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by

P.M. Higgins and A. Umar

0 Introduction The study of various (finite) semigroups of transformations makes a significant contribution to semigroup theory, just as the study of the finite symmetric and alternating groups forms an important part of group theory. Arguably, one of the most successful class of semigroups of transformations is that of the order-decreasing (also called extensive in the literature, [8]) transformations, and since their first appearance in [8] several articles containing interesting results have been published, see for example [**10** & **11**]. These semigroups are not regular, however, they are abundant (in the sense of Fountain [2]), $\mathcal{R}$−trival semigroups. Thus, from the point of view of abundant semigroup theory one would like to have a more general representative class of abundant semigroups of mappings, that is, one which is not $\mathcal{R}$− trival. It is this search that gives rise to the class of semigroups in the title. For standard concepts in semigroup theory, see for example [4] or [6].

1 Preliminaries For a given (partial) mapping or transformation α : $Y \subseteq X \longrightarrow X$ we denote its set of fixed points by $F(\alpha) = \{x \in Y : x\alpha = x\}$, its domain Y by $Dom\alpha$ and its image set by $Im\alpha$. If $Dom\alpha = X$ then α is called a *full* or *total* mapping, otherwise it is *strictly partial*. Let $T(X)$ and $P(X)$ be the full transformation and partial transformation semigroups on a set X, respectively. An *idempotent* mapping ϵ in $P(X)$ is characterized by the property that every element in its image set is a fixed point, that is, ϵ is idempotent if and only if $F(\epsilon) = Im\epsilon$. Now for an arbitrary subset, of X say

V we define a weak V-stabilizer map as $\alpha \in T(X)$ such that $V\alpha \subseteq V$. (We drop the adjective "weak" if $V\alpha = V$.) The set of all weak V-stablizer maps (for some $V \subseteq X$) will be denoted by $W(V)$. Then $W(V)$ is a semigroup since for all α, β in $W(V)$, we have $V\alpha\beta = (V\alpha)\beta \subseteq V\beta \subseteq V$, showing that the product $\alpha\beta$ is in $W(V)$ also. The following notation from [7] is quite helpful: let

$$\alpha = \begin{pmatrix} A_i \\ a_i \end{pmatrix} \ (\, i \in I \,)$$

where $Im\alpha = \{a_i : i \in I\}$, $A_i\alpha = a_i$ (or $A_i = a_i\alpha^{-1}$) and $\bigcup_{i\in I} A_i = X$. The $A_i's$ are called the blocks of α.

We now state some facts about weak V-stabilizer mappings that will be useful in later sections.

Lemma 1.1. *Let* $\alpha \in W(V)$. Then $a_i\alpha^{-1} \cap V \neq \emptyset$ if and only if $a_i \in V$. *Equivalently,* $A_i\alpha^{-1} \cap V \neq \emptyset$ if and only if $a_i \in V$.

Lemma 1.2. *Let* $\alpha \in W(V)$. *Then for every* v *in* $V\alpha$ *there exists* v' *in* V *such that* $v'\alpha = v$.

2 Abundant Semigroups: Green's Equivalances First we characterize the Green's equivalences on $W(V)$.

Lemma 2.1. *Let* $\alpha, \beta \in W(V)$. *Then*

(1) $(\alpha, \beta) \in \mathcal{R}$ *if and only if* $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$ *and* $V\alpha^{-1} = V\beta^{-1}$;
(2) $(\alpha, \beta) \in \mathcal{L}$ *if and only if* $Im\alpha = Im\beta$ *and* $V\alpha^{-1} = V\beta^{-1}$;
(3) $(\alpha, \beta) \in \mathcal{H}$ *if and only if* $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}, Im\alpha = Im\beta$, $V\alpha^{-1} = V\beta^{-1}$ *and* $V\alpha = V\beta$.

