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Explicit limit distributions for immigration-branching diffusions

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Abstract

The discrete time Markov branching diffusion in which offspring move at random in a bounded space X and allowing immigration of particles from an external source is considered. Asymptotic behavior of the non-extinction probability and the expected size of the population are studied when first moment functional of the immigration process decreases regularly. The obtained approximation is used to describe a class of limit distributions for the vector of population sizes in disjoint subsets of the space X.

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1 Introduction

We consider a population evolving as a branching process where the offspring move at random on a bounded space X. Typical models that have been studied by many authors are branching random walks and branching diffusions. In branching random walks the offspring jump when they are

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born, and retain their position until they die or reproduce. In branching diffusions, which we are going to deal with, the offspring move throughout their lives according to diffusions, independently of each other and of the family tree, each offspring starting from the location of its birth. Each particle at the end of its life independently of others generates a population whose members may be located at any point of X. The lifetime of a particle is defined as the hitting time of either of two barriers ∂ and Δ. If the particle hits ∂, it is instantaneously replaced by the empty population, that is disappers. If the particle hits Δ, then it is replaced by a population of new particles distributed in X according to a probability law.

In addition a random number of new particles produced by an external source may immigrate into the population and initial positions of these new particles may also be arbitrary points of X. The state of the process at any given time is characterized by the total number n of particles at present and their positions $x_1, x_2, ..., x_n \in X$. The development of the process in the state space $\hat{X}$ of all finite populations of particles located in X is Markovian. In other words, the motion of a particle up to the time of reproduction is a Markov process on X.

Let (X, U) be a measurable space, $X^{(n)}$ be the symmetrization of the direct product of n copies of X (see Moyal(1961)). We put $X^{(0)} = \theta$, where θ is some extra point,

$$\hat{X} = \bigcup_{n=0}^{\infty} X^{(n)}$$

and denote $\hat{U}$ the σ-algebra on $\hat{X}$ induced by U. An element $x \in X$ we understand as position, energy or type of a particle diffusing in the bounded space. Then, it is clear that $\hat{X}$ is the space of all possible populations and θ denotes an empty population.

We denote $\boldsymbol{\beta}$ as the Banach algebra of all bounded, complex-valued U-measurable functions ξ on X with supremum norm

$$\| \xi \| = \sup_{x \in X} | \xi(x) |$$

and put $\boldsymbol{\beta}_+ = \{\xi \in \boldsymbol{\beta}: \xi > 0\}$. Let ∇ be the open unit ball in $\boldsymbol{\beta}$ and $\bar{\nabla}$ its closure.

For $\hat{x} \in \hat{X}$ and $\xi \in \bar{\nabla}$ we define

$$\hat{x}[\xi] = \begin{cases} 0 & \text{if } \hat{x} = \theta \\ \sum_{i=1}^{n} \xi(x_i) & \text{if } \hat{x} = < x_1, ..., x_n > \end{cases} \tag{1}$$

For example, if $\xi(x) = 1_A(x), A \subset X$, then $\hat{x}_t[1_A]$ is the number of particles in A at time t. Introduce the union operation for populations $\hat{x}, \hat{y} \in \hat{X}$ as following:

$$\hat{x} + \hat{y} = \begin{cases} \theta & \text{if } \hat{x} = \hat{y} = \theta \\ < x_1, ..., x_n, y_1, ..., y_n > & \text{otherwise,} \end{cases} \tag{2}$$

that is, if $\hat{x} =< x_1, ..., x_n >, \hat{y} =< y_1, ..., y_n >$. Then the Markov branching diffusion $\{\hat{x}_t, P^{\{}\hat{x}\}$ is defined by the relation

$$\hat{x}_{t+s} = \sum_{i=1}^{\hat{x}_t[1]} \hat{x}_{t+s}^{t,i}, t, s > 0 \tag{3}$$

with $\hat{x}_{t+s}^{t,i}, i = 1, 2, ..., \hat{x}_t[1]$ are conditionally independent, given $\mathcal{F}_t := \sigma(\hat{x}_u, u \leq t)$ and

$$P^{\hat{x}}(\hat{x}_{t+s}^{t,j} \in \hat{A} | \mathcal{F}_t) = P^{<x_j>}(\hat{x}_s \in \hat{A})$$

a.s.$P^{\hat{x}}$, $\hat{x}_t =< x_1, ..., x_{\hat{x}_t[1]} >$. Here and later on $P^{\hat{x}}$ and $E^{\hat{x}}$ denote the probability and expectation when the initial population is $\hat{x}$ and $\mathcal{F}_t$ is the σ-algebra generated by the branching diffusion ut to time t.

Let $t \in N_0 = \{0, 1, ...\}$ and $\hat{y}_t =< y_{t1}, ..., y_{tl} >$ be the population of immigrating at time t particles, where $y_{tj} \in X$ is the position (or energy) of j-th particle from $\hat{y}_t$. If we denote $\hat{x}_t^{<y_{ki}>}$ the branching diffusion initiated by particle y_{ki}, then the discrete time immigration-branching diffusion can be given by the following relation:

$$\hat{Z}_t = \sum_{k=0}^{t} \sum_{i=1}^{\hat{y}_k[1]} \hat{x}_{t-k}^{<y_{ki}>}. \tag{4}$$

In the literature many papers have been published on branching or immigration branching diffusions. Mathematical foundations of a general theory of stochastic population processes were given by Moyal (1962). Existence and probabilistic constructions of branching diffusions are due to Ikeda, Nagasawa and Watanabe (1965, 1966) (see also Conner(1961, 1967), Savits (1969), Hering(1973, 1978), Asmussen and Hering(1977), Kageyama and Ogura (1980)). Convergence problems for systems of critical branching Markov chains are considered by Cox (1994). It should be noted the book by Asmussen and Hering(1983) as a convenient source on branching diffusions.

In that book, in particular, limit theorems for the continuous and discrete time immigration-branching diffusions with non-stationary immigration have been proven. Note that the limit distributions obtained there have no explicit form.

As it is known from the theory of branching processes the behavior of the process with immigration strictly depends on the expected number of immigrating particles (see Rahimov (1995), Ch III, for example). Therefore it seems preferable to study separately processes in which the first moment functional of the immigration process decreases, increases or is periodical and assuming, in addition, that it is a regularly varying function of the time. It allows to obtain results describing properties of the process more precisely. In particular it is possible to find explicit limit distributions for the vector of population sizes in disjoint subsets A_i of the space X for arbitrary measurable decomposition $\{A_i, i = 1, 2, ...j\}$ of X. In this paper we study the asumptotic behaviour of the probability of "non-extinction" and of the first moment functional of the branching diffusion with decreasing immigration. In the last section of the article we will prove explicit limit theorems for the process. The proofs use a mixture of analitic and probabilistic techniques. Namely they will be carried out in the following scheme. First using analitic methods we prove limit theorems for some "partial processes" counting only descendants of particles immigrated in the beginning of the process or immigrated "recently". Then from these theorems by direct probabilistic arguments obtain limit distributions for the basic process. Since properties of the process and the results in the cases of discrete and continuous time are similar, we restrict ourselves by considering only discrete time processes.

