the normal distribution. The multivariate t-distribution has also fatter tail and can, therefore, characterize rates of return on common stocks.

The estimation of the scale matrix of the multivariate normal distribution was considered by Olkin and Selliah (1977) under a weighted squared error loss function. Joarder (1995) considered the estimation of the scale matrix of the multivariate t-distribution under a squared error loss function. Since the class of elliptical distributions accommodates multivariate normal as well as multivariate t-distribution, we feel it is important to generalize the estimation strategy for the class of elliptical distributions given by

$$\int_0^\infty \frac{|\tau^2\Sigma|^{-1/2}}{(2\pi)^{p/2}}\, exp\left(-\frac{1}{2}\,(x-\theta)'(\tau^2\Sigma)^{-1}(x-\theta)\right) h(\tau)d\tau. \tag{1.1}$$

Here $x = (x_1, x_2, \ldots, x_p)'$ is a p (≥ 2)-dimensional column vector, θ an unknown vector of location parameters and Σ an unknown positive definite matrix of scale parameters while $h(\tau)$ is the probability density function (p.d.f.) of a non-discrete (degenerate or continuous) random variable τ. Many distributions having p.d.f. constant on the hyper-ellipse

$$(x-\theta)'\Sigma^{-1}(x-\theta) = c^2$$

may be generated by varying $h(\tau)$ and hence these distributions are known as elliptical distributions.

The density in (1.1) is the joint p.d.f. of a class of elliptical distributions. It is also known as compound normal distributions or scale mixture of normal distributions. The above distribution has a mean vector $\theta = (\theta_1, \theta_2, \cdots, \theta_p)'$ and covariance matrix $\gamma_2\Sigma$ where $\gamma_2 = E(\tau^2) > 0$ is assumed to be known, whenever it exists.

In this paper we consider the estimation of Σ, its trace and its inverse under suitable quadratic loss functions. The rest of the paper is organized as follows. Section 2 proposes the estimators of Σ. The expressions for the risk of the estimators are provided in section 3. The properties of the class of proposed estimators and its comparison with the usual estimator are also given in the same section. An analysis of relative risk is presented in section 4. Some important characteristics of the scale matrix are also considered for estimation in section 5.

2 The Proposed Estimation Strategy

Let the p-dimensional ($p \geq 2$) random vectors (not necessarily independent) $X_1, X_2, \ldots, X_N$ have the joint probability density function (p.d.f.)

$$\int_0^\infty \frac{|\tau^2\Sigma|^{-N/2}}{(2\pi)^{Np/2}}\, exp\left(\frac{-1}{2}\sum_{j=1}^{N}(x_j-\theta)'(\tau^2\Sigma)^{-1}(x_j-\theta)\right) h(\tau)d\tau \tag{2.1}$$

where $x_j = (x_{1j}, x_{2j}, \ldots, x_{pj})'$, $j = 1, 2, \ldots, N$ is a p-dimensional column vector, θ an unknown vector of location parameters and Σ an unknown positive definite matrix of scale parameters. The observations $X_1, X_2, \ldots, X_N$ are independent only if τ is degenerate at the point unity in which case the joint p.d.f. in (2.1) denotes the p.d.f. of the product of N independent multivariate normal distributions each being $N_p(\theta, \Sigma)$. Further, if ν/τ^2 has a χ^2_ν distribution, then the p.d.f. in (2.1) defines a joint multivariate t-distribution of $X_1, X_2 \ldots, X_N$ each having mean vector θ and the covariance matrix $\Sigma_\nu = \nu\Sigma/(\nu-2)$. The joint multivariate t-distribution has been considered, among others, by Zellner (1976), Sutradhar and Ali (1989) and Joarder and Ahmed (1996).

The scale matrix Σ is usually estimated, especially in the multivariate normal case (a special case of the model in (1.1)), by multiples of the sample sum of product matrix A. For example an unbiased estimator of Σ for the model in (1.1) when $\nu/\tau^2 \sim \chi^2_\nu$ is given by $(\nu-2)A/(\nu n)$, $n = N - 1$ (Anderson and Fang, 1990, p. 208).