Proof. (1). Let $(\alpha, \beta) \in \mathcal{R}$, then $(\alpha, \beta) \in \mathcal{R}_{T(X)}$ so that $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$. Moreover, there exist $\delta, \gamma \in W(V)$ such that $\alpha = \beta\delta$ and $\beta = \alpha\gamma$. Now let $x \in V\alpha^{-1}$ then $x\alpha \in V$ which in turn implies that $x\beta = x\alpha\gamma \in$

$V\gamma \subseteq V$ and so $x \in V\beta^{-1}$. Thus $V\alpha^{-1} \subseteq V\beta^{-1}$. Similarly, we can show that $V\beta^{-1} \subseteq V\alpha^{-1}$. Hence $V\alpha^{-1} = V\beta^{-1}$, as required.

Conversely, let $\alpha = \begin{pmatrix} A_i \\ a_i \end{pmatrix}$, $\beta = \begin{pmatrix} A_i \\ b_i \end{pmatrix}$ $(i \in I)$ and suppose also that $V\alpha^{-1} = V\beta^{-1}$. Notice that if $a_i \in V$ then it follows that $b_i\beta^{-1} = a_i\alpha^{-1} \subseteq V\alpha^{-1} = V\beta^{-1}$ and so $b_i \in V$. Similarly, we can show that if $b_i \in V$ then $a_i \in V$. Thus we deduce that $a_i \in V$ if and only if $b_i \in V$. Now define δ, γ in $W(V)$, respectively by

$$a_i\delta = b_i, \qquad x\delta = x(x \neq a_i);$$
$$b_i\gamma = a_i, \qquad x\gamma = x(x \neq b_i).$$

Then it is clear that $\delta, \gamma \in W(V)$ and $\alpha\delta = \beta$, $\beta\gamma = \alpha$ and so $(\alpha, \beta) \in \mathcal{R}$, as required.

(2.) Let $(\alpha, \beta) \in \mathcal{L}$, then $(\alpha, \beta) \in \mathcal{L}_{T(X)}$ so that $Im\alpha = Im\beta$. Moreover, there exist δ, γ in $W(V)$ such that $\alpha = \delta\beta$ and $\beta = \gamma\alpha$. Thus

$$V\alpha = V\delta\beta \subseteq V\beta \text{ and } V\beta = V\delta\alpha \subseteq V\alpha.$$

Hence $V\alpha = V\beta$, as required.

Conversely, let $\alpha = \begin{pmatrix} A_i \\ c_i \end{pmatrix}$, $\beta = \begin{pmatrix} B_i \\ c_i \end{pmatrix}$ $(i \in I)$ and suppose also that $V\alpha = V\beta$. Now define δ, γ in $W(V)$, respectively by

$$A_i\delta = a_i \in c_i\beta^{-1}, \;\; B_i\gamma = b_i \in c_i\alpha^{-1} \;\; (i \in I)$$

where a_i, b_i are chosen from $c_i\beta^{-1} \cap V$, $c_i\alpha^{-1} \cap V$, respectively if $c_i \in V$, which ensures that $\delta, \gamma \in W(V)$, by Lemma 1.1. Moreover,

$$A_i\delta\beta = a_i\beta = c_i = A_i\alpha, \;\; B_i\gamma\alpha = b_i\alpha = c_i = B_i\beta \;\; (i \in I)$$

showing that $(\alpha, \beta) \in \mathcal{L}$.

(3.) The result follows directly from (1) and (2) above.

Theorem 2.2. *Let $W(V)$ be the semigroup of all weak V-stabilizer mappings and let $\alpha, \beta \in W(V)$. Then*

(1) $(\alpha, \beta) \in \mathcal{D}$ *if and only if*

$$|Im\alpha| = |Im\beta|, |V\alpha| = |V\beta| \text{ and } |Im\alpha \backslash V| = |Im\beta \backslash V|;$$

(2) $\mathcal{D} = \mathcal{J}$.

