King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 242

April 1999

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Global existence and blow up in solutions of a semilinear wave equation with mixed boundary conditions.

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Abstract

We consider the semilinear wave equation $u_{tt} - u_{xx} = f(u, u_t)$ associated with mixed boundary conditions and prove a global existence, as well as, a blow up result.

Keywords : wave equation, blow up, solution energy, global existence, dissipative, antidissipative.

Subject classification 35 L45

1. Introduction

In [6] the following problem

$$
\begin{aligned}
&u_{tt} - u_{xx} = F(x,t,u,u_x,u_t), \qquad x \in I = (0, \;\; 1), \qquad t > 0 \\
&u_x(0,t) - u(0,t) = u_x(1,t) + u(1,t) = 0, \qquad t > 0 \qquad\qquad (1.1)\\
&u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad\qquad x \in I.
\end{aligned}
$$

has been considered and a local existence, as well as an asymptotic expansion result have been established. The authors used a classical contraction argument to prove their theorems. For this purpose they introduced a space $W(M,T)$, which consists of all measurable functions v satisfying

$$v \in L^\infty\left([0,T);\; H^2(I)\right),\; v_t \in L^\infty\left([0,T);\; H^1(I)\right),\; v_{tt} \in L^\infty\left([0,T);\; H^0(I)\right), \quad (1.2)$$

$$||v||^2_{L^\infty([0,T);\; H^2(I))} + ||v_t||^2_{L^\infty([0,T);\; H^1(I))} + ||v_{tt}||^2_{L^\infty([0,T);\; H^0(I))} \leq M^2,$$

and made some assumptions on F and the initial data; More precisely they require that

(H1) $u_0 \in H^2(I)$ and $u_1 \in H^1(I)$, with

$$u_0'(0) - u_0(0) = u_0'(1) + u_0(1) = 0,\; u_1'(0) - u_1(0) = u_1'(1) + u_1(1) = 0 \qquad (1.3)$$

(H2) $F \in C^1\left([0,1] \text{ x } [0,\infty) \text{ x } \mathbb{R}^3\right).$

They also defined the following equivalent norm on the space $H^1(I)$

$$\| v \|^2_{H^1(I)} = v^2(0) + \int_0^1 v_x^2(x)dx \tag{1.4}$$

and proved the following :

Theorem 1. *Let (H1) and (H2) hold. Then there exist $M > 0$ (large enough) and $T > 0$ such that the problem (1.1) has a unique weak solution $u \in W(M,T)$.*

Remark 1.1. By using the result of [9], we conclude that

$$u \in \bigcap_{k=0}^{2} C^k \left([0, \ T); H^{2-k}(I)\right). \tag{1.5}$$

As the authors pointed out, their result is a relative generalization of [1]. It is also worth mentioning that (1.1), for different forms of the function F and different types of boundary conditions, has been dicussed by Aregba and Hanouzit [2], Nguyen and Alain [7] and many others (See [6] for more references).

In this paper, we consider the above problem and show that the solution can exist globally for small enough initial data if the forcing term F takes certain form and it blows up for some other forms. Our method of proving the blow up is due to Levine [4], [5]. This work is divided into three sections. In the first one, we establish the global existence result. The other two sections are devoted to the blow up in the dissipative and the antidissipative cases.

2. Global Existence.

In this section, we assume that $F(x,t,u,u_x,u_t) := f(u,u_t)$, where f satisfies the following conditions

(GL1) $f \in C^2(\mathbb{R}^2)$

(GL2) $f(0,0)=0, \quad f_u(0,0)=0, \qquad f_{u_t}(0,0)=-\alpha, \ \alpha>0.$

Therefore problem (1.1) takes the form

$$\begin{aligned} &u_{tt}-u_{xx}=f(u,u_t),. \qquad x\in I, \qquad t>0 \\ &u_x(0,t)-u(0,t)=u_x(1,t)+u(1,t)=0, \qquad t>0 \\ &u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \qquad x\in I. \end{aligned} \tag{2.1}$$