The maximum likelihood estimation of Σ was studied by Anderson, Fang and Hsu (1986) when $(X_1, X_2, \ldots, X_N)$ follows a broader class of elliptical distributions than (1.1). In our case the maximum likelihood estimator is given by A/N (Anderson and Fang, 1990, p. 205). However, the most important properties of maximum likelihood estimators follows from the usual assumption of independence of sample observations which is not necessarily true in model (1.1). Therefore an alternative method of estimation is considered here following Dey (1988) who developed simultaneous estimators of the eigenvalues of the covariance matrix of the multivariate normal distribution by shrinking sample eigenvalues towards their geometric mean.

We now propose a class of estimators for the scale matrix of a class of elliptical distributions having joint p.d.f. given by (2.1). In particular, we consider the following two classes of estimators of Σ.

2.1 Class of Usual Estimators

First, let us define the sample sum of product matrix

$$A = \sum_{j=1}^{N} (X_j - \bar{X})(X_j - \bar{X})',$$

where $\bar{X} = (\bar{X}_1, \bar{X}_2, \cdots, \bar{X}_p)'$ and $\bar{X}_i = \sum_{j=1}^{N} X_{ij}/N, \quad i = 1, 2, \cdots, p$. The *class of usual estimators (CUE)* of Σ denoted by $\tilde{\Sigma}$ is

$$\tilde{\Sigma} = c_o A,$$

where c_0 is a fixed positive constant.

2.2 Class of Improved Estimators

We propose a *class of improved estimators (CIE)* of Σ denoted by $\hat{\Sigma}$ and is given by

$$\hat{\Sigma} = c_o A - c|A|^{1/p} I,$$

where c is chosen such that $\hat{\Sigma}$ is positive definite and I is an identity matrix.

Let Σ^* be any estimator of Σ. To appraise the statistical properties of estimators, we explore a quadratic loss function:

$$\mathcal{L}(\Sigma^*, \ \Sigma) = tr[(\Sigma^* - \Sigma)^2]. \tag{2.2}$$

In estimating Σ by Σ^*, the risk function is defined as usual by taking expectation over the loss function, i.e.,

$$R(\Sigma^*,\ \Sigma) = E\mathcal{L}[\Sigma^*,\ \Sigma)]. \tag{2.3}$$

An estimator Σ^* is said to dominate another estimator Σ^o if, $R(\Sigma^*;\Sigma) \leq R(\Sigma^o;\Sigma)$ for all Σ. If, in addition, $R(\Sigma^*;\Sigma) < R(\Sigma^o;\Sigma)$ for at least some Σ, then Σ^* strictly dominates Σ^o.

In the light of the above definition, we prove a dominance theorem that the class of proposed estimators $\hat{\Sigma}$ of Σ dominates the class of usual estimators $\tilde{\Sigma}$ of Σ in the sense of having smaller risk i.e.

$$R(\hat{\Sigma},\ \Sigma) = E[tr(\hat{\Sigma} - \Sigma)^2] < R(\tilde{\Sigma},\ \Sigma) = E[tr(\tilde{\Sigma} - \Sigma)^2].$$

We now sketch proofs of three important lemmas that will be required in the sequel. These lemmas are generalizations of some lemmas of Joarder and Ali (1992) and Joarder (1998). It follows from (1.1) that $X|\tau \sim N_p(\theta, \tau^2\Sigma)$ and consequently from (2.1)

$$A|\tau \sim \mathcal{W}_p(n,\ \tau^2\Sigma),\ n = N-1 \tag{2.4}$$

i.e. for given τ, the random matrix A has the usual Wishart distribution with parameters n and $\tau^2\Sigma$. Thus, the p.d.f. of the Wishart matrix based on the class of elliptical distributions in (2.1) is given by

$$m(A) = \int_0^\infty \frac{|\tau^2\Sigma|^{-n/2}|A|^{(n-p-1)/2}}{2^{np/2}\ \Gamma_p(n/2)} \exp\left[-\frac{1}{2}tr\ ((\tau^2\Sigma)^{-1}A)\right] h(\tau)d\tau, \tag{2.5}$$

where $A > 0,\ n \geq p$ and

$$\Gamma_p(\alpha) = \pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma\left(\alpha - \frac{1}{2}(i-1)\right),\ \alpha > \frac{p-1}{2}. \tag{2.6}$$