Proof. (1). Let $(\alpha, \beta) \in \mathcal{D}$, then $(\alpha, \beta) \in \mathcal{D}_{T(X)}$ so that $|Im\alpha| = |Im\beta|$. Moreover, there exist δ in $W(V)$ such that $\alpha\mathcal{R}\delta\mathcal{L}\beta$. Thus, by Lemma 2.1, we have $V\alpha^{-1} = V\delta^{-1}$, $Im\delta = Im\beta$ and $V\delta = V\beta$. Also, there exist $\eta_1, \eta_2, \eta_3, \eta_4$ in $W(V)$ such that

$$\alpha = \delta\eta_1, \ \ \delta = \alpha\eta_2, \ \ \delta = \eta_3\beta, \ \beta = \eta_4\delta.$$

Therefore,

$$|V\alpha| = |V\delta\eta_1| \leq |V\delta| = |V\beta| \text{ and } |V\beta| = |V\eta_4\delta| \leq |V\delta| = |V\alpha\eta_2| \leq |V\alpha|$$

and so $|V\alpha| = |V\beta|$. Similarly, we have

$$\begin{aligned} |Im\alpha \backslash V| = |Im\delta\eta_1 \backslash V| &\leq |Im\delta\eta_1 \backslash V\eta_1| \\ &\leq |(Im\delta \backslash V)\eta_1| \\ &\leq |Im\delta \backslash V| \\ &= |Im\beta \backslash V|, \end{aligned}$$

$$\begin{aligned} |Im\beta \backslash V| = |Im\eta_4\delta \backslash V| \leq |Im\delta \backslash V| &= |Im\alpha\eta_2 \backslash V| \\ &\leq |Im\alpha\eta_2 \backslash V\eta_2| \\ &\leq |(Im\alpha \backslash V)\eta_2| \\ &\leq |Im\alpha \backslash V|. \end{aligned}$$

Thus, $|Im\alpha\backslash V| = |Im\beta\backslash V|$.

Conversely, suppose that $|Im\alpha| = |Im\beta|$, $|V\alpha| = |V\beta|$ and $|Im\alpha\backslash V| = |Im\beta\backslash V|$. Let θ be a bijection from $Im\alpha$ onto $Im\beta$ such that $(V\alpha)\theta = V\beta$ and $(Im\alpha\backslash V)\theta = Im\beta\backslash V$. Notice that $V\theta \subseteq V$ and $V\theta^{-1} \subseteq V$. Now define δ in $W(V)$ by

$$x\delta = (x\alpha)\theta \ \ (x \in X).$$

Then clearly $V\delta = (V\alpha)\theta = V\beta$, $Im\delta = (Im\alpha)\theta = Im\beta$ and so $(\delta, \beta) \in \mathcal{L}$, by Lemma 2.1. It is also clear that

$$\begin{array}{ccc} x\alpha = y\alpha & \text{if and only if} & (x\alpha)\theta = (y\alpha)\theta \\ \text{i.e.,} & \text{if and only if} & x\delta = y\delta \end{array}$$

and so $\delta \circ \delta^{-1} = \alpha \circ \alpha^{-1}$. Moreover, let $x \in V\delta^{-1}$ then

$$x \in V\theta^{-1}\alpha^{-1} \subseteq V\alpha^{-1}$$

and so $V\delta^{-1} \subseteq V\alpha^{-1}$. Similarly, we can show that $V\alpha^{-1} \subseteq V\delta^{-1}$. Thus $V\alpha^{-1} = V\delta^{-1}$ and so $(\alpha, \delta) \in \mathcal{R}$. Hence $(\alpha, \beta) \in \mathcal{D}$.