2 Basic assumptions

Let for $\hat{x} \in \hat{X}, \hat{A} \in \hat{U}, s, t \in N_0$

$$P_t(\hat{x}, \hat{A}) = P\{\hat{x}_{t+s} \in \hat{A} | \hat{x}_s = \hat{x}\}$$

be the branching transition function of $\hat{x}_t$. For $\eta \in \bar{\nabla}$ and $\hat{x} \in \hat{X}$ we define

$$\tilde{\eta}(\hat{x}) = \begin{cases} 1, & \hat{x} = \theta \\ \prod_{i=1}^{n} \eta(x_i), & \hat{x} =< x_1, ..., x_n > \end{cases} \quad (5)$$

Then the generating functional

$$F_t(\hat{x}, \eta) = \int_{\hat{X}} \tilde{\eta}(\hat{y}) P_t(\hat{x}, d\hat{y})$$

of $P_t(\hat{x}, .)$ is well-defined on $\bar{\nabla}$.

It is known (Asmussen, Hering(1983), p.139) that the mapping $F_t : \bar{\nabla} \mapsto \bar{\nabla}$ defined by

$$F_t[\eta](x) := F_t(< x >, \eta), \eta \in \bar{\nabla}, x \in X$$

satisfies the semigroup relation

$$F_{t+s}[\eta] = F_s[F_t[\eta]], t, s \geq 0, \eta \in \bar{\nabla}$$

and it is called a generating semigroup.

Assume that there exist a linear-bounded functional of ξ on $\boldsymbol{\beta}$ defined by the relation

$$M_t(\hat{x}, \xi) = \int_{\hat{X}} \hat{y}[\xi] P_t(\hat{x}, d\hat{y}).$$

It is also known, Asmussen, Hering (1983), that for any $s, t \in N_0$

$$M_{t+s} = M_t M_s$$

that is $\{M_t\}$ is a semigroup. It is called the moment semigroup of the process.

Now we introduce so called (M) and (R) assumptions which are important for our further considerations.

Assumption (M). The moment semigroup $\{M_t\}$ can be represented as

$$M_t = \rho^t P + \Delta_t, t \geq 0, \tag{6}$$

where $\rho \in (0, \infty), P\xi = \Phi^\star[\xi]\varphi, \xi \in \boldsymbol{\beta}$, with $\Phi^\star : \boldsymbol{\beta} \mapsto \boldsymbol{C}$ is linear-bounded, non-negative on $\boldsymbol{\beta}_+, \varphi \in \boldsymbol{\beta}_+$; further $\Delta_t : \boldsymbol{\beta} \mapsto \boldsymbol{\beta}$ such that for all $t > 0$

$$P\Delta_t\xi = \Delta_t P\xi = 0, \xi \in \boldsymbol{\beta},$$

$$-\gamma_t P\xi \leq \Delta_t\xi \leq \gamma_t P\xi, \xi \in \boldsymbol{\beta}_+$$

with $\gamma_t : (0, \infty) \mapsto \boldsymbol{R}_+$ satisfying $\rho^{-t}\gamma_t \downarrow 0, t \uparrow \infty$.

When X is a finite set and M_t is primitive, satisfaction of (M) follows from the well-known Perron's theorem (see for details Sevastyanov (1971), Ch

IV). In the general case it can also be satisfied by a wide class of branching diffusions. Examples of such processes can be seen in Hering (1978).

Suppose $\{M_t\}$ exists as a semigroup of bounded operators. Then

$$1 - F_t[\eta] = M_t[1-\eta] - R_t(\eta)[1-\eta] \tag{7}$$

where the mapping $R_t(.)[\eta] : \bar{\nabla} \otimes \boldsymbol{\beta} \mapsto \boldsymbol{\beta}$ is non-increasing in the first variable and linear-bounded in the second, such that

$$0 = R_t(1)\xi \le R_t(\eta)\xi \le M_t\xi,$$

$$(\eta, \xi) \in \bar{\nabla}_+ \otimes \boldsymbol{\beta}.$$

Assumption (R). For every $t > 0$, there exist a mapping $g_t : \bar{\nabla}_+ \mapsto \boldsymbol{\beta}$ such that

$$R_t(\xi)[1-\xi] = g_t[\xi]\rho^t \Phi^\star[1-\xi]\varphi, i \in \bar{\nabla}_+,$$

$$\lim_{\|1-\xi\| \to \infty} \| g_t[\xi] \| = 0.$$

Under the assumption (M) the (R) is automatically satisfied if X is finite. In the general case it may follow from (M) under some restrictions on φ and $\Phi^\star$. Some sufficient conditions for fulfillment of (R) are given, for example, in the book by Asmussen and Hering(1983, p.160).

We also assume that the second moment functional of $\hat{x}_t$

$$M_2^{(2)}(< x >, \xi, \eta) = E^{<x>}\hat{x}_t^{(2)}[\xi, \eta]$$

is finite. Here

$$\hat{x}^{(2)}[\xi, \eta] = \begin{cases} 0, & \hat{x}[1] \le 1, \\ \sum\sum_{i \ne j} \xi(x_i)\eta(x_j), & \hat{x} =< x_1, ..., x_n >, n \ge 2. \end{cases}$$

We denote

$$\mu = \frac{1}{2t}\Phi^\star[M_t^{(2)}[\varphi]], M_t^{(2)}[\xi](x) = M_t^{(2)}(< x >, \xi, \xi)$$

It is known that

$$q(x) = \lim_{t \to \infty} F_t[0](x)$$

exists for any $x \in X$, and given (M) with $\rho = 1$ the quantity μ is constant as a function of t.

3 Approximation of the non-extinction probability

Now we proceed to consider the immigration-branching diffusion. Let F_k^I be the generating functional of $P\{\hat{y}_k \in \hat{A}\}, \hat{A} \in \hat{U}$. If F_k^I has a bounded first moment functional

$$M_k^I[\xi] = \int_{\hat{X}} \hat{y}[\xi] P\{\hat{y}_k \in d\hat{y}\},$$

then it follows that (7) holds for F_k^I. Assume that $\alpha(t) = M_t^I[\varphi]$ is a regularly varying function as $t \to \infty$ and

$$r(\varepsilon) = \sup_{s>0} \frac{R_s^I((1-\varepsilon)1)[\varphi]}{\alpha(s)} \to 0, \varepsilon \to 0. \tag{8}$$

Condition (8) holds, for example, if the second factorial moment

$$M_k^{I(2)}[\xi, \eta] = \int_{\hat{X}} \hat{y}^{(2)}[\xi, \eta] P\{\hat{y}_k \in d\hat{y}\}$$

of the immigration process is finite as a symmetric bilinear-bounded functional on $\beta^{(2)} = \beta \otimes \beta$ and

$$\sup_k \frac{M_k^{I(2)}[1,1]}{M_k^I[\varphi]} < \infty.$$

In the case of stationary immigration process, it is clear that, the state $\theta \in \hat{X}$ is a reflecting screen for the process. If the immigration process has an decreasing intensity, then the state θ may reflect or may also absorb the process. Hence it is important to know the behavior of $P\{\hat{Z}_t \neq \theta\}$ as $t \to \infty$. The first result gives us conditions under which the above probability tends to 1, to zero or to a positive limit less than 1.