Theorem 2. *Let (GL1) and (GL2) be satisfied. Then there exists a positive constant δ such that for any intial data satisfying (H1) and*

$$||u_0||_2^2+||u_1||_1^2 \le \delta^2,. \tag{2.2}$$

the problem (2.1) has a unique global solution

$$u \in \bigcap_{k=0}^{2} C^k\left([0, \ \infty); H^{2-k}(I)\right).$$

Proof. to establish the global existence, it suffices to show that

$$Sup \ \{ \ ||u(.,t)||_2^2+||u_t(.,t)||_1^2, \ \ 0\le t<T \ \} \tag{2.3}$$

remains bounded independantly of T . For this aim, we set

$$\varepsilon(t) := \int_0^1 (u_t^2 + u_x^2 + u_{xx}^2 + u_{xt}^2 + u_{tt}^2)(x,t)dx$$

$$+ 2\alpha \int_0^t \int_0^1 (u_t^2 + u_{tt}^2)(x,s)dxds + u^2(0,t) + u_t^2(0,t),$$

$$\varepsilon_0 := \int_0^1 (u_1^2 + u_0'^2 + u_0''^2 + u_1'^2 + u_2^2)(x)dx + u_0^2(0) + u_1^2(0) + u_0^2(1) + u_1^2(1)$$

where

$$u_2 = u_0'' + f(u_0, u_1),$$

and

$$Q(t) := Sup\{|u(x,s)| + |u_t(x,s)|, \quad 0 \le x \le 1, \quad 0 \le s < t\}$$

Remark 2.1. Note that $u_2 \in H^0(I)$ by virtue of (H1), (GL1), and (GL3) and $Q(t) \le \sqrt{\varepsilon(t)}$ by well-known Sobolev inequalities.

We first rewrite equation (2.1) as

$$u_{tt} - u_{xx} = -\alpha u_t + h(u, u_t), \qquad x \in I, \qquad t > 0, \tag{2.4}$$

where

$$h(u_t) = \alpha u_t + f(u, u_t),$$

then multiply it by u_t, and integrate over I x $(0,t)$, to get

$$\int_0^1 (u_t^2 + u_x^2)(x,t)dx + u^2(0,t) + u^2(1,t) + 2\alpha \int_0^t \int_0^1 u_t^2(x,s)dxds \tag{2.5}$$

$$= \int_0^1 (u_1^2 + u_0'^2)(x)dx + u_0^2(0) + u_0^2(1) + 2\int_0^t \int_0^1 h(u, u_t)u_t(x, s)dxds.$$

Next, we apply the difference operator

$$\Delta_\xi W(x,t) := W(x, t+\xi) - W(x,t)$$

to equation (2.4) to obtain

$$\Delta_\xi u_{tt} - \Delta_\xi u_{xx} = -\alpha \Delta_\xi u_t + \Delta_\xi h(u, u_t). \tag{2.6}$$

By multiplying (2.6) by $\Delta_\xi u_t$, integrating over I x $(0, t)$, using integrating by parts, dividing by ξ^2, and letting ξ go to zero, we get

$$\int_0^1 (u_{tt}^2 + u_{xt}^2)(x,t)dx + u_t^2(0,t) + u_t^2(1,t) + 2\alpha \int_0^t \int_0^1 u_{tt}^2(x,s)dxds \tag{2.7}$$

$$= \int_0^1 (u_2^2 + u_1'^2)(x)dx + u_1^2(0) + u_1^2(1) + 2\int_0^t \int_0^1 (h(u, u_t))_t \; u_{tt}(x,s)dxds.$$

To handle the last term in (2.5), we exploit Taylor expansion of h about $(0,0)$; i.e :

$$h(u, u_t) = \frac{1}{2}u^2 h_{uu}(\lambda u, \lambda u_t) + uu_t h_{uu_t}(\lambda u, \lambda u_t) + \frac{1}{2}\; u_t^2 h_{u_t u_t}(\lambda u, \lambda u_t), \quad 0 < \lambda < 1;$$

hence we have

$$|2\int_0^t \int_0^1 h(u, u_t)u_t(x,s)dxds.| \le MQ(t)\varepsilon(t), \tag{2.8}$$

where $M > 1$ is generic positive constant depending on the upper bounds of the partial derivatives, up to second order, of f on $[0,\ 1] \times [0,\ 1]$. By estimating the last term in (2.7), in a similar way, and combining (2.5),(2.7), and (2.8), using (GL2), we arrive at

$$\int_0^1 (u_t^2 + u_x^2 + u_{tt}^2 + u_{xt}^2)(x,t)dx + 2\alpha \int_0^t \int_0^1 (u_t^2 + u_{tt}^2)(x,s)dxds \tag{2.9}$$

$$+u^2(0,t) + u^2(1,t) + u_t^2(0,t) + u_t^2(1,t) \leq \varepsilon_0 + MQ(t)\varepsilon(t).$$

