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On Mutually Normal Operators

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Abstract

In this paper, we introduce the mutually normal relation, m, between operators acting on a Hilbert space. We show that many properties are shared by mutually normal operators. We give some characterizations in terms of the relation m.

0 Introduction.

Let $L(H)$ be the algebra of all bounded linear operators acting on a Hilbert space H. If $T \in L(H)$ then T is normal if and only if $T^*T = TT^*$. If $T, F \in L(H)$ then we say that T and F are mutually normal, TmF, if the following hold:

$$TT^* = F^*F; \ \ T^*T = FF^*. \qquad (*)$$

In the first section we investigate some properties of the mutually normal relation. In the second section we prove that many properties are shared by mutually normal operators. In the third section we show that under certain conditions, general mutually normal operators become normal operators. In the fourth and last section we give characterizations of partial isometries and unitary operators in terms of the mutually normal relation.

1 In the first section we investigate some properties of the mutualy normal relation.

Proposition 1.0 *If T, F and S are in $L(H)$ then the following facts follow immediately from* $(*)$

(i) TmT^*.

(ii) TmO *if, and only if,* $T = O$.

(iii) TmT *if, and only if,* T *is normal.*

(iv) If T, F *are unitary operators then* TmF.

(v) TmF *if, and only if,* FmT.

(vi) If TmF *and* FmS *then* TmS^*.

The following result shows that some properties are shared by mutually normal operators.

Proposition 1.1 *Let* $T, F \in L(H)$ *such that* TmF*, then:*

(i) If T *is normal then so is* F.

(ii) If T *is unitary then so is* F.

(iii) If T *is binormal* (TT^* *commutes with* T^*T) *then so is* F.

(iv) If T *is a partial isometry then so is* F.

(v) If T *is compact then so is* F.

(vi) If T is essentially normal ($T^*T - TT^*$ is compact) then so is F.

(vii) $\|T\| = \|F\|$ and in particular if T is a contraction then so is F.

(viii) If T is seminormal (T or T^* is hyponormal) then so is F.

Proof. (i), (ii) and (iii) follow immediately from $(*)$.

(iv) Since T is a partial isometry, T^*T is a projection which implies that TT^* is a projection. Since TmF, $TT^* = F^*F$. Thus F^*F is a projection which means that F is a partial isometry.

(v) Since T is compact, T^*T is compact ([4], p. 427). Since TmF, FF^* is compact. Thus $(F^*)^*F^*$ is compact which implies that F^* is compact. Thus $F = (F^*)^*$ is compact.

(vi) Since T is essentially normal, $T^*T - TT^*$ is compact. Thus $FF^* - F^*F$ is compact. Since any scalar multiple of a compact operator is compact, $F^*F - FF^*$ is compact which means that F is essentially normal.

(vii) Since $TmF, T^*T = FF^*$ and $TT^* = F^*F$. Since for any $T \in L(H)$, $\|T^*T\| = \|T\|^2$ and $\|T^*T\| = \|TT^*\|, \|T\|^2 = \|F\|^2$ which implies that $\|T\| = \|F\|$. Since T is a contraction, $\|T\| \leq 1$ which implies that $\|F\| \leq 1$. Thus F is a contraction.

(viii) Since T is seminormal then T is hyponormal or T^* is hyponormal i.e. $T^*T \geq TT^*$ or $TT^* \geq T^*T$. Since TmF then $FF^* \geq F^*F$ or $F^*F \geq FF^*$, which means that F or F^* is hyponormal. Thus F is seminormal.

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In section two we give some general results involving the mutual normality relation.

Proposition 2.1 *If $T, F \in L(H)$ such that TmF then TF and FT are normal operators.*

Proof. Since $TmF, TT^* = F^*F$ and $T^*T = FF^*$. Using these two equations we get

$$\begin{aligned}(TF)^*(TF) &= F^*T^*TF \\ &= F^*FF^*F \\ &= TT^*TT^* \\ &= TFF^*T^* \\ &= (TF)(TF)^*.\end{aligned}$$

Thus TF is normal. Similarly we show that FT is normal.

If T and F are partial isometries then TF is not necessarily a partial isometry. In the following proposition we show that the product of two mutually normal operators is a partial isometry even if only one of the two operators is a partial isometry.