Theorem 1 *Let (M) , (R) and (8) be satisfied,* $\rho = 1, 0 < \mu < \infty, q = 1, \alpha(t) \to 0, t \to \infty$.
1^o. *If* $\alpha(t) \ln t \to 0$, *then*

$$\lim_{t\to\infty} P\{\hat{Z}_t \neq \theta\} = 0.$$

2^o. *If* $\alpha(t) \ln t \to C \in (0, \infty)$, *then*

$$\lim_{t\to\infty} P\{\hat{Z}_t \neq \theta\} = 1 - \exp\{-\frac{C}{\mu}\}.$$

3^o. *If* $\alpha(t)\ln t \to \infty$, *then*

$$\lim_{t\to\infty} P\{\hat{Z}_t \neq \theta\} = 1.$$

Proof. First we prove part 2^o. If we denote $H_t[\xi]$ the generating functional of the process $\hat{Z}_t$, then it follows from (4) that

$$H_t[\xi] = \prod_{k=0}^{t} F_k^I[F_{t-k}[\xi]]. \tag{9}$$

We consider the sum

$$A = \sum_{k=0}^{t} M_k^I[1 - F_{t-k}[0]] = A_1 + A_2 + A_3, \tag{10}$$

where $A_i, i = 1,2,3,$ are sums with respect to $0 \le k < t/\ln t, t/\ln t \le k \le t - \ln t$ and $t - \ln t < k \le t$.

Further we use the following results from Asmussen, Hering (1983,pp. 184, 202).

Lemma 1 *If (M) and (R) are satisfied and* $q = 1$, *then for every* $t > 0$ *there exist a mapping* $h_t : \bar{\nabla}_+ \to \boldsymbol{\beta}$ *such that*

$$1 - F_t[\xi] = (1 + h_t[\xi])\Phi^\star[1 - F_t[\xi]]\varphi, \xi \in \bar{\nabla}_+, \tag{11}$$

where $\lim_{t\to\infty} \| h_t[\xi] \| = 0$ *uniformly in* $\xi \in \bar{\nabla}_+$.

The next result is a generalization of so called basic lemma in the theory of Galton-Watson processes (see Jagers(1983), p.25).

Lemma 2 *If (M) and (R) are satisfied,* $0 < \mu < \infty, \rho = 1, q = 1$ *then for any* $\delta > 0$

$$\lim_{n\to\infty} \frac{1}{n\delta}\left\{\Phi^\star[1 - F_{n\delta}[\xi]]^{-1} - \Phi^\star[1-\xi]^{-1}\right\} = \mu \tag{12}$$

uniformly in $\xi \in \bar{\nabla}_+ \bigcap \{\Phi^\star[1-\xi] > 0\}$.

It follows from (11) and (12) that

$$(n+1)(1 - F_n[0]) = (1 + \varepsilon_n)\frac{\varphi}{\mu}, \tag{13}$$

where $\lim_{n\to\infty} \| \varepsilon_n \| = 0$.

Since M_t^I is a linear-bounded functional, using (13), we have

$$A_1 = \sum_{k\in E_1} \frac{1}{t-k+1} M_k^I[(1+\varepsilon_{t-k})\frac{\varphi}{\mu}],$$

where $E_1 = \{k : 0 \le k < t/\ln t\}$. Using here the simple estimates

$$1- \| \varepsilon \| \le 1+\varepsilon \le 1+ \| \varepsilon \| \qquad (14)$$

we obtain that A_1 is non-greater than

$$\mu^{-1}(1+\sup_{k\in E_1} \| \varepsilon_{t-k} \|) \sum_{k\in E_1} \alpha(k)(t-k+1)^{-1}$$

which tends to zero as $t \to \infty$.

Now we considerA_2. Again using (13) and (14), we obtain

$$A_2 \le \mu^{-1}(1+\sup_{k\in E_2} \| \varepsilon_{t-k} \|) \sum_{k\in E_2} \alpha(k)(t-k+1)^{-1},$$

where $E_2 = \{k : t/\ln t < k \le t - \ln t\}$. Since under the our conditions

$$\lim_{t\to\infty} \sum_{k\in E_2} \alpha(k)(t-k+1)^{-1} = C,$$

we have

$$\limsup_{t\to\infty} A_2 \le \frac{C}{\mu}.$$

If we use the left side of (14), we obtain that $\liminf A_2$ is non-less than C/μ. Thus A_2 tends to C/μ as $t \to \infty$.

By similar arguments it can be verified that A_3 is non-greater than

$$const \cdot \frac{1}{\ln t} \sum_{k=0}^{[\ln t]} (k+1)^{-1}$$

which tends to zero as $t \to \infty$.

If we use (7) written for F_t^I, we have that under the condition (8)

$$\lim_{t\to\infty} \sum_{k=0}^{t} \{1 - F_k^I[F_{t-k}[0]] = \lim_{t\to\infty} A. \qquad (15)$$

Thus assertion of the part 2^o follows from (15) and the simple relation

$$\ln H_t[0] = \sum_{k=0}^{t} \ln F_k^I[F_{t-k}[0]].$$

Now we prove part 1^o. For any $\varepsilon > 0$ and fixed μ we can find such a $C > 0$ that

$$1 - \exp\{-\frac{C}{\mu}\} < \varepsilon.$$

Let $\hat{Z}_t^\star$ be the immigration-branching diffusion with the same branching transition function and with immigration process such that $\alpha^\star(t)\ln t \to 0$. Then it is clear that for any $\varepsilon > 0$

$$\limsup_{t\to\infty} P\{\hat{Z}_t^\star \neq \theta\} \leq \lim_{t\to\infty} P\{\hat{Z}_t \neq \theta\} \leq \varepsilon.$$

From the last relation we have the assertion of the part 1^o.

Part 3^o can also be derived from part 2^o. To do this one has to choose $C > O$ such that

$$\exp\{-\frac{C}{\mu}\} < \varepsilon$$

and have to compare the "non-extinction" probability of the process $\hat{Z}_t$ and a new process whose immigration process satisfies conditions of the part 3^o. The theorem is proved.

We now consider the asymptotic behavior of the non-extinction probability in the case 1^o i.e. when $\alpha(t)\ln t \to 0, t \to \infty$. Introduce the following notation:

$$\alpha(t) = M_t^I[\varphi] = \frac{l(t)}{t^\alpha + 1}, \alpha \geq 0, a(t) = \sum_{k=0}^{t} \alpha(k)$$

$$\beta(t) = P^{<x>}\{\hat{x}_t \neq \theta\}, b(t) = \sum_{k=0}^{t} \beta(k).$$

Here $l(t)$ is a slowly varying as $t \to \infty$ function. It follows from (13) that under the our assumptions

$$t\mu\beta(t) \sim \varphi, t \to \infty. \tag{16}$$

Note that $a(t)$, when $\alpha \geq 1$, and $b(t)$ are slowly varying functions. Therefore, it follows from the Karamata representation of slowly varying functions that there exist positive, integer valued functions $L_i(t), i = 1, 2$, such that $L_i(t) \to \infty, L_i(t) = o(t), t \to \infty$ and (see also Lemma 3 below)

$$a(L_1(t)) \sim a(t), b(L_2(t)) \sim b(t), t \to \infty. \qquad (17)$$

Introduce the following "partial" processes:

$$\hat{Z}_t^{(j)} = \sum_{k \in I_j^t} \sum_{i=1}^{\hat{y}_k(1)} \hat{x}_{t-k}^{<y_{ki}>},$$

where

$$I_1^t = \{k \in N : 0 \le k \le L_1(t)\}, I_3^t = \{k \in N : t - L_2(t) \le k \le t\},$$

$$I_2^t = \{k \in N : L_1(t) + 1 \le k \le t - 1 - L_2(t)\}.$$

It is clear that

$$\hat{Z}_t = \hat{Z}_t^{(1)} + \hat{Z}_t^{(2)} + \hat{Z}_t^{(3)} \qquad (18)$$

Theorem 2 *Let assumptions of Theorem 1 are satisfied. Then*
1^o. *If* $\alpha \ge 1$, *then*

$$P\{\hat{Z}_t^{(1)} \ne \theta\} \sim (\mu t)^{-1} a(t),$$

if $\alpha < 1$, *then*

$$P\{\hat{Z}_t^{(1)} \ne \theta\} = o(\frac{a(t)}{t});$$

2^o.