By using (2.1), (GL1), and (GL2), we have the estimate

$$\int_0^1 u_{xx}^2(x,t)dx \leq M\varepsilon_0 + MQ(t)\varepsilon(t). \tag{2.10}$$

Finally a combination of (2.9) and (2.10) leads to

$$\varepsilon(t) \leq M\varepsilon_0 + MQ(t)\varepsilon(t), \tag{2.11}$$

provided that

$$0 \leq u \leq 1, \quad 0 \leq u_t \leq 1, \quad x \in I, \ 0 \leq t < T. \tag{2.12}$$

To this end, we choose $k > 0$ so that $M\sqrt{k} < 1/4$ and δ so that $M\,\delta^2 < k/4$. Therefore we conclude, from (2.11), that if $\varepsilon(t) \leq k$ for some t in $[0,\ T)$ then $\varepsilon(t) \leq k/3$; hence by continuity we have $\varepsilon(t) \leq k \ \forall \ t \in [0,\ T)$ and (2.12) is satified provided that $\varepsilon(0) \leq \frac{1}{3}k$. Of course this can be done by choosing δ small enough. Therefore (2.3) holds which implies that $T = \infty$ (See [8]).

Remark 2.1. The global existence can be obtained even if (GL1) is replaced by $f \in C^2(\Omega)$, where Ω is a neighbourhood of $(0,0)$.

3. Blow up for dissipative equations.

In this section we study the case where $f(u,u_t) = -u_t + b|u|^p u$, $b > 0$, and show that, under appropriate conditions on the initial data and p, the solution collapses in finite time. We thus consider

$$u_{tt} - u_{xx} = -u_t + b|u|^p u, \quad x \in I = (0, \ 1), \quad t > 0, \tag{3.1}$$

together with the initial and boudary conditions (1.1) and require, in addition to (H1), that

(B1) $\quad p > \sqrt{2}$

(B2) $\quad \int_0^1 u_0(x)u_1(x)dx > 0$

(B3) $\quad \int_0^1 [u_1^2(x) + | u_0'(x)|^2]dx - \frac{2b}{p+2}\int_0^1 |u_0(x)|^{p+2}dx + u_0^2(0) + u_0^2(1) \leq 0.$

Theorem 3. *Let (H1),(B1) - (B3) be satisfied. Then the solution of (3.1), with initial and boundary conditions (1.1), blows up in finite time.*

To prove this theorem, we first establish two lemmas. For this aim we define the formal energy of the solution

$$E(t) := \int_0^1 [u_t^2 + {u_x}^2 - \frac{2b}{p+2}|u|^{p+2}](x,t)dx + u^2(0,t) + u^2(1,t). \tag{3.2}$$

By multiplying equation (3.1) by u_t and integrating over I, we easily see that

$E'(t) = -\int_0^1 u_t^2(x,t)dx$; hence $E(t) \leq E(0) \leq 0$ by (B3).

Lemma 3.1. *Assume that (H1),(B1) - (B3) hold. Then*

$$F(t) := \frac{1}{2}\int_0^1 u^2(x,t)\,dx > 0, \quad \forall\; t \geq 0. \tag{3.3}$$

Proof. We differentiate F twice to get

$$F'(t) = \int_0^1 u(x,t)u_t(x,t)\,dx, \quad F''(t) = \int_0^1 [uu_{tt} + u_t^2](x,t)\,dx.$$

Straightforward computations, using equation (3.1) and integration by parts, yield

$$F''(t) = \int_0^1 [u_t^2 - u_x^2 + b|u|^{p+2} - uu_t](x,t)\,dx - u^2(0,t) - u^2(1,t) \tag{3.4}$$

$$= -(\frac{p}{2}+1)E(t) + \frac{p}{2}[u^2(0,t)+u^2(1,t)] + (2+\frac{p}{2})\int_0^1 u_t^2(x,t)\,dx$$

$$+\frac{p}{2}\int_0^1 u_x^2(x,t)\,dx - \int_0^1 uu_t(x,t)\,dx$$

$$\geq -\,(\frac{p}{2}+1)E(t) + \frac{p}{2}[u^2(0,t)+u^2(1,t)] \;+\frac{p}{2}\int_0^1 u_x^2(x,t)\,dx$$

$$+(2+\frac{p}{2}-\frac{1}{2\gamma})\int_0^1 u_t^2(x,t)\,dx - \frac{\gamma}{2}\int_0^1 u^2(x,t)\,dx, \quad \forall\,\gamma > 0,$$

by Young's inequality. We then use

$$\int_0^1 u^2(x,t)dx \leq 2\,u^2(0,t) + 2\int_0^1 u_x^2(x,t)dx \tag{3.5}$$

to arrive at

$$\begin{aligned} F''(t) \;\geq\; & -(\frac{p}{2}+1)E(t) + (\frac{p}{2}-\gamma)u^2(0,t) + \frac{p}{2}u^2(1,t) \\ & + (2+\frac{p}{2}-\frac{1}{2\gamma})\int_0^1 u_t^2(x,t)\,dx + (\frac{p}{2}-\gamma)\int_0^1 u_x^2(x,t)\,dx \end{aligned}$$