Lemma 2.1 Let the sum of products matrix (Wishart matrix) A have the p.d.f. given by (2.5). Then the rth moment of $|A|$ is given by

$$E(|A|^r) = 2^{pr}\ \frac{\Gamma_p(n/2+r)}{\Gamma_p(n/2)}\ |\Sigma|^r\gamma_{2pr}$$

where r is any real number and $\gamma_{2pr} = E(\tau^{2pr}) > 0$ (assumed to exist).

Proof. It follows from the mixture representation in (2.4) and Muirhead (1986) that for any integer r

$$E(|A|^r) = E[E(|A|^r|\tau)] = E\left[2^{pr}\ \frac{\Gamma_p(n/2+r)}{\Gamma_p(n/2)}\ |\tau^2\Sigma|^r\right]$$

and hence the proof.

Lemma 2.2 Consider the p.d.f. given by (2.5). Then for any real number k satisfying $n+2k>0$ and $\gamma_{2kp+2} = E(\tau^{2kp+2}) > 0$ (assumed to exist), the following result holds:

$$E[|A|^k A] = 2^{kp}\ (n+2k)\ \frac{\Gamma_p(n/2+k)}{\Gamma_p(n/2)}\ |\Sigma|^k\ \Sigma\ \gamma_{2kp+2}.$$

Proof. Since $A|\tau \sim \mathcal{W}_p(n,\ \tau^2\Sigma)$, it follows from Dey (1988, p 140) that for any real number k

$$E[|A|^k A] = E\left[E\left(|A|^k A|\tau\right)\right] = E\left[2^{kp}\ (n+2k)\ \frac{\Gamma_p(n/2+k)}{\Gamma_p(n/2)}\ |\tau^2\Sigma|^k\ (\tau^2\Sigma)\right]$$

and hence the proof.

Lemma 2.3 Let A have the p.d.f. given by (2.5). Then we have

$$E[(trA)^2] = n\gamma_4[n\ (tr\Sigma)^2 + 2\ tr(\Sigma^2)]$$

provided $\gamma_4 = E(\tau^4)$ exists.

Proof. It follows from Anderson (1958, p. 161) that for Wishart matrix $A = ((a_{ik}))$ we have

$$E(a_{ii}a_{kk}|\tau) = n^2(\tau^2\ \sigma_{ii})(\tau^2\sigma_{kk}) + 2n(\tau^2\sigma_{ik})^2\quad (i,k=1,2,\cdots,p)$$

so that

$$\begin{aligned} E[(trA)^2|\tau] &= \sum_{i=1}^{p} E(a_{ii}^2) + 2\sum_{i(<k)=1}^{p}\sum_{k=1}^{p} E(a_{ii}a_{kk}) \\ &= \tau^4\left[n^2\sum_{i=1}^{p}\sigma_{ii}^2 + 2n\sum_{i=1}^{p}\sigma_{ii}^2\right. \\ &\left. +\ 2n^2\sum_{i(<k)=1}^{p}\sum_{k=1}^{p}\sigma_{ii}\sigma_{kk} + 4n\sum_{i(<k)=1}^{p}\sum_{k=1}^{p}\sigma_{ik}^2\right]. \end{aligned}$$

Rearranging, we have

$$E[(trA)^2|\tau] = n^2\ (tr\ \tau^2\ \Sigma)^2 + 2n\ tr(\tau^2\ \Sigma^2).$$

Recalling the mixture representation of the Wishart matrix given by (2.5), we have

$$E[(trA)^2] = E[E[(trA)^2|\tau]] = E[n^2(tr\ \tau^2\Sigma)^2 + 2n\ tr(\tau^2\Sigma)^2]$$

and the proof then follows.