(2.) Let $(\alpha, \beta) \in \mathcal{J}$ then there exist $\delta_1, \delta_2, \xi_1, \xi_2$ in $W(V)$ such that

$$\alpha = \delta_1\beta\delta_2, \ \ \beta = \xi_1\alpha\xi_2.$$

It now follows that $|Im\alpha| = |Im\beta|$ and

$$|V\alpha| = |V\delta_1\beta\delta_2| \leq |V\beta|, \ |V\beta| = |V\xi_1\alpha\xi_2| \leq |V\alpha|.$$

So, $|V\alpha| = |V\beta|$. Moreover,

$$\begin{aligned} |Im\alpha\backslash V| &= |Im\delta_1\beta\delta_2\backslash V| \leq |Im\beta\delta_2\backslash V| \leq |Im\beta\delta_2\backslash V\delta_2| = |Im\beta\backslash V|, \\ |Im\beta\backslash V| &= |Im\xi_1\alpha\xi_2\backslash V| \leq |Im\alpha\xi_2\backslash V| \leq |Im\alpha\xi_2\backslash V\xi_2| = |Im\alpha\backslash V|. \end{aligned}$$

Thus $|Im\alpha\backslash V| = |Im\beta\backslash V|$, and the proof is now complete.

Recall from [11] that a subset U of a semigroup S is called a left (right) inverse ideal of S if for all u in U there exist u' in S such that $u = uu'u$ and $u'u \in U$ $(uu' \in U)$, and a left and right inverse ideal is called an inverse ideal. Then we have

Lemma 2.3. *$W(V)$ is an inverse ideal of $T(X)$.*

Proof. For a given $\alpha \in W(V)$, define α' in $T(X)$ by

$$x\alpha' = \begin{cases} a_x \in x\alpha^{-1} \cap V & \text{if} \quad x \in Im\alpha \cap V \\ b_x \in x\alpha^{-1} & \text{if} \quad x \in Im\alpha \backslash V \\ x & \text{otherwise.} \end{cases}$$

Then

$$x\alpha\alpha'\alpha = a_{x\alpha}\alpha = x\alpha \quad (\text{ if } x\alpha \in V),$$

$$x\alpha\alpha'\alpha = b_{x\alpha}\alpha = x\alpha \quad (\text{ if } x\alpha \notin V).$$

Notice also that $(V\alpha)\alpha' \subseteq V\alpha \subseteq V$ and $(V\alpha')\alpha \subseteq V\alpha \subseteq V$. Thus $\alpha\alpha', \alpha'\alpha \in E(W)$ and so $W(V)$ is an inverse ideal as required.

Hence by **[11, Lemma 3.1.9]** and **[1, Lemmas 10.55 and 10.56]**, we deduce the following result:

Lemma 2.4 *$W(V)$ is a (non-regular) abundant semigroup, and for α , $\beta \in W(V)$ we have*

(1) $(\alpha, \beta) \in \mathcal{R}^*$ *if and only if* $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$;
(2) $(\alpha, \beta) \in \mathcal{L}^*$ *if and only if* $Im\alpha = Im\beta$;
(3) $(\alpha, \beta) \in \mathcal{H}^*$ *if and only if* $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$ *and* $Im\alpha = Im\beta$.

As in [10] to characterize the relation $\mathcal{D}^*$ we consider the relation $\mathcal{K}$ on $W(V)$ defined by the rule that

$$(\alpha, \beta) \in \mathcal{K} \text{ if and only if } |Im\alpha| = |Im\beta|.$$

Then obviously, $\mathcal{L}^*, \mathcal{R}^*$ and $\mathcal{D}^* \subseteq \mathcal{K}$. We now have

Lemma 2.5. *On the semigroup $W(V)$, we have $\mathcal{D}^* = \mathcal{R}^* o \mathcal{L}^* o \mathcal{R}^*$.*