Proposition 2.2 *If $T, F \in L(H)$ such that TmF and T is a partial isometry then TF is a partial isometry.*

Proof. Since T is a partial isometry,

$$T^*T = T^*TT^*T. \tag{1}$$

Since TmF,

$$T^*T = FF^*. \tag{2}$$

Using (1) and (2) above we obtain

$$\begin{aligned}(TF)^*(TF)(TF)^*(TF) &= F^*T^*TFF^*T^*TF \\ &= F^*T^*TT^*TT^*TF \\ &= F^*T^*TT^*TF \\ &= F^*T^*TF \\ &= (TF)^*(TF).\end{aligned}$$

Thus $(TF)^*(TF)$ is a projection which means that TF is a partial isometry.

Proposition 2.3 *If $T, F \in L(H)$ are positive and mutually normal then $T = F$.*

Proof. Since T, F are positive, $T = T^*$ and $F = F^*$. Thus $T^2 = TT^* = T^*T$ and $F^2 = F^*F$. Since $TmF, TT^* = F^*F$ which implies that $T^2 = F^2$. Thus $T = F$.

Proposition 2.4 *If $T, F \in L(H)$ are isometries and TmF then T and F are unitaries.*

Proof Since T, F are isometries $T^*T = I$ and $F^*F = I$. Since $TmF, T^*T = FF^*$ and $TT^* = F^*F$. Thus $F^*F = FF^* = I$ and $T^*T = TT^* = I$ which means that T and F are unitary operators.

Proposition 2.5 *If T, F are two invertible operators in $L(H)$ then TmF if, and only if, $T^{-1}mF^{-1}$.*

Proof. Suppose first that TmF then $T^*T = FF^*$ and $TT^* = F^*F$. Using these two equations, we get $(T^{-1})^*(T^{-1}) = (T^*)^{-1}(T^{-1}) = (TT^*)^{-1} = (F^*F)^{-1} = (F^{-1})(F^*)^{-1} = (F^{-1})(F^{-1})^*$. Similarly one can show that $(T^{-1})(T^{-1})^* = (F^{-1})^*(F^{-1})$. Thus $T^{-1}mF^{-1}$. Suppose now that $T^{-1}mF^{-1}$, then $(T^{-1})^{-1}m(F^{-1})^{-1}$. Thus TmF.

Definition 2.1 The numerical range, $W(T)$, of an operator $T \in L(H)$ is the set of all complex numbers of the form $\langle Tx, x\rangle$, where x varies over all vectors on the unit sphere i.e. $\|x\| = 1$. The numerical radius, $w(T)$, of T is defined by $w(T) = \sup\{|\lambda| : \lambda \in W(T)\}$.

Proposition 2.6 *If $T, F \in L(H)$ such that TmF then $w(T^*T) = w(F^*F)$.*

Proof. Since TmF then following the proof of Proposition 1.1 (vii), we have $\|T\|^2 = \|F\|^2$. Since T^*T is selfadjoint, it is normal and thus $w(T^*T) = \|T^*T\|$ ([3], p. 117), but $\|T^*T\| = \|T\|^2$, so that $w(T^*T) = \|T\|^2$, and thus $w(T^*T) = w(F^*F)$.

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In this section we show that if T and F are mutually normal operators in $L(H)$ then under certain conditions T and F become normal.

Proposition 3.1 *If $T, F \in L(H)$ such that $TF = FT$ and TmF, then T and F are normal operators.*

Proof. Since $TF = FT, T^*F^* = F^*T^*$. Thus we have

$$TFF^*T^* = FTT^*F^*. \tag{3}$$

Since TmF, $T^*T = FF^*$ and $TT^* = F^*F$ which when we substitute in (3) we get $TT^*TT^* = FF^*FF^* = T^*TT^*T$. Thus we have

$$(TT^*)^2 = (T^*T)^2. \tag{4}$$

Since T^*T and TT^* are positive, then from (4), we conclude that that $TT^* = T^*T$ which implies that T is normal. The normality of F follows from Proposition 1.1 (i).

Proposition 3.2 *If $T, F \in L(H)$ are hyponormal operators such that TmF, then T and F are normal.*

Proof. Since T is hyponormal, $T^*T \geq TT^*$. Since $TmF, T^*T = FF^*$ and $TT^* = F^*F$. Thus $FF^* \geq F^*F$. Since F is hyponormal, $F^*F \geq FF^*$. Thus $FF^* = F^*F$ which means that F is normal and by Proposition 1.1(i), T is normal.