3 Main Results

In the following theorem we prove that the class of proposed estimators of the scale matrix of a class of elliptical

distributions dominates the class of usual estimators under certain conditions.

Theorem 3.1 Consider the class of elliptical distributions given by (2.1). Then with respect to the loss function given by (2.2) the class of proposed estimators $\hat{\Sigma}$ dominates the class of usual estimators $\tilde{\Sigma}$, for any c satisfying

$$\begin{aligned} & d < c < 0 \\ & \text{where } d = \left(c_0\frac{np+2}{p} - \gamma\right)\frac{\Gamma_p(n/2+1/p)}{\Gamma_p(n/2+2/p)}, \quad n = N-1 \\ & \text{with } c_0 < \frac{p\gamma}{np+2}, \\ & \text{or, } 0 < c < d \end{aligned} \tag{3.1}$$

where d is given by (3.1) with $c_0 > p\gamma/(np+2)$ and γ by

$$\gamma = \gamma_2/\gamma_4,\ \gamma_i = E(\tau^i), \quad i = 1,2,3,4 \text{ (assumed to exist).} \tag{3.2}$$

Proof. Let $D(\Sigma,\ c) = R(\hat{\Sigma},\ \Sigma; c) - R(\tilde{\Sigma},\ \Sigma)$. Then it is easy to show that

$$\begin{aligned} D(\Sigma,\ c) &= E\left[tr(\hat{\Sigma}-\Sigma)^2 - tr(\tilde{\Sigma}-\Sigma)^2\right] = Etr\left[\hat{\Sigma}^2 - \tilde{\Sigma}^2 - 2(\hat{\Sigma}-\tilde{\Sigma})\Sigma\right] \\ &= E\left[-2cc_0|A|^{1/p}tr(A) + c^2p|A|^{2/p} + 2c|A|^{1/p}tr(\Sigma)\right]. \end{aligned}$$

It then follows from Lemma 2.1 and Lemma 2.2 that

$$\begin{aligned} D(\Sigma,\ c) &= 4p\ \gamma_4 \frac{|\Sigma|^{2/p}}{\Gamma_p(n/2)}\ c \\ &\times \left[\left(-c_0\frac{np+2}{p}+\gamma\right)\Gamma_p(n/2+1/p)\ \frac{tr(\Sigma/p)}{|\Sigma|^{1/p}}+\Gamma_p(n/2+2/p)\ c\right] \\ &= 4p\ \gamma_4 \frac{\Gamma_p(n/2+2/p)}{\Gamma_p(n/2)}\ |\Sigma|^{2/p}\ c\left(c-\frac{tr(\Sigma/p)}{|\Sigma|^{1/p}}\ d\right) \qquad (3.3) \end{aligned}$$

where d is given by (3.1) and γ_4 by (3.2). But this is equivalent to the equation (2.9) of Joarder and Ahmed (1996) and hence following arguments therein we find that $\hat{\Sigma}$ dominates $\tilde{\Sigma}$ if $d < c < 0$ or $0 < c < d$. However,

$$d < 0 \text{ if and only if } c_0 < p\gamma/(np+2)$$

$$\text{while } d > 0 \text{ if and only if } c_0 > p\gamma/(np+2).$$

Hence the class of proposed estimators $\hat{\Sigma}$ dominates the class of usual estimators $\tilde{\Sigma}$ if c satisfies the conditions mentioned in the theorem. Thus the theorem is proved.

It may be remarked that for the case when $c_0 = p\gamma/(np+2)$, we have $d = 0$. In this case it is seen from (3.3) that $D(\Sigma, c) \geq 0$ so that there exists no proposed estimator $\hat{\Sigma}$ dominating the usual estimator $\tilde{\Sigma}$. However, $D(\Sigma, c) = 0$ only if $c = 0$ in which case the two estimators do coincide.

We now find explicit expressions for the risk functions of the class of usual estimators and the class of proposed estimators of the scale matrix of the class of elliptical distributions.