Proof. Let $\alpha = \begin{pmatrix} A_i \\ a_i \end{pmatrix}$, $\beta = \begin{pmatrix} B_i \\ b_i \end{pmatrix}$ $(i \in I)$ so that $|Im\alpha| = |Im\beta|$. Recall from Lemma 1.1 that $A_i \cap V \neq \emptyset \neq B_i \cap V$ implies $a_i, b_i \in V$. Now define δ, γ in $W(V)$, respectively, by

$$A_i\delta = \begin{cases} a_i & \text{if} \quad A_i \cap V \neq \emptyset \\ b_i & \text{if} \quad B_i \cap V \neq \emptyset \\ a_i & \text{otherwise} \end{cases} \qquad B_i\gamma = \begin{cases} a_i & \text{if} \quad A_i \cap V \neq \emptyset \\ b_i & \text{if} \quad B_i \cap V \neq \emptyset \\ a_i & \text{otherwise.} \end{cases}$$

Now clearly, $\delta, \gamma \in W(V)$ and $\alpha \mathcal{R}^* \delta \mathcal{L}^* \gamma \mathcal{R}^* \beta$, by Lemma 2.4 and so $\mathcal{K} \subseteq \mathcal{R} o \mathcal{L}^* o \mathcal{R}^*$. However, it is clear that $\mathcal{R}^* o \mathcal{L}^* o \mathcal{R}^* \subseteq \mathcal{K}$. Thus $\mathcal{K} = \mathcal{R}^* o \mathcal{L}^* o \mathcal{R}^*$. Now from the inequalities

$$\mathcal{D}^* \subseteq \mathcal{K} = \mathcal{R}^* o \ \mathcal{L}^* o \mathcal{R}^* \subseteq \mathcal{D}^*$$

we deduce the result of the lemma.

Corollary 2.6. *Let $\alpha, \beta \in W(V)$. Then $(\alpha, \beta) \in \mathcal{D}^*$ if and only if $|Im\alpha| = |Im\beta|$.*

Using the same techniques as in [**10, Section 2**] we can prove the next result.

Theorem 2.7. *On the semigroup $W(V)$, $\mathcal{D}^* = \mathcal{J}^*$.*

3 The Finite Case

Let $X_n = \{1, 2, \ldots, n\}$, let $V_k = \{a_1, a_2, \ldots, a_k\} \subseteq X_n$ for some $k \leq n$ and let

$$W_n(V_k) = \{\alpha \in T(X_n) : V_k\alpha \subseteq V_k\} \tag{3.1}$$

be the semigroup of all weak A_k-stabilizer finite mappings of X_n. (In the finite case we shall denote T(X) and P(X) by T_n and P_n, respectively.) Then we have

Theorem 3.1. *Let $W_n(V_k)$ be as defined in* (3.1). *Then*

$$|W_n(V_k)| = k^k n^{n-k}$$

Proof. The condition $V_k\alpha \subseteq V_k$ implies that there are k^k possible ways of mapping the elements of V_k. For the remaining $n-k$ elements of $X_n \backslash V_k$ there is no restriction and so each has n degrees of freedom, so that there are n^{n-k} possible ways of mapping these elements. Hence the result follows.

Theorem 3.2. *Let $W_n(V_k)$ be as defined in* (3.1) *and $E(W_n(V_k))$ be its set of idempotents. Then*

$$|E(W_n(V_k))| = \sum_{t+s=1}^{n} \binom{k}{t} \binom{n-k}{s} t^{k-t} (t+s)^{n-k-s}$$

Proof. Let $F(\epsilon) = Im\,\epsilon = \{a_1\, a_2 \ldots, a_t, b_1, b_2, \ldots, b_s\}, a_i \in V_k, b_j \in X_n \backslash V_k$ and $r = t + s$. We can choose $\{a_2, a_2, \ldots, a_t\}$ from V_k in $\binom{k}{t}$ ways, and $\{b_1, b_2, \ldots, b_s\}$ from $X_n \backslash V_k$ in $\binom{n-k}{s}$ ways. Moreover, the remaining $k - t$ elements of $V_k \backslash \{a_1, a_2, \ldots, a_t\}$ can be mapped into $\{a_1, a_2, \ldots, a_t\}$ in t^{k-t} ways, while the remaining $n-k-s$ elements of $X_n \backslash [V_k \cup \{b_1, b_2, \ldots, b_s\}]$ are unrestricted and so can be mapped into $Im\,\epsilon$ in $(t+s)^{n-k-s}$ ways. Now taking the sum over $t + s$ from 1 to n yields the required result.

