King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 246

August 1999

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ABSTRACT. The problem of uniform approximants subject to Hermite interpolatory constraints is considered with an alternate approach. Uniqueness and the convergence aspects of this problem are also discussed. Our apprroach is based on a work of P. Kirchberger and a generalization of Weierstrass approximation theorem.

1. Introduction

Let π_m denote the set of all polynomials of degree $\le m$. Let $[a,b]$ be a closed finite interval and $C[a,b]$ the space of real continuous functions on $[a,b]$ with uniform norm

$$\|h\|_\infty = \max_{x\in[a,b]} |h(x)| .$$

Here we discuss uniform approximation of a prescribed $f \in C[a,b]$ by the polynomials that are also Hermite interpolants to a set of given data at a finite number of pre-assigned points in the interval $[a,b]$. More precisely, we consider the following problem due to Loeb *et al* [6]:

Problem (P). *Suppose that k, m and $n_i, i = 1,2,...k$, are positive integers with $m \ge (\sum_{i=1}^k n_i) - 1$ and that $\{u_i\}_{i=1}^k$ is a subset of $[a,b]$ satisfying $a \le u_1 < u_2 < ... < u_k \le b$. Then the problem is to find a best uniform approximant to a given $f \in C[a,b]$ from the class*

$$\Phi_{m,\beta} = \left\{\phi \in \pi_m : \phi^{(j)}(u_i) = \beta_{ij}, 1 \le i \le k, 0 \le j \le n_i - 1\right\} \tag{1.1}$$

where

$$S_\beta := \left\{\beta_{ij} : 1 \le i \le k, 0 \le j \le n_i - 1\right\} \tag{1.2}$$

is a subset of the real numbers with $\beta_{i0} = f(u_i)$.

This problem originates from the work of S. Paszkowski [8,9] who studied it by imposing the interpolatory conditions only on the values of interpolating polynomials at

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AMS Classification: 41A05, 41A10, 41A29, 41A52

k distinct points of $[a,b]$ in the sense of Lagrange interpolation. In doing this, he followed the classical Tchebycheff approach of approximating a continuous function by elements of an n-dimensional Haar subspace. In [1] F. Deutsch discussed Paszkowski's results [8, Theorems 2 and 5] with a different method . Deutsch's work bases on a characterization theorem of best approximation that involves extreme points of the closed unit ball in the dual of the underlying space [2, Corollary 2.6]. Later, H.L.Loeb *et al* [6] extended the work of Deutsch by constraining the uniform approximants with Hermite interpolatory conditions. In fact, they discussed the Problem (P) through the notions of the n-dimensional extended Haar subspace of order ν of $C[a,b]$ [4] and the generalized weight functions [7]. They also established a convergence result by giving a generalization of a theorem of de La Vallée Poussin [3, p.77].

In the present paper we continue the study of the Problem (P). Our approach for its solution is based on the work of P. Kirchberger [5] that deals with extreme values of error function. Uniqueness and convergence problems are also addressed in our work. In doing this, we take into account an extension of Weierstrass approximation theorem[11].

2. NOTATIONS AND REFORMULATION OF PROBLEM (P)

For the sake of convenience, we set $I_k := \{1,2,...,k\}, N_i := \{0,1,...,n_i-1\}$, $s := \sum_{i\in I_k} n_i$ and

$$W(x) := \prod_{i\in I_k}(x-u_i)^{n_i}. \tag{2.1}$$

The notation $H_{s-1}(x,S_\beta)$,where S_β is given in (1.2), will stand for the polynomial of degree $\le s-1$ that satisfies the following conditions:

$$H_{s-1}^{(j)}(u_i,S_\beta) = \beta_{ij}, \forall i\in I_k, \forall j \in N_i.$$

For every $f\in C[a,b]$ we define

$$f_{H,\beta}(x) := f(x) - H_{s-1}(x,S_\beta) \tag{2.2}$$

The error funtion corresponding to $f,g\in C[a,b]$ and the set of its extreme points in $[a,b]$ will be denoted respectively as follows:

$$\left.\begin{array}{r} e_{f,g}(x) := |f(x)-g(x)| \\ crit(e_{f,g}) := \left\{x\in[a,b] : |e_{f,g}(x)| = \|e_{f,g}\|_\infty\right\} \end{array}\right\}. \tag{2.3}$$

Let π_m^* denote the $m-s+1$ dimensional subspace of π_m generated by the polynomials $x^jW(x), j=0,1,2...,m-s$, where W is given by (2.1) The following remark gives an explicit representation of the elements of $\Phi_{m,\beta}$ (cf (1.1):

Remark 1. *A typical element ϕ^*in the approximating class $\Phi_{m,\beta}$ is of the form*

$$\phi^*(x) = H_{s-1}(x, S_\beta) + q^*(x)$$

where $q^* \in \pi^*_m$. This shows that $\phi^* \in \Phi_{m,\beta}$ is a best approximant to f from the class $\Phi_{m,\beta}$ if and only if $q^* \in \pi^*_m$ is a best approximant to $f_{H,\beta}$ from the class π^*_m. In particular, if $m = s-1$ then $H_{s-1}(x, S_\beta)$ will be the best approximation to f. For this obvious reason we shall assume that $m \geq s$ in rest of the paper.