By choosing $\gamma = \frac{p}{2}$ we deduce

$$F''(t) \geq (2+\frac{p}{2}-\frac{1}{p})\int_0^1 u_t^2(x,t)\,dx \geq 0, \quad \forall\, p \geq \sqrt{6}-2 \tag{3.6}$$

which shows that $F'(t)$ is increasing; consequently $F'(t) \geq F'(0) > 0$ by (B2). Hence (3.3) is established.

Next we set $G(t) := F^{-\beta}(t)$ and state

Lemma 3.2. *Assume that (H1),(B1) - (B3) hold and* $0 < \beta \leq (p^2-2)/4p$. *Then*

$$G'(t) < 0, \qquad G''(t) \leq 0, \qquad \forall\, t \geq 0 \tag{3.7}$$

Proof. Differentiation of G twice gives

$$G'(t) = -\beta F^{-(\beta+1)}(t)F'(t), \quad G''(t) = -\beta F^{-(\beta+2)}(t)Q(t) \tag{3.8}$$

where

$$Q(t) = F(t)F''(t) - (\beta+1)F'^2(t)$$

$$= \frac{1}{2}\left(\int_0^1 u^2(x,t)\,dx\right)F"(t) - (\beta+1)\left[\int_0^1 u(x,t)u_t(x,t)\,dx\right]^2.$$

By using (3.6) and Holder inequality, we obtain

$$Q(t) \geq \frac{1}{2}\int_0^1 u^2(x,t)\,dx(2+\frac{p}{2}-\frac{1}{p})\int_0^1 u_t^2(x,t)\,dx - (\beta+1)\int_0^1 u^2(x,t)dx\int_0^1 u_t^2(x,t)\,dx$$

$$\geq \frac{1}{2}\int_0^1 u^2(x,t)\,dx(\frac{p}{2}-\frac{1}{p}-2\beta)\int_0^1 u_t^2(x,t)\,dx \geq 0,$$

which implies that $G"(t) \leq 0$; hence $G'(t) < 0$, $\forall\ t$ since $G'(0) < 0$. Therefore (3.7) is established.

Proof of the theorem. Taylor expansion of G, using (3.7), yields

$$G(t) \leq G(0) + t\,G'(0), \qquad \forall t, \tag{3.9}$$

which shows that $G(t)$ must vanish at a time $t_m \leq -G(0)/G'(0)$. Consequently $F(t)$ must become infinite at time t_m.

Remark 3.1. The above calulations, not only prove the blow up but also give an upper bound to the blow up time

$$t^* = \frac{4p}{(p^2-\ 2)}\frac{\int_0^1 u_0^2(x)dx}{\int_0^1 u_0u_1(x)dx}$$

.

Remark 3.2. Note that no assumption has been made on the size of the initial data. In fact the blow up takes place even for small data provided that (H1), (B2), and

(B3) are satisfied. However, a strong nonlinearity condition ($p > \sqrt{2}$) on the source term is needed to overcome the damping effect of u_t.

Remark 3.3. One might investigate a situation where $p > 0$ with large initial data. In fact a close problem has been discussed in [3], where the authors claimed a blow up result but the argument they used is completely false.

4. Blow up for antidissipative equations.

In this section we consider the situation where $f(u, u_t) = u_t + b|u|^p u$, $b > 0$, and show that, under weaker conditions on the nonlinearity of the source tern, the solution blows up in finite time; hence equation (1.1) takes the form

$$u_{tt} - u_{xx} = u_t + b|u|^p u, \quad x \in I, \qquad t > 0. \tag{4.1}$$

We also require, in addition to (H1), that

(A1) $\quad p > 0$

(A2) $\quad \int_0^1 u_0(x)[u_1(x) - \frac{1}{2}u_0(x)]dx > 0$

(A3) $\quad \int_0^1 [u_1^2 + u_0'^2 - u_0u_1](x)dx - \frac{2b}{p+2}\int_0^1 |u_0|^{p+2}(x)dx + u_0^2(0) + u_0^2(1) \leq 0.$