$$P\{\hat{Z}_t^{(3)} \ne \theta\} \sim \mu^{-1} \alpha(t) \ln t;$$

3^o.

$$P\{\hat{Z}_t \ne \theta\} \sim \frac{1}{\mu t} a(t) + \frac{1}{\mu} \alpha(t) \ln t.$$

Remark. It should be noted that part 3^o of the theorem shows that event $\{\hat{Z}_t \ne \theta\}$ may occur because of either $\{\hat{Z}_t^{(1)} \ne \theta\}$ or $\{\hat{Z}_t^{(2)} \ne \theta\}$ and, when $\alpha(t) \sim t^{-1} const$ these two possibilities are asymptotically equiprobable.

Proof of Theorem 2. First we prove part 1^o. It follows from (7) and (13) that

$$0 \le 1 - F_k^I[F_{t-k}[0]] \le \frac{1}{t-k+1} M_k^I[(1 + \varepsilon_{t-k}) \frac{\varphi}{\mu}] \to 0 \qquad (19)$$

as $t \to \infty$ uniformly in $k \in I_1$. Therefore

$$-\sum_{k\in I_1} \ln F_k^I[F_{t-k}[0]] \sim \sum_{k\in I_1}(1 - F_k^I[F_{t-k}[0]]).$$

Using (7), (13) and that M_k^I and R_k^I are linear-bounded functionals we obtain that the last sum as $t \to \infty$ is equivalent to

$$\delta_t = \frac{1+ \parallel \varepsilon_t \parallel}{\mu} \sum_{k\in I_1} \frac{M_k^I[\varphi]}{t-k+1}[1 - r(k,t)],$$

where

$$r(k,t) = \frac{R_k^I[F_{t-k}[0]][\varphi]}{M_k^I[\varphi]}.$$

Taking into account the choice of $L_1(t)$ and condition (8) we have that δ_t is equivalent to $(\mu t)^{-1}a(t)$ when $\alpha \geq 1$. If $\alpha < 1$, then $a(L_1(t)) = o(a(t))$, and thus in this case $\delta_t = o(t^{-1}a(t))$. The assertion of part 1^0 follows from this and the simple relation $1 - e^{-x} \sim x, x \to 0$.

Let us prove part 2^o. It is not difficult to see that (19) holds as $t \to \infty$ uniformly in $k \in I_3$ also. Thus we have to consider

$$A = \sum_{k\in I_3}(1 - F_k^I[F_{t-k}[0]]) = A_1 + A_2, \tag{20}$$

where

$$A_1 = \sum_{k\in I_3} M_k^I[1 - F_{t-k}[0]],$$

$$A_2 = \sum_{k\in I_3} R_k^I(F_{t-k}[0])[1 - F_{t-k}[0]].$$

Using again (13), (14) and that M_k^I is linear-bounded functional we obtain that $A_1 \sim \mu^{-1}\alpha(t)\ln t, t \to \infty$. Now we consider A_2. Since R_k^I is linear-bounded functional, using (13) and (14) we have that it is non-greater than

$$const \cdot \sum_{k\in I_3} \frac{1}{t-k+1} R_k^I(F_{t-k}[0])[\varphi].$$

It follows from condition (8) that for any $\varepsilon > 0$ there exist $\delta > 0$ such that $r(\Delta) < \varepsilon$ for all $\Delta < \delta$. Since $F_k[0] \to 1$, there exist $m \in N$ such that $1 - F_{t-k}[0] < \delta$ for $t - k > m$. Now we partition I_3 as $I_3 = I_3^{(1)} \bigcup I_3^{(2)}$, where

$$I_3^{(1)} = \{k : t - L_2(t) \leq k \leq t - m\}, I_3^{(2)} = \{k : t - m < k \leq t\}.$$

Then we have

$$\sum_{k\in I_3^{(1)}} (t-k+1)^{-1} R_k^I(F_{t-k}[0])[\varphi] \le \varepsilon\alpha(t)\ln t.$$

Since $R_t^I(\eta)\xi \le M_t^I\xi$, the second part of the sum is non-greater than

$$\sum_{k\in I_3^{(2)}} (t-k+1)^{-1}\alpha(k) = o(\alpha(t)\ln t)$$

as $t\to\infty$ for any fixed m. Therefore we have $A_2 = o(\alpha(t)\ln t), t\to\infty$. Thus we obtain the assertion of part 2^o from (20) and estimates for A_1 and A_2. Now we prove part 3^o. First we consider

$$P_2 = P\{\hat{Z}_t^{(2)} \ne \theta\} = 1 - \prod_{k\in I_2} F_k^I[F_{t-k}[0]].$$

Using simple inequality $\prod_i(1-\varepsilon_i) \ge 1 - \sum_i \varepsilon_i, \varepsilon_i \ge 0$ and relation (7) we obtain that P_2 is non-greater than

$$\sum_{k\in I_2} M_k^I[1 - F_{t-k}[0]].$$

Since M_k^I is a linear functional we have from (13) and (14) that

$$P_2 \le \frac{1}{\mu}(1 + \sup_{k\in I_2} \| \varepsilon_{t-k} \| \sum_{k\in I_2} (t-k+1)^{-1} M_k^I[\varphi].$$

It follows from properties $M_k^I[\varphi]$ that there exist $C(\alpha) > 0$ such that

$$\min_{0\le k\le t}\left\{\frac{t-k+1}{t+1} + \frac{M_t^I[\varphi]}{M_k^I[\varphi]}\right\} \ge C(\alpha) > 0.$$

Therefore

$$\sum_{k\in I_2} \frac{1}{t-k+1}\alpha(k) \le \frac{1}{C(\alpha)}\left\{\frac{1}{t}\sum_{k\in I_2}\alpha(k) + \alpha(t)\sum_{k\in I_2}\frac{1}{t-k}\right\},$$

where

$$\frac{1}{t}\sum_{k\in I_2}\alpha(k) \le \frac{const}{t}[a(t) - a(L_1(t))] = o(\frac{1}{t}a(t)),$$

$$\alpha(t)\sum_{k\in I_2}\frac{1}{t-k}\le const\cdot\alpha(t)[b(t)-b(L_2(t))]=o(\alpha(t)\ln t).$$

Thus we can conclude that

$$P_2=o(\frac{1}{t}a(t)+\alpha(t)\ln t). \tag{21}$$

Further we use the following equality

$$\{\hat{Z}_t\ne\theta\}=\bigcup_{i=1}^{3}\{\hat{Z}_t^{(i)}\ne\theta\} \tag{22}$$

where events on the right hand side are independent. If we denote $P_i, i = 1,2,3$ the probability of the i-th event, then

$$P\{\hat{Z}_t\ne\theta\}=\sum_{i=1}^{3}P_i-\sum_{i\ne j}P_iP_j+P_1P_2P_3. \tag{23}$$

The assertion of the part 3^o follows from (23), if we use results of parts 1^o and 2^o and relation (21). Theorem 2 is proved.