Theorem 3.2 The risk function of the class of proposed estimators $\hat{\Sigma}$, and the class of usual estimators $\tilde{\Sigma}$, are respectively given by

$$\begin{aligned} R(\hat{\Sigma}, \Sigma; c) &= 4p\gamma_4 \frac{\Gamma_p(n/2+2/p)}{\Gamma_p(n/2)}\ |\Sigma|^{2/p}\ c\left(c-\frac{tr(\Sigma/p)}{|\Sigma|^{1/p}}\ d\right) \\ &+ [1+nc_0\gamma_4\ (c_0(n+1)-2\gamma)]\ tr(\Sigma^2)+nc_0^2\ \gamma_4\ (tr\Sigma)^2, \qquad (3.4) \end{aligned}$$

and

$$R(\tilde{\Sigma}, \Sigma) = [1+nc_0\ \gamma_4\,(c_0(n+1)-2\gamma)]\,tr(\Sigma^2)+nc_0^2\ \gamma_4(tr\Sigma)^2, \qquad (3.5)$$

where d is given by (3.1) while γ and γ_4 are given by (3.2).

Proof. The risk function of the class of usual estimators is

$$\begin{aligned} R(\tilde{\Sigma},\Sigma) &= E[tr(\tilde{\Sigma}-\Sigma)^2] = c_0^2\, E[tr(A^2)] - 2c_0\, tr(\Sigma E(A)) + tr(\Sigma^2) \\ &= c_0^2\, E\left[E[tr(A^2|\tau)]\right] - 2c_0\, tr[\Sigma E(E(A|\tau))] + tr(\Sigma^2) \\ &= nc_0^2\, E(\tau^4)\left[(n+1)tr(\Sigma^2) + (tr\Sigma)^2\right] - 2nc_0\, E(\tau^2)tr(\Sigma^2) + tr(\Sigma^2) \\ &= tr(\Sigma^2)\,[1 + nc_0\,\gamma_4\,((n+1)c_0 - 2\gamma)] + nc_0^2\,\gamma_4\,(tr\Sigma)^2 \end{aligned}$$

where the second moment of usual Wishart matrix is used from Srivastava and Khatri (1979, p. 99). The risk function of the class of proposed estimators is given by

$$R(\hat{\Sigma},\Sigma;c) = D(\Sigma,c) + R(\tilde{\Sigma},\Sigma)$$

where $D(\Sigma,c)$ and $R(\tilde{\Sigma},\Sigma)$ are given by (3.3) and (3.5). This completes the proof.

4 Relative Risk Analysis

To compare the risks of the two classes of estimators $\hat{\Sigma}$ and $\tilde{\Sigma}$, we use the *relative risk (RR)* defined by

$$RR(\hat{\Sigma}:\tilde{\Sigma};c) = \frac{R(\hat{\Sigma},\Sigma)}{R(\tilde{\Sigma},\Sigma;c)} \tag{4.1}$$

where $0 \le RR(\hat{\Sigma}:\tilde{\Sigma};c) \le 1$ for the choices of c given by Theorem 3.1. The RR in relation (4.1) is a parabola in c. Theorem 3.1 provides a range of values of c where the proposed estimator dominates the usual estimator. Thus, we have a problem of minimization of parabola in c on restricted sets $\{c:\ d < c < 0\}$ or $\{c:\ 0 < c < d\}$. However, neither of these two sets is closed and consequently we do not have unique solution for c. Note that the unrestricted minimization occurs at

$$c_m = \left(\frac{tr(\frac{\Sigma}{p})}{|\Sigma|^{\frac{1}{p}}}\right)\frac{d}{2},$$

but this is not usable in practice since Σ is unknown.

In Figure 1, we plot the RR against c. For computational sake we assumed that ν/τ^2 in (2.1) has a χ^2_ν distribution so that the p.d.f. in (2.1) defines a joint multivariate t-distribution. We have also assumed that $n = 10,\ \nu = 5, c_0 = 1/(n+1)$ and

$$\Sigma = \begin{pmatrix} 30 & 12 & 7 \\ 12 & 8 & 3 \\ 7 & 3 & 4 \end{pmatrix}.$$

For this special case, it follows from Theorem 3.1 that

$$0 < c < 0.141737 \ (\approx d) \tag{4.2}$$

Note that since Σ is assumed to be known, we can check that $c_m = 0.16288445$ which is near d. On the other hand, when Σ is unknown one may choose a value of c closer to d keeping in mind that there is no way one can compare this with c_m.