Remark. Notice that if $k = n$, then $s = 0$ and $r = t$. Thus the formulae in Theorems 3.1 and 3.2 reduced to those for T_n in [9].

The *core* of a semigroup S is defined as the subsemigroup of S generated by all the idempotents of S. To characterize the core of $W_n(V_k)$ we introduce the following two conditions:

(C_1) $V_k\alpha = V_k \Rightarrow v\alpha = v$ for all $v \in V_k$;
(C_2) $T\alpha = T\,(= X_n \backslash V_k) \Rightarrow t\alpha = t$ for all $t \in T$.

Then we have

Theorem 3.3. *Let $W_n(V_k)$ be as defined in* (3.1.) *Then α in $W_n(V_k)$ is expressible as a product of idempotents in $W_n(V_k)$ if and only if α satisfies* (C_1) *and* (C_2).

For the direct half it is enough to prove the following lemma:

Lemma 3.4. *Let $\alpha \in E^m (m > 0)$. Then α satisfies (C_1) and (C_2).*

Proof. First we prove the result for the case $m = 1$, i.e., every idempotent in $W_n(V_k)$ satisfies (C_1) and (C_2). Let $\epsilon^2 = \epsilon$ be an arbitrary idempotent in $W_n(V_k)$ and suppose that $V_k\epsilon = V_k$, then for every v in $V_k\epsilon$ there exists v' in V_k such that $v'\epsilon = v$, by Lemma 1.2. So $v = v'\epsilon = v'\epsilon^2 = (v'\epsilon)\epsilon = v\epsilon$. Thus ϵ satisfies (C_1). Similarly, if $T\epsilon = T = X_n \backslash V_k$ we can show that $t\epsilon = t$, for all $t \in T$, i.e., ϵ satisfies (C_2).

Next let $\alpha = \epsilon_1\epsilon_2 \dots \epsilon_m (m > 1)$, where ϵ_i are idempotents in $W_n(V_k)$. Then if $V_k\alpha = V_k$ and $T\alpha = T$, it follows that

$$|V_k| = |V_k\alpha| \leq |V_k\epsilon_i| \leq |V_k| \text{ and } |T| = |T\alpha| \leq |T\epsilon_i| \leq |T|$$

for all $i \in \{1, 2, \dots, m\}$. Therefore $|V_k| = |V_k\epsilon_i|$ and $|T\epsilon_i| = |T|$, which implies that $V_k\epsilon_i = V_k$ and $T\epsilon_i = T$. The former is obvious, while for the latter suppose that $T\epsilon_i \neq T$. Then either $\epsilon_i|T$ is not 1-1 and so $\alpha|T$ is not 1-1, which is a contradition; or there exists $v \in T' = V_k$ such $t\epsilon_i = v$ for some $t \in T$. Now if $t = t_0\epsilon_1\epsilon_2 \dots \epsilon_{i-1} (t_0 \in T)$, it follows that $t_0\alpha = (t_0\epsilon_1, \epsilon_2 \dots \epsilon_{i-1})\epsilon_i \dots \epsilon_m = (t\epsilon_i)\epsilon_{i+1} \dots \epsilon_m = v\epsilon_{i+1} \dots \epsilon_m \in V_k = T'$ and so $T\alpha \neq T$, which is a contradiction. Hence we deduce, that

$$v\epsilon_i = v \ \ (v \in V_k) \ \text{ and } \ t\epsilon_i = t \ \ (t \in T),$$

and so $v\alpha = v$ (for all $v \in V_k$) and $t\alpha = t$ (for all $t \in T$) proving that $\alpha = \epsilon_1\epsilon_2 \dots \epsilon_m (m > 1)$ satisfies (C_1) and (C_2).