4 Expected size of population

We first proof the following result.

Lemma 3 *If $L(x)$ is a slowly varying as $x\to\infty$ function, then there is another slowly varying function $l(x)\sim L(x), x\to\infty$ for which it is possible to find function $\lambda_l(x)\to\infty, \lambda_l(x)=o(x)$ such that as $x\to\infty$*

$$\sup_{u\in\Delta(x)}\mid\frac{l(u)}{l(x)}-1\mid\to 0, \Delta(x)=[\frac{x}{\lambda_l(x)},x]. \tag{24}$$

In some sense the Lemma 3 is an extension of so called uniform convergence theorem for slowly varying functions.

Proof. We use the following well-known representation of slowly varying functions $L(x) = C(x)A(x)$, where

$$A(x) = \exp\left\{\int_B^x \frac{\varepsilon(u)}{u} du\right\},$$

$\varepsilon(u)$ is a continuous function on $[B, \infty)$ such that $\varepsilon(u) \to 0, u \to \infty$ and $C(x) \to C \in (0, \infty)$ as $x \to \infty$. It is clear that as $l(x)$ we can take $l(x) = CA(x)$. Let $\lambda(x) = \lambda_l(x)$ be such that $\lambda(x) \to \infty, \lambda(x) = o(x)$ as $x \to \infty$ and

$$\delta(x) = \sup_{u \in \Delta(x)} \mid \varepsilon(u) \mid \ln \lambda(x) \to 0$$

Since

$$\sup_{u \in \Delta(x)} \mid \int_u^x \frac{\varepsilon(v)}{v} dv \mid \leq \delta(x)$$

using the simple estimate $\mid e^\alpha - 1 \mid \leq \mid \alpha \mid e^{|\alpha|}$ we obtain that $\mid A(u)/A(x) - 1 \mid \to 0$ as $x \to \infty$ uniformly in $u \in \Delta(x)$. The lemma is proved.

Now we consider the asymptotic behavior of the expectation of the process $\hat{Z}_t$. We obtain from (3) and (4) that

$$E\hat{Z}_t[\xi] = \sum_{k=0}^{t} E\hat{x}_{t-k}^{\hat{y}_k}[\xi] = \sum_{k=0}^{t} E\hat{y}_k[(M_{t-k}\xi)(y)],$$

where $(M_t\xi)(y) = M_t(< y >, \xi)$ and

$$E\hat{y}_k[(M_{t-k}\xi)(y)] = \sum_{\nu=1}^{n} E(M_{t-k}\xi)(y_{k\nu}),$$

if $\hat{y}_k =< y_{k1}, ... y_{kn} >$ and equals zero if $\hat{y}_k = \theta$. Therefore under the condition (M) we have:

$$\Phi^\star(\xi) I_- \leq E\hat{Z}_t[\xi] = \sum_{k=0}^{t} M_k^I[M_{t-k}\xi] \leq \Phi^\star(\xi) I_+,$$

where

$$I_\pm = \sum_{k=0}^{t} (1 \pm \gamma_{t-k})\alpha(k).$$

Recall that $\alpha(t)$ is a regularly varying function, that is it has the form given before Theorem 2:

$$\alpha(t) = \frac{l(t)}{t^\alpha + 1}, \alpha \geq 0,$$

where $l(t)$ is a slowly varying function as $t \to \infty$. Consequently there exists a function $\lambda_l(t) \to \infty, \lambda_l(t) = o(t), t \to \infty$ satisfying (24).

Now we will consider

$$I_\pm = \sum_{i=1}^{3} \sum_{k \in B_i} (1 \pm \gamma_{t-k}) \alpha(k), \tag{25}$$

where $B_1 = \{k : 0 \leq k \leq t/\lambda_l(t)\}, B_2 = \{k : t/\lambda_l(t) < k \leq t - \ln t\}$ and $B_3 = \{k : t - \ln t < k \leq t\}$. First we consider the case $\alpha < 1$. It follows from the choice of $\lambda_l(t)$ that as $t \to \infty$

$$\sum_{k \in B_2} (1 \pm \gamma_{t-k}) \alpha(k) \sim \alpha(t) t \tag{26}$$

On the other hand the sum over B_3 is non-greater than

$$const \cdot \alpha(t) \sum_{k \in B_3} k^{-\alpha} = O(\alpha(t) \ln t).$$

Now we consider the sum over B_1. Let $\delta > 0$ be such that $\alpha + \delta < 1$. It follows from a property of slowly varying functions (see Seneta (1976), p. 20) that

$$\sup_{0 \leq k \leq r} l(k) t^\delta \sim l(r) r^\delta, r \to \infty.$$

Then it is clear that for some positive constant C and sufficiently large t

$$\sum_{k \in B_1} (1 \pm \gamma_{t-k}) \alpha(k) \leq C l(t) t^\delta \left\{ \frac{t}{\lambda_l(t)} \right\}^{1-\alpha-\delta},$$

and consequently the sum over the set B_1 is $o(t\alpha(t))$ as t$\to \infty$. Thus we can formulate the following result.

Theorem 3 *If (M) satisfied with $\rho = 1, q = 1$, then*

$$E\hat{Z}_t[\xi] \sim \Phi^*[\xi] t \alpha(t), t \to \infty, \tag{27}$$

when $\alpha < 1$ *and*

$$E\hat{Z}_t[\xi] \sim \Phi^*[\xi] \sum_{k=0}^{t} \alpha(k), t \to \infty, \tag{28}$$

when $\alpha \geq 1$ *and in the latest case the limit is finite if* $\alpha > 1$ *and tends to infinity as a slowly varying function if* $\alpha = 1$.

Note that,if $\xi = 1_A, A \subset X$, then (27) and (28) give asymptotics of the expected number of particles in the set A and, if $A = X$, we have the asymptotics for the expected total number of particles at time t.

5 The limit theorems

Now we turn our attention to limit theorems for the vector of population sizes in disjoint subsets of the space X. Let $\{A_i, i = 1, ..., j\}$ be a measurable decomposition of X, that is

$$X = \cup_{i=1}^{j} A_i, A_l \cap A_m = \phi.$$

If $\chi(A)$ is the indicator function of the set $A \subseteq X$, then it is clear that $\hat{Z}_t[\chi(A)]$ is the number of particles in A at time t.

Theorem 4 *Let (M), (R) and (8) be satisfied,* $\rho = 1, 0 < \mu < \infty$ *and* $q = 1$. *If* $\alpha(t) \to 0$ *such that* $\alpha(t) \ln t \to \infty$, *then for every finite measurable decomposition* $\{A_i, i = 1, ..., j\}$ *of X*

$$\lim_{t \to \infty} P\left\{ (\frac{\hat{Z}_t[\chi(A_i)]}{t})^{\alpha(t)/\mu} \leq x_i, i = 1, ..., j \right\} = P\{\nu_1 \leq x_1, ..., \nu_j \leq x_j\},$$

where $\nu_1 = ... = \nu_j$ *with probability 1 and* ν_k *has the uniform distribution on* $[0, 1]$.