Many heavier-tailed elliptical distributions may be generated by varying $h(\tau)$ in (2.1). The estimation strategy developed in this paper is true for the entire class of elliptical distributions defined by (2.1). The above method is applied to the estimation of different characteristics of scale matrix in the following section.

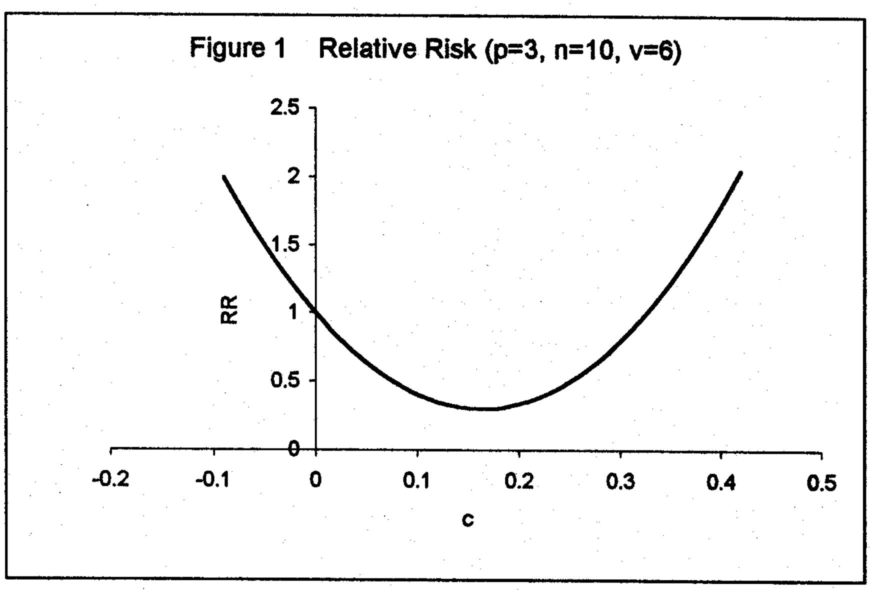

Figure 1: Relative Risk for $p = 3,\ n = 10,\ \nu = 6$.

5 Estimation of Some Characteristics of the Scale Matrix

In the above sections we developed estimation strategies for the scale matrix of the scale matrix of the class of elliptical distributions given by (1.1). Estimation method for some important characteristics of scale matrix are sketched below.

5.1 Estimation of the Trace of Scale Matrix

Consider the estimation of the trace of scale matrix i.e. $\delta = tr\Sigma$ with the quadratic loss function:

$$\mathcal{L}(u(A),\ \delta) = tr[(u(A) - \delta)^2] \tag{5.1}$$

where $u(A)$ is any suitable estimator of δ. In estimating δ by $u(A)$, the risk function is defined as usual by taking expectation over the loss function, i.e.,

$$R(u(A),\ \delta) = E\mathcal{L}[u(A),\ \delta)]. \tag{5.2}$$

Let the usual and the proposed estimators be $\tilde{\delta} = c_o\ trA$ and $\hat{\delta}a = c_0 trA - cp|A|^{-1/p}$ respectively, where c_o is a fixed positive number and c is such that the proposed estimator is positive. The risk difference of the estimators is

$$\begin{aligned} D(\delta, c) &= E[(\hat{\delta} - \delta)^2 - (\tilde{\delta} - \delta)^2] &(5.3)\\ &= E[\hat{\delta}^2 - \tilde{\delta}^2 - 2(\tilde{\delta} - \delta)\delta]. &(5.4)\end{aligned}$$