Theorem 4. *Let (H1),(A1) - (A3) be satisfied. Then the solution of (4.1), with initial and boundary conditions (1.1), blows up in finite time.*

Proof. We set

$$v(x,t) := u(x,t)e^{-t/2}$$

and substitute in (4.1) to get

$$v_{tt} - v_{xx} - \frac{1}{4}v = b|v|^p v e^{pt/2}, \quad x \in I, \quad \forall\, t > 0. \tag{4.2}$$

By multiplying (4.2) by v_t and integrating over I, we obtain

$$E'(t) = \frac{-bp}{p+2} e^{pt/2} \int_0^1 |v|^{p+2}(x,t)dx \leq 0, \qquad \forall\, t \geq 0,$$

where

$$E(t) = \int_0^1 [v_t^2 + v_x^2 - \frac{1}{4}v^2](x,t)dx - \frac{2b}{p+2} e^{pt/2} \int_0^1 |v|^{p+2}(x,t)dx + v^2(0,t) + v^2(1,t).$$

By using (A3) and noting that

$$v_0 = u_0, \quad v_1 = u_1 - \frac{1}{2}\, u_0,$$

we conclude that

$$E(t) \leq E(0) \leq 0, \qquad \forall\, t \geq 0.$$

As in the previous section, we set $F(t) := \frac{1}{2}\int_0^1 v^2(x,t)\, dx$ and differentiate twice, with respect to t, to get $F'(t) = \int_0^1 v(x,t)v_t(x,t)\, dx$ and

$$F''(t) = \int_0^1 [vv_{tt} + v_t^2](x,t)\, dx = (2 + \frac{p}{2}) \int_0^1 v_t^2 \tag{4.3}$$

$$+\frac{p}{2}\int_0^1 v_x^2 - \frac{p}{8}\int_0^1 v^2 + \frac{p}{2}[v^2(0,t) + v^2(1,t)] - (1 + \frac{p}{2})E(t).$$

Again by using (3.5), we easily arrive at

$$F''(t) \geq (2+\frac{p}{2})\int_0^1 v_t^2 + \frac{p}{4}\int_0^1 v_x^2 + \frac{p}{4}v^2(0,t) + \frac{p}{2}v^2(1,t) - (1+\frac{p}{2})E(t) \geq 0, \qquad (4.4)$$

which implies that $F'(t) \geq F'(0) > 0$ by virtue of (A2) and (4.4). Therefore $F(t) > 0$, $\forall\, t \geq 0$.

Next we define $G(t) := F^{-\beta}(t),\ 0 < \beta \leq \frac{p}{4}$, and differentiate twice to get

$$G'(t) = -\beta F^{-(\beta+1)}(t)F'(t), \quad G''(t) = -\beta F^{-(\beta+2)}(t)Q(t) \qquad (4.5)$$

where

$$Q(t) = F(t)F''(t) - (\beta+1)F'^2(t)$$

$$= \frac{1}{2}\int_0^1 v^2\{(2+\frac{p}{2})\int_0^1 v_t^2 + \frac{p}{2}\int_0^1 v_x^2 - \frac{p}{8}\int_0^1 v^2$$

$$+\frac{p}{2}[v^2(0,t)+v^2(1,t)] - (1+\frac{p}{2})E(t)\} - (\beta+1)[\int_0^1 vv_t]^2$$

$$\geq \frac{1}{2}\int_0^1 v^2[(\frac{p}{2}-2\beta)\int_0^1 v_t^2 + \frac{p}{4}\int_0^1 v_x^2 + \frac{p}{4}v^2(0,t) + \frac{p}{2}v^2(1,t)] \geq 0.$$

Therefore we conclude, from (4.5), that $G''(t) \leq 0$ and $G'(t) < 0$, $\forall\, t \geq 0$. Finally the same argument as in the dissipative case completes the proof.

Remark 4.1. Note that, contrary to the dissipative case, no strong nonlinearity assumption on the source term is needed. In fact we may have blow up even for small intial data provided they satisfy (H1), (A2), and (A3)..

Remark 4.2. The above calulations give an upper bound to the blow up time

$$t^* = \frac{4 \int_0^1 u_0^2(x)dx}{p \int_0^1 u_0(u_1 - u_0)(x)dx}$$

Acknowledgement The author would like to express his sincere thanks to KFUPM for its support.

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