Proof. It follows from the condition $\alpha(t) \ln t \to \infty$ that $\alpha(t)$ is a slowly varying function as $t \to \infty$. Thus there is a function $\lambda_\alpha(t)$ for which the extended uniform convergence theorem holds (see Lemma 3). If $0 < x_i < 1, i = 1, ..., j$ and $x = \min\{x_1, ..., x_j\}$, then the function $L(t) = tx^{\mu/\alpha(t)} \to \infty, L(t) = o(t)$ as $t \to \infty$. Now we consider the sum

$$\sum_{k=0}^{t} M_k^I[1 - F_{t-k}[\xi_t]], \xi_t = \exp\left\{ -\frac{\sum_{i=1}^{j} \lambda_i \chi(A_i)}{tx_i^{\mu/\alpha(t)}} \right\},$$

where $\lambda_i \geq 0, i = 1, ..., j$. We decompose the sum as following

$$A = A_1 + A_2 + A_3 \tag{29}$$

where $A_i, i = 1, 2, 3$ are sums of the terms when k belongs to $K_1 = \{k : 0 \leq k < t/\lambda_\alpha(t)\}, K_2 = \{k : t/\lambda_\alpha(t) \leq k < t - L(t)\}$ and $K_3 = \{k : t - L(t) \leq k \leq t\}$ respectively. In this case we obtain from (11) and (12) the following relation

$$1 - F_n[\xi] = \frac{\varphi}{\mu n + \Phi^\star[1-\xi]^{-1}}(1 + \varepsilon_n^\star[\xi]), \tag{30}$$

where $\| \varepsilon_n^\star \| \rightarrow 0$ as $n \rightarrow \infty$ uniformly in $\xi \in \bar{\nabla}_+ \bigcap \{\Phi^\star[1-\xi] > 0\}$. The following estimates are also true for any ξ from the above set:

$$1 - \| \varepsilon_n^\star[\xi] \| \leq 1 + \varepsilon_n^\star[\xi] \leq 1 + \| \varepsilon_n^\star[\xi] \| \tag{31}$$

First we consider A_2. Using (30), right side of (31) and Lemma 3 we get

$$\limsup_{t\rightarrow\infty} A_2 \leq \limsup_{t\rightarrow\infty} \alpha(t) \sum_{k\in K_2} \left\{\mu k + \Phi^\star[1-\xi_t]^{-1}\right\}^{-1}.$$

It follows from the choice of $L(t)$ and linearity of $\Phi^\star$ that

$$\lim_{t\rightarrow\infty} L(t)\Phi^\star[1-\xi_t] = \delta(E_0) \equiv \sum_{i\in E_0} \lambda_i \Phi^\star[\chi(A_i], \tag{32}$$

where $E_0 = \{i : x_i = x\}$. Thus we have from the above estimate that

$$\limsup_{t\rightarrow\infty} A_2 \leq \limsup_{t\rightarrow\infty} \frac{\alpha(t)}{\mu} \ln(1 + \mu\delta(E_0)x^{-\mu/\alpha(t)}) = -\ln x.$$

If we use the left side of (31), by the same arguments obtain that the lim inf of A_2 is non-less than $-\ln x$. Consequently $A_2 \rightarrow -\ln x$ as $t \rightarrow \infty$.

Using (30) and right side of (31) we can see that $\limsup \lambda_\alpha(t) A_1 < \infty$, that is A_1 tends to zero as $t \rightarrow \infty$.

Now we consider A_3. Again using (30), (31) and the fact that $\alpha(t)$ slowly varies, we obtain, for some positive constant C, the following estimate

$$A_3 \leq C\alpha(t) \sum_{k\in K_3} \{\mu(t-k) + \Phi^\star[1-\xi_t]^{-1}\}^{-1} < C\delta(E_0)\alpha(t),$$

which shows that A_3 also tends to zero as $t \to \infty$. Thus it follows from (29) that $A \to -\ln x$ as $t \to \infty$.

Using relation (7) written for $F_k^I[\xi]$ we have

$$\sum_{k=0}^{t}\{1 - F_k^I[F_{t-k}[\xi_t]]\} = A - \sum_{k=0}^{t} R_k^I[F_{t-k}[\xi_t]][1 - F_{t-k}[\xi_t]].$$

The relations (9), (29) and the simple approximation $\ln(1-x) \sim -x, x \to 0$, gives us that

$$\lim_{t\to\infty} \ln H_t[\xi_t] = -\lim_{t\to\infty} A = \ln x.$$

Now we use the standard arguments allowing to obtain the limit theorem with large deviations from the behavior of the Laplace transform (see [9], for example). The number $0 < x < 1$ is the Laplace transform of the random vector $\tau = (\tau_1, ..., \tau_j)$ such that $P\{\tau = \mathbf{0}\} = \mathbf{x}, \mathbf{P}\{\cup_{i=1}^{j}\{\tau_i = \infty\}\} = \mathbf{1} - \mathbf{x}$ and $P\{\cap_{i=1}^{j}\{\tau_i < \infty\}, \tau \neq \mathbf{0}\} = 0$. Hence by the continuity theorem for Laplace transforms

$$\lim_{t\to\infty} P\left\{\hat{Z}_t[\chi(A_i)]t^{-1}x_i^{-\mu/\alpha(t)} \le y_i, i = 1, ..., j\right\} = x$$

for any finite positive numbers $y_1, ..., y_j$. The assertion of the theorem follows from the last relation if we put $y_1 = ... = y_j = 1$. The theorem is proved.

The following theorem gives a limit distribution when $\alpha(t) \sim C(\ln t)^{-1}$.

Theorem 5 *Let (M), (R) and (8) be satisfied, $\rho = 1, 0 < \mu < \infty$ and $q = 1$. If $\alpha(t) \to 0$ such that $\alpha(t)\ln t \to C \in (0, \infty)$ as $t \to \infty$, then for every finite measurable decomposition $\{A_i, i = 1, ...j\}$ of X*

$$\lim_{t\to\infty} P\left\{\frac{\hat{Z}_t[\chi(A_i]^{\alpha(t)/\mu} - 1}{e^{C/\mu} - 1} \le x_i, i = 1, ..., j\right\} = P\{\nu_i \le x_i, i = 1, ..., j\},$$

where $\nu_1 = ... = \nu_j$ with probability 1 and

$$P\{\nu_i \le x\} = e^{-C/\mu} + x(1 - e^{-C/\mu}), 0 \le x \le 1.$$

Proof. We consider the sum from (29) with

$$\xi_t = \exp\{-\sum_{i=1}^{j} \lambda_i \chi(A_i/C_t(x_i))\},$$

where $C_t(x_i) = (1 + x_i(e^{C/\mu} - 1))^{\mu/\alpha(t)}, 0 < x_i < 1$. It is not difficult to see that under the conditions of Theorem 5 $C_t(x_i) \to \infty, C_t(x_i) = o(t)$ as $t \to \infty$. If we decompose the sum as in (29) with $\lambda_\alpha(t) = \ln t$ and $L(t) = \ln\ln t$, using again (30) and (31) obtain that as $t \to \infty$

$$A_2 \sim \frac{\alpha(t)}{\mu} \ln \frac{\mu t \Phi^\star[1-\xi_t]+1}{\mu\Phi^\star[1-\xi_t]\ln\ln t + 1}. \tag{33}$$

Taking into account the fact that

$$\lim_{t\to\infty} C_t(x)\Phi^\star[1-\xi_t] = \delta(E_0),$$

where $x = \min\{x_1, ..., x_j\}$ and $\delta(E_0)$ the same as in (32), we have from (33) that A_2 tends to $C\mu^{-1} - \ln(1 + x(e^{C/\mu} - 1))$. The same arguments as in the proof of Theorem 4 shows that A_1 and A_3 tend to zero as $t \to \infty$. Thus

$$\lim_{t\to\infty} H_t[\xi_t] = e^{-C/\mu} + x(1 - e^{-C/\mu}).$$

The assertion of the theorem follows from the last relation by continuity theorem for the Laplace transforms. The theorem is proved.