But

$$\begin{aligned} \hat{\delta}^2 &= \left(\tilde{\delta} - cp|A|^{1/p}\right)^2 \\ &= \tilde{\delta}^2 - 2\tilde{\delta}cp|A|^{1/p} + c^2p^2|A|^{2/p}\end{aligned}$$

so that

$$D(\delta, c) = -2cpE\left(\tilde{\delta}|A|^{2/p}\right) + c^2p^2E\left(|A|^{2/p}\right) + 2cpE\left(\delta|A|^{1/p}\right)$$

which is p times $D(\delta, c)$ given by relation (3.3). Thus, the dominance picture is established. By the use of Lemma 2.3, the risk function of the usual estimator is seen to be

$$\begin{aligned} R(\tilde{\delta}, \delta) &= c_0^2\ E[(trA)^2] - 2c_0\ E[trA]\ \delta + \delta^2 \\ &= [nc_0\ (nc_0\gamma_4 - 2\gamma_2) + 1]\ \delta^2 + 2nc_0^2\ \gamma_4 tr(\Sigma^2). \end{aligned} \tag{5.5}$$

The risk function $R(\hat{\delta}, \delta; c)$ of the proposed estimator is then given by

$$R(\hat{\delta}, \delta; c) = D(\delta, c) + R(\tilde{\delta}, \delta)$$

where $D(\delta, c) = pD(\Sigma, c)$ given by (3.3). The trace of the scale matrix of a multivariate t-model which is a special case of our model in (1.1) is considered in Joarder and Beg (1998).

5.2 Estimation of Inverted Scale Matrix

Next consider the estimation of inverted scale matrix $\Psi = \Sigma^{-1}$ with the quadratic loss function:

$$\mathcal{L}(u(A),\ \Psi) = tr[(u(A) - \Psi)^2] \tag{5.6}$$

where $u(A)$ is any suitable estimator of Ψ. In estimating Ψ by $u(A)$, the risk function is defined as usual by taking expectation over the loss function, i.e.,

$$R(u(A),\ \Sigma) = E\mathcal{L}[u(A),\ \Psi)]. \tag{5.7}$$

Let the usual estimator and the proposed estimators be denoted by $\tilde{\Psi} = c_o A^{-1}$ and

$$\hat{\Psi} = c_0 A^{-1} - c|A|^{-1/p} I$$

respectively, where c_0 is a fixed positive number and c is such that the proposed estimator is positive definite.

Theorem 5.1 Consider the class of elliptical distributions given by (2.1). Then with respect to the loss function given by (5.4) the class of proposed estimators $\hat{\Psi}$ dominates the class of usual estimators $\tilde{\Psi}$, for any c satisfying

$$d < c < 0$$

where $d = 4\left(\frac{c_0}{n-2/p-p-1} - \frac{\gamma_{-2}}{\gamma_{-4}}\right)\frac{\Gamma_p(n/2-1/p)}{\Gamma_p(n/2-2/p)}, \quad n = N-1$

with $c_o < (n-2/p-p-1)\ \frac{\gamma_{-2}}{\gamma_{-4}}$

or, $0 < c < d$,

where d is given by (5.6) with

$c_o > (n-2/p-p-1)\ \frac{\gamma_{-2}}{\gamma_{-4}}$

and γ_i is defined by (3.2).

Proof. Let $D(\Psi,\ c) = R(\hat{\Psi},\ \Psi; c) - R(\tilde{\Psi},\ \Psi)$. Then it is easy to show that

$$\begin{aligned} D(\Psi,\ c) &= E\left[tr(\hat{\Psi}-\Psi)^2 - tr(\tilde{\Psi}-\Psi)^2\right] = Etr\left[\hat{\Psi}^2 - \tilde{\Psi}^2 - 2(\hat{\Psi}-\tilde{\Psi})\Psi\right] \\ &= E\left[-2c_0\ c|A|^{1/p}tr(A^{-1}) + c^2p|A|^{-2/p} + 2c|A|^{-1/p}tr\Psi\right]. \end{aligned}$$