Definition 3.1 $T, F \in L(H)$ are called metrically equivalent if and only if $\|Tx\| = \|Fx\|$ for all $x \in H$.

It can be shown that T and F are metrically equivalent if, and only if, $T^*T = F^*F$.

Proposition 3.3 *If $T, F \in L(H)$ such that TmF then T and F are metrically equivalent if, and only if, T and F are normal.*

Proof. Suppose first that T and F are metrically equivalent, then $T^*T = F^*F$. Since $TmF, T^*T = FF^*$ which implies that $F^*F = FF^*$. Thus F is normal. Hence, by Proposition 1.1(i), T is normal.

Conversely, suppose that T and F are normal operators then $T^*T = TT^*$ and $F^*F = FF^*$. Since $TmF, F^*F = TT^* = T^*T$. Thus T and F are metrically equivalent.

Corollary 3.1 *If $T, F \in L(H)$ such that T is normal and TmF, then $T = UF$ for some unitary operator U.*

Proof. Since T is normal and TmF, then by Proposition 1.1(i), F is normal. Thus, by Proposition 3.3, T and F are metrically equivalent. Let $U : F(H) \rightarrow T(H)$ be defined by $UF(x) = T(x)$ for all $x \in H$, then one can show that U is linear and bijective. Now $\|UF(x)\| = \|Tx\|$ and since T and F are metrically equivalent $\|Tx\| = \|Fx\|$ for all $x \in H$. Thus $\|UF(x)\| = \|Fx\|$ which means that U is isometric. Thus, by ([1], Theorem 1, p. 145), U is unitary and $T = UF$.

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In section four, we give characterizations of partial isometries and unitary operators in terms of the mutually normal relation.

Proposition 4.1 *If $T \in L(H)$ then T is unitary if, and only if, TmT^{-1}.*

Proof. Suppose first that T is unitary then T^{-1} is also unitary. Thus, by Proposition 1.0(iv), TmT^{-1}.

Now suppose that TmT^{-1} then $T^*T = (T^{-1})(T^{-1})^* = (T^*T)^{-1}$. Thus $(T^*T)^2 = I$. Since T^*T is always positive, $T^*T = I$. Similarly one can show that $TT^* = I$. Thus $T^* = T^{-1}$ which implies that T is unitary.

Definition 4.1 The Moore-Penrose pseudoinvers A^+ of an operator $A \in L(H)$ is characterized by the following equations

$$A = AA^+A,\ A^+ = A^+AA^+,\ (AA^+)^* = AA^+,\ (A^+A)^* = A^+A. \qquad (**)$$

It is obvious that A^+A and AA^+ are projections.

Theorem 4.1 *If $T, F \in L(H)$ have Moore-Penrose pseudoinverses T^+ and F^+ respectively, then*

(i) $(T^+)^+ = T$

(ii) $(T^*)^+ = (T^+)^*$

(iii) $(TT^*)^+ = (T^+)^*T^+$ *and* $(T^*T)^+ = T^+(T^+)^*$.

Proof. ([2], p. 8).

Proposition 4.1 *If* $T \in L(H)$ *then* T *is a partial isometry if, and only if,* TmT^+.

Proof. Suppose that T is a partial isometry, then by ([1], p. 153)

$$T = TT^*T. \tag{5}$$

Since $T = TT^+T$,

$$T^* = T^*TT^+. \tag{6}$$

Using (5) and (6) above we get

$$\begin{aligned} TT^* &= TT^*TT^*TT^+ \\ &= TT^*TT^+ \\ &= TT^+. \end{aligned}$$

Thus $TT^* - TT^+ = 0$ which implies that $T(T^* - T^+) = 0$. Since T is one-to-one, $T^* - T^+ = 0$ which means that $T^* = T^+$. Since TmT^* for any $T \in L(H)$, TmT^+.

Conversely, suppose that TmT^+ then $TT^* = (T^+)^*T^+$ which implies that $TT^*TT^* = TT^*(T^+)^*T^+ = T(T^+T)^*T^+ = TT^+TT^+ = TT^+$. Thus $(TT^*)^2$ is a projection which implies that TT^* is a projection, or T is partially isometric.

Proposition 4.2 $T \in L(H)$ *is unitary if, and only if,* TmI.

Proof. Let TmI then $T^*T = TT^* = I$. Thus T is unitary. Suppose now that T is unitary, then, by Proposition 1.0(iv), TmI.

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