We proceed a further study in the following scheme. We consider the "partial processes" introduced in Section 3. It is clear from the definition that $Z_t^{(1)}$ and $Z_t^{(3)}$ are branching diffusions generated by the particles immigrated in the beginning of the process and by the recent immigrants, respectively. First, we prove limit theorems for partial processes using analytic technique as in the proof of theorems 4 and 5. After that we deduce the explicit limit theorem for the basic process from those results by the direct probability arguments.

We denote by $H_t^{(i)}[\xi]$ generating functionals of processes $\hat{Z}_t^{(i)}, i = 1, 2, 3$, that is

$$H_t^{(i)}[\xi] = \prod_{k\in I_i} F_k^I[F_{t-k}[\xi]].$$

Theorem 6 *Let $(M), (R)$ and (8) be satisfied, $\rho = 1$, $0 < \mu < \infty$, $q = 1$. If $\alpha(t) \to 0$ such that $\alpha(t)\ln t \to 0$, $\alpha \geq 1$, then for any finite measurable decomposition $\{A_i, i = 1, \ldots, i\}$ of X*

$$\lim_{t\to\infty} P\left\{\frac{\hat{Z}_t^{(1)}[\chi[A_i]]}{t} \leq x_i,\ i = 1, \ldots, j | \hat{Z}_t^{(1)} \neq \theta\right\} =$$
$$= P\left\{\nu_i\Phi^\star[\chi(A_i)] \leq x_i, i = 1, \ldots, j\right\},$$

where $\nu_1 = \cdots = \nu_j$ *with probability 1 and*

$$P\{\nu_i \le x\} = \begin{cases} 0, & x < 0 \\ 1 - e^{-x/\mu}, & x \ge 0 \end{cases}$$

Theorem 7 *Let* $(M), (R)$ *and (8) be satisfied,* $\rho = 1,\ 0 < \infty, q = 1$. *If* $\alpha(t) \to 0$ *such that* $\alpha(t) \ln t \to 0$, *then for any finite measurable decomposition* $\{A_i, i = 1, \dots, j\}$ *of* X

$$\lim_{t\to\infty} P\left\{ \frac{\ln \hat{Z}_t^{(3)}[\chi(A_i)]}{\ln t} \le x_i, i = 1, \dots, j | \hat{Z}_t^{(3)} \ne \theta \right\} = P\{\nu_i \le x_i, i = 1, \dots, j\}$$

where $(\nu_1, \dots, \nu_j)$ *are the same as in Theorem 4.*

Proof of Theorem 6. We consider the sum

$$A = \sum_{k\in I_1} \left(1 - F_k^I[F_{t-k}[\xi_t]]\right), \xi_t = \exp\left\{-t^{-1}\sum_{i=1}^{j} \lambda_i \chi(A_i)\right\},$$

where $\lambda_i \ge 0, i = 1, \dots, j$ and I_1 is the set defined in Section 3. Using (7) we can write

$$A = A_1 + A_2 \tag{34}$$

where

$$A_1 = \sum_{k\in I_1} M_k^I[1 - F_{t-k}[\xi_t]], A_2 = \sum_{k\in I_1} R_k^I(F_{t-k}[\xi_t])[1 - F_{t-k}[\xi_t]].$$

Let us consider A_1. Using relations (30), (31) and the definition of the set I_1, we get that A_1 is equivalent as $t \to \infty$ to

$$a(L_1(t))\left\{\mu t + \Phi^\star[1 - \xi_t]^{-1}\right\}^{-1}$$

where $a(t)$ is defined in the Section 3. Since $\Phi^\star[\cdot]$ is linear-bounded and nonnegative on β_+ (see Assumption (M)), $t\Phi^\star[1 - \xi_t] \to \delta(E)$ as $t \to \infty$, where $E = \{1, \dots, j\}$ and δ is the same as in (32). Thus from here and from (17) we conclude that

$$A_1 \sim \frac{\delta(E)}{1 + \mu\delta(E)} a(t) t^{-1},\ t \to \infty. \tag{35}$$

Now, we consider A_2. Since $R_k^I(\cdot)[\cdot]$ is a non-increasing function of the first argument and a linear bounded functional with respect to the second argument and using relations (30), (31) we find that for some positive constant C and sufficiently large t

$$A_2 \leq C \sup_k \frac{R_k^I((1-\varepsilon_t)1)[\varphi]}{\alpha(k)} t^{-1} a(t),$$

where $\varepsilon_t > 0$ such that $\varepsilon_t \to 0$ as $t \to \infty$. The last inequality shows that under condition (8), $A_2 = o(t^{-1}a(t))$ as $t \to \infty$.

It is not difficult to see that the Laplace transform of the conditional distribution in Theorem 6 equals

$$1 - \frac{1 - H_t^{(1)}[\xi_t]}{P\{\hat{Z}_t^{(1)} \neq \theta\}}.$$

Using the approximation $\ln(1-x) \sim -x$, $x \to 0$ we have that $1 - H_t^{(1)}[\xi_t] \sim A$, $t \to \infty$ and, consequently, taking into account (34), (35) and part 1^0 of Theorem 2, we obtain that the limit of the above above Laplace transform is $(1 + \mu\delta(E))^{-1}$. The theorem is proved.

Proof of Theorem 7. First we consider the sum

$$C_t(m) = \sum_{k=0}^{m} \left\{\mu k + \Phi^\star[1-\xi_t]^{-1}\right\}^{-1}, \ \xi_t = \exp\left\{-\sum_{i=1}^{j} \lambda_i \chi(A_i) t^{-x_i}\right\}.$$

It is not difficult to see that

$$C_t(t) - C_t(L_2(t)) \leq b(t) - b(L_2(t))$$

and the last difference has the order $o(\ln t)$ according to relation (17). If we denote $q(x,t) = \{\mu x + \Phi^\star[1-\xi_t]^{-1}\}^{-1}$, then $C_t(t)$ can be written as follows:

$$C_t(t) = \int_0^{t-1} q(x,t)dx + C^{(1)}(t), \tag{36}$$

where

$$0 \leq C^{(1)}(t) \leq \sum_{k=0}^{t-1} [q(k)t) - q(k+1,t)]$$

and the sum on the right side tends to zero as $t \to \infty$. Since $t^x \Phi^\star[1-\xi_t] \to \delta(E_0)$, where $x = \min\{x_1, \dots, x_j\}$ and $\delta(E_0)$ the same as in (32), we find that the integral in (36) is equivalent to $\mu^{-1} \ln(1 + \mu\delta(E_0)t^{1-x})$ as $t \to \infty$. Hence we have

$$C_t(L_2(t)) \sim \frac{1-x}{\mu} \ln t, t \to \infty. \tag{37}$$

Now we consider

$$C^{(2)}(t) = \sum_{k \in I_3} \|\varepsilon^\star_{t-k}[\xi_t]\| q(t-k, t),$$

where $\varepsilon^\star_t[\cdot]$ is the functional from (30). Since $\|\varepsilon^\star_k[\xi]\| \to 0$ as $k \to \infty$ uniformly in $\xi \in \overline{\nabla}_+ \cap \{\Phi^\star[1-\xi] > 0\}$, there exists a positive integer m such that $\|\varepsilon^\star_{t-k}[\xi_t]\| < \varepsilon$ for $t - k \geq m$.