It follows from Lemma 2.1 and a result of Dey (1988, p. 141) that

$$\begin{aligned} D(\Psi,\ c) &= -2c_oc\left[\frac{2^{-1}}{n-2/p-p-1}\ \frac{\Gamma_p(n/2-1/p)}{\Gamma_p(n/2)}\ |\Psi|^{1/p}\ tr\Psi\ \gamma_{-4}\right] \\ &+ c^2p\left[2^{-2}\ \frac{\Gamma_p(n/2-2/p)}{\Gamma_p(n/2)}\ |\Psi|2/p\ \gamma_{-4}\right] \\ &+ 2c\left[2^{-1}\ \frac{\Gamma_p(n/2-1/p)}{\Gamma_p(n/2)}\ |\Psi|^{1/p}\ tr\Psi\ \gamma_{-2}\right]\ tr\Psi \\ &= \frac{p\gamma_{-4}}{4}\ \frac{\Gamma_p(n/2-2/p)}{\Gamma_p(n/2)}\ |\Psi|^{2/p}\ c\ \left(c - \frac{tr\Psi/p}{|\Psi|^{1/p}}\ d\right) \end{aligned}$$

where d is given by (5.6) and γ_i by (3.2). But the risk difference in (5.7) is similar to (3.3) and the rest of the proof is obvious from Theorem 3.1.

6 Conclusion

The present paper is based on a paper by Joarder and Ahmed (1996) who did not derive expressions for the risk functions of estimators. This paper finds expressions for the exact risk functions of the estimators of scale matrix for a broader class of elliptical distributions. Moreover the robustness of the estimation of some characteristics of scale matrix has been discussed.

Acknowledgements

Thanks are due to the editor and a team of referees for their valuable comments, which led to improvements in the paper. The first author acknowledges the excellent research facilities provided by King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia.

References

Anderson, T.W. (1958). *An Introduction to Multivariate Statistical Analysis.* Wiley, New York.

Anderson, T.W.; Fang, K.T. and Hsu, H. (1986). Maximum likelihood estimates and likelihood-ratio criteria for multivariate elliptically contoured distributions. *Canad. J. Statist.*, **14**, 55–59.

Anderson, T.W. and Fang, K.T. (1990). Inference in multivariate elliptically contoured distribution based on maximum likelihood. *Statistical Inference in Elliptically and Related Distributions* (K.T. Fang and T.W. Anderson, eds), 201–216, Allerton Press, New York.

Dey, D.K. (1988). Simultaneous estimation of eigenvalues. *Ann. Inst. Statist. Math.*, **40**, 137–147.

Joarder, A.H. (1995). Estimation of the scale matrix of a multivariate t-model. *J. Stat. Res.*, **29**, 55–66.

Joarder, A.H. (1998). Some useful Wishart expectations based on the multivariate t-model. To appear in *Statist. Papers.*

Joarder, A.H. and Ahmed, S.E. (1996). Estimation of the characteristics roots of the scale matrix. *Metrika*, **44**, 259–267.

Joarder, A.H. and Ali, M.M. (1992). On some generalized Wishart expectations. *Commun. Statist. - Theor. Meth.*, **21**, 283–294.

Joarder, A.H. and Beg, G.K. (1998). Estimation of the trace of the scale matrix of a multivariate t-model under a squared error loss function. To appear in *Statistica.*

Muirhead, R. J. (1986). A note on some Wishart expectations. *Metrika*, **33**, 247–251.

Olkin, I. and Selliah, J.B. (1977). Estimating covariance in a multivariate normal distribution. *Statistical Decision Theory and Related Topics II,* (eds. S.S. Gupta and D. Moore), 313–326, Academic Press, New York.

Srivastiva, M.S. and Khatri, C.G. (1979). *An Introduction to Multivariate Statistics.* North-Holland, New York.

Sutradhar, B.C. and Ali, M.M. (1989). A generalization of the Wishart distribution for the elliptical model and its moments for the multivariate t model. *J. Mult. Anal.,* **29**, 155–162.

Zellner, A. (1976). Bayesian and non-Bayesian analysis of the regression model with multivariate Student-t error term. *J. Amer. Statist. Assoc.,* **71**, 400–405 (correction, **71**, 1000).