To prove the converse of Theorem 3.3 we utilize [**5, Theorem I**] and [**3, Lemma 2.7**]. But first we define a mapping $\bar{\ } : W_n(V_k) \to PT_{n-k}$ (where for convenience we take the base set for PT_{n-k} to be $T \cup \{0\} = \{0, k+1, k+2, \ldots, n\}$, regarding PT_{n-k} as embedded in T_{n-k+1} in the Vagner [**12**] fashion) as follows:

$$i\overline{\alpha} = \begin{cases} i\alpha & \text{if} \quad i\alpha \in T \\ 0 & \text{if} \quad i\alpha \notin T \end{cases}$$

Then it is routine to verify that is a surjective homomorphism.

Now let α (in $W_n(V_k)$) satisfy (C_1) and (C_2). Then $\alpha' = \alpha|V_k \in \langle E(T_k)\rangle$, by (C_1). So $\alpha' = \epsilon_1'\epsilon_2' \ldots \epsilon_m'$ for some $\epsilon_i' \in E(T_k)$ by [**5, Theorem I**]. Let ϵ_i in T_n be such that $\epsilon_i|V_k = \epsilon_i'$ and $\epsilon_i|T = 1|T$. Then $\epsilon_i \in E(W_n)$.

Next, write

$$J = \{t \in T : t\alpha \in V_k\} = \{t_1, t_2, \ldots, t_p\} \ \text{(for some } p > 0)$$

and for each t in J define β_t in $E(W_n)$ with $S(\beta_t) = \{t\}$ and $t\beta_t = t\alpha$.

Finally, since α satisfies $(C_2), \overline{\alpha}$ can be written as a product of idempotents $\overline{\alpha} = \overline{\gamma}_1\overline{\gamma}_2 \ldots \overline{\gamma}_m$ for some $\overline{\gamma}_i$ in $E(PT_{n-k})$, by [**3, Lemma 2.7**]. Moreover, by finiteness we take γ_i in $E(W_n)$ where $\gamma_i|T = \overline{\gamma_i}$ and $\gamma_i|V_k = 1|V_k$. Thus,

$$\alpha = \epsilon_1\epsilon_2 \ldots \epsilon_t \cdot \beta_{t_1}\beta_{t_2} \ldots \beta_{t_p} \cdot \gamma_1\gamma_2 \ldots \gamma_m,$$

showing that $\alpha \in E(W_n)$.

We complete this article by finding the order of the core of $W_n(V_k)$.

Theorem 3.5. *Let $W_n(V_k)$ be as defined in* (3.1) *and let $\langle E(W_n)\rangle$ be its core. Then*

$$|\langle E(W_n)\rangle| = k^k n^{n-k} - (k! - 1)n^{n-k} - [(n-k)! - 1]k^k + (k! - 1)[(n-k)! - 1]$$

Proof. For $i = 1, 2$ let

$$Y_i = \{\alpha \in W_n(V_k) : \alpha \text{ satisfies } (C_i)\}$$
$$\overline{Y}_i = \{\alpha \in W_n(V_k) : \alpha \text{ violates } (C_i)\}.$$

Then

$$|Y_1 \cap Y_2| = |W_n(V_k)| - |\overline{Y}_1| - |\overline{Y}_2| + |\overline{Y}_1 \cap \overline{Y}_2|.$$

Computing the orders of $\overline{Y}_1, \overline{Y}_2$ and $\overline{Y}_1 \cap \overline{Y}_2$ yields the required result.

Remark. Notice that if $n = k$, the formula in Theorem 3.5 reduces to $n^n - n! + 1$, which is the order of $\langle E(T_n) \rangle$.

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