We partition I_3 as $I_3 = I_3^{(1)} \cup I_3^{(2)}$, where $I_3^{(1)} = \{k : t - L_2 \leq k \leq t - m\}$ and $I_3^{(2)} = \{t - m < k < t\}$. It follows from the choice of m that the sum of terms with $k \in I_3^{(1)}$ is less than $\varepsilon C_t(L_2(t))$ and the sum of terms with $k \in I_3^{(2)}$ tends to zero as $t \to \infty$ for any fixed m. Thus using (30) and (31), we obtain that as $t \to \infty$

$$A_1 \equiv \sum_{k \in I_3} M_k^I[1 - F_{t-k}[\xi_t]] \sim \alpha(t) C_t(L_2(t)). \tag{38}$$

The same arguments as in the proof of Theorem 6, if we take into account (37) and (38), give us that under condition (8), $1 - H_t^{(3)}[\xi_t] \sim \mu^{-1}(1 - x)\alpha(t) \ln t$ as $t \to \infty$. Consequently, using part 2^0 of Theorem 2, we conclude that the Laplace transform of the conditional distribution in Theorem 7 tends to $x = \min\{x_1, \dots, x_j\}$ as $t \to \infty$. How from this fact the assertion of the theorem follows can be shown as in the proof of Theorem 5. The theorem is proved.

We now formulate the limit theorem for the basic branching diffusion. It turns out that the form of limiting distributions and the normalizing functions depends on the behavior at infinity of the function

$$\Theta(t) = t\alpha(t) \ln t / a(t).$$

Theorem 8 *Let $(M), (R)$ and (8) be satisfied, $\rho = 1, 0 < \mu < \infty, q = 1, \alpha(t) \ln t \to 0, t \to \infty$ and $\{A_i, i = 1, \dots, j\}$ be a finite measurable decomposition of X.*

1⁰. If $\liminf_{t\to\infty} \Theta(t) > 0$*, then for any* $x_i \in [0,1]$, $i = 1, \dots, j$, $x = \min\{x_1, \dots, x_j\}$ *as* $t \to \infty$

$$P\left\{\frac{\ln \hat{Z}_t[\chi[A_i]]}{\ln t} = \le x_i, i = 1, \dots, j | \hat{Z}_t \ne \theta\right\} = \frac{x\Theta(t)}{1+\Theta(t)}(1+o(1)).$$

2⁰. If $\limsup_{t\to\infty} \Theta(t) < \infty$*, then for any* $x_i \ge 0, i = 1, \dots, j$ *as* $t \to \infty$

$$P\left\{\frac{\hat{Z}_t[\chi[A_i]]}{t} \le x_i, i = 1, \dots, j | \hat{Z}_t \ne \theta\right\} = \frac{\Theta(t) + G(x_1, \dots, x_j)}{1+\Theta(t)}(1+o(t)),$$

where $G(x_1, \dots, x_j)$ *is the limiting distribution in Theorem 6.*

Examples. 1^0. If $\alpha < 1$, then $\Theta(t) \to \infty$ and we obtain from part 1^0 of the theorem that the limit distribution is the same as in Theorem 7. If $\alpha > 1$, then $\Theta(t) \to 0$ and it follows from part 2^0 of the theorem that the limit distribution is the same as in Theorem 6.

2^0. Let now $\alpha = 1$, $l(t) \equiv C_0 \in (0, \infty)$. Then $\Theta(t) \to 1$ as $t \to \infty$. In this case it follows from parts 1^0 and 2^0 of Theorem 8 that the limit distributions there have atoms of the same mass 2^{-1} at points 1 and zero respectively. The limit Theorems 6 and 7 obtained for the partial processes explain the cause of the appearance of these atoms.

Proof of Theorem 8. Let $B_i = B_i(t)$ be the event that i-th partial process is not equal to θ at time t. We consider the following events

$$C_i = B_i \cap \overline{B}_j \cap \overline{B}_k, C_{ij} = B_i \cap B_j \cap \overline{B}_k, C_{ijk} = B_i \cap B_j \cap B_k,$$

where $i, j, k = 1, 2, 3, i \ne j, i \ne k, j \ne k$.

If we denote $D = \{\hat{Z}_t \ne \theta\}$, then it is not difficult to see that

$$D = \bigcup_{i=1}^{3} C_i \cup \bigcup_{\substack{i,j=1\\ i\ne j}}^{2} C_{ij} \cup C_{123}. \tag{39}$$

Since the events on the right side of (39) are disjoint, for any event A we have

$$P\{A|D\} = \sum_{i=1}^{3} P\{A|C_i\}\frac{P\{C_i\}}{P(D)} + \sum_{\substack{i,j=1\\ i\ne j}}^{2} \frac{P\{A \cap C_{ij}\}}{P\{D\}} + \frac{P\{A \cap C_{123}\}}{P\{D\}}. \tag{40}$$

It follows from the definition of the process that the partial processes $\hat{Z}_t^{(i)}, i = 1,2,3$, are independent. Therefore, using the assertion of Theorem 1, we get, under the conditions of Theorem 8, that

$$P\{C_i\} \sim P\{B_i\},\ P\{C_{ij}\} \sim P\{B_i\}P\{B_j\},\ P\{C_{123}\} \sim P\{B_1\}P\{B_2\}P\{B_3\}.$$

Let us prove part 1^0. In order to do it we put in (40) that

$$A = \left\{ \frac{\ln \hat{Z}_t[\chi(A_i)]}{\ln t} \leq x_i,\ \ i = 1, \ldots, j \right\}, 0 < x_i < 1.$$

Using Theorems 6 and 7 we have that $P\{A|C_3\} \to x = \min\{x_1, \ldots, x_j\}$ and $P\{A|C_1\} \to 0$ as $t \to \infty$. On the other hand, as it was shown in the proof of Theorem 2, $P\{C_2\} \sim P\{B_2\} = o(P\{D\})$, $t \to \infty$. Thus the first sum in (40) is equivalent to $xP\{B_3\}/P\{D\}$ as $t \to \infty$.

Taking into account results of Theorem 2 again, we see that $P\{C_{ij}\} = o(P\{D\})$ and $P\{C_{123}\} = o(P\{D\})$ as $t \to \infty$. Hence, it follows from these relations and from (40) that as $t \to \infty$

$$P\{A|D\} = \frac{x\Theta(t)}{1 + \Theta(t)} + o(1)$$

which gives the assertion of part 1^0 when liminf $\Theta(t) > 0$. Part 1^0 of the theorem is proved.

Now we prove part 2^0. We now put in (40)

$$A = \left\{ t^{-1}\hat{Z}_t[\chi(A_i)] \leq x_i, i = 1, \ldots, j \right\}.$$

In this case we obtain from Theorems 6 and 7 that $P\{A|C_1\}$ tends to $G(x_1, \ldots, x_j)$ and $P\{A|C_3\} \to 1$ as $t \to \infty$.

It follows from the condition limsup$\Theta(t) < \infty$ that $\alpha \geq 1$. Consequently we can use part 1^0 of Theorem 2 and thus, the first sum in (40) as $t \to \infty$ is equivalent to

$$\frac{G(x_1, \ldots, x_j) + \Theta(t)}{1 + \Theta(t)}.$$

The theorem is proved.

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