In view of the above remark, the Problem (P) can be reformulated as

Problem (P*). *For a given function $f \in C[a,b]$, find a best approximation to $f_{H,\beta}$ (cf (2.2)) in the uniform norm from the class π^*_m.*

3. CHARACTERIZATION OF BEST APPROXIMATION

This section deals with a necessary and sufficient condition for a solution of the Problem (P*). We note that every $q \in \pi^*_m$ can be expressed as

$$q(x) = W(x)R_q(x) \tag{3.1}$$

where $R_q(x)$ is a polynomial of degree at most $m-s$. For an $f \in C[a,b]$ and a $q \in \pi^*_m$, we set

$$E_{f_{H,\beta},q}(x) := \frac{f_{H,\beta}(x)}{W(x)} - R_q(x). \tag{3.2}$$

An alternate form of the characterization theorem [6, Theorem 3.1] that solves (P) may be stated as follows:

Theorem 1. *Let $f \in C[a,b]$ such that $f_{H,\beta} \notin \pi^*_m$. Then q^* is a best uniform approximant to $f_{H,\beta}$ from the class π^* if and only if there exist N points $\alpha_i \in crit(e_{f_{H,\beta},q^*})$ satisfying the following conditions:*

a. $N = m - s + 2$

b. $a \leq \alpha_1 < \alpha_2 < ... < \alpha_N \leq b$

c. $sgn(E_{f_{H,\beta},q^*}(\alpha_i)) = (-1)^{i+1} sgn(E_{f_{H,\beta},q^*}(\alpha_1)), \forall i = 2, 3, ..., N.$

Our method of proof is based on the following Lemma which may be found in the standard texts of approximation theory e.g. [9, Lemma 7.1]:

Lemma 1. *Let Y be a linear subspace of $C[a,b]$ and let $h \in C[a,b]$. Then $g^* \in Y$ is a best uniform approximant to f in Y if and only if there does not exist any $g \in Y$ such that*

$$\{h(x) - g^*(x)\}\, g(x) > 0$$

for all $x \in crit(e_{f,g^*})$ (cf (2.2)).

Remark 2. *If we set $Y = \pi_m^*$ and $h = f_{H,\beta}$ in the above Lemma then the necessary and sufficient condition for $q^* \in \pi_m^*$ to be a best approximation to $f_{H,\beta}$ is that there does not exist any $p \in \pi_m^*$ such that*

$$E_{f_{H,\beta},q^*}(x) R_p(x) > 0$$

for all $x \in crit(e_{f_{H,\beta},q^*})$ where $R_p(x)$ and $E_{f_{H,\beta},q^*}$ are respectively given in (3.1) and (3.2). To justify this , it is enough to note that

$$\{f_{H,\beta}(x) - q^*(x)\}\, p(x) = E_{f_{H,\beta},q^*}(x) R_p(x) W^2(x).$$

Remark 3. *If $f_{H,\beta} \notin \pi_m^*$ then $W(x)$ is a non-vanishing function on the compact set $crit(e_{f_{H,\beta},p})$ regardless of the choice of $p \in \pi_m^*$. Consequently, $f_{H,\beta}$ is continuous as well as nowhere zero on $crit(e_{f_{H,\beta},p})$.*

4. Proof of Theorem 1

If q^* is not a best approximation to $f_{H,\beta}$ then by Remark 2, there exists $p \in \pi_m^*$ such that

$$E_{f_{H,\beta},q^*}(x) R_p(x) > 0 \tag{4.1}$$

as x ranges over $crit(e_{f_{H,\beta},q^*})$. We note that R_p being a polynomial of degree $\leq m-s$ (cf (3.1)) changes sign at most at $m-s$ places. Therefore, it follows from (4.1) that $E_{f_{H,\beta},q^*}(x)$ can not change sign more than $m-s$ times as x ranges over $crit(e_{f_{H,\beta},q^*})$. This contradicts the condition (c) of Theorem 1 as $N = m-s+2$.

Conversely, assume that there are N points $\alpha_i \in crit(e_{f_{H,\beta},q^*}), i = 1,2,...,N$, satisfying the conditions (b) and (c) of Theorem 1 but $N \leq m-s+1$. For each $i = 1,2,3,...,N-1$, fix a point $w_i \in (\alpha_i, \alpha_{i+1})$ such that

$$\left.\begin{array}{c} u_j \notin [w_i, \alpha_{i+1}],\ j = 1,2,...,k, \\ (w_i, \alpha_{i+1}] \text{ and } crit(e_{f_{H,\beta},q^*}) \text{ are disjoint,} \\ sgn(E_{f_{H,\beta},q^*}(w_i)) = sgn(E_{f_{H,\beta},q^*}(\alpha_{i+1})). \end{array}\right\} \tag{4.2}$$

The choice of w_i, as required above, directly follows from Remark 3. Now we set

$$\tilde{p}(x) := sgn(E_{f_{H,\beta},q^*}(\alpha_1)) W(x) \prod_{i=1}^{N-1} (w_i - x).$$

Then $\tilde{p} \in \pi_m^*$ with $R_{\tilde{p}}(x) = sgn(E_{f_{H,\beta},q^*}(\alpha_1)) \prod_{i=1}^{N-1}(w_i - x)$. We claim that

$$E_{f_{H,\beta},q^*}(\alpha) R_{\tilde{p}}(\alpha) > 0 \text{ for all } \alpha \in crit(e_{f_{H,\beta},q^*}).$$

This can be seen by restricting α to each set $(\alpha_i, \alpha_{i+1}] \cap crit(e_{f_{H,\beta},q^*})$ for $i = 0, 1, ..., N-1$, where $\alpha_0 = a$, and then using the condition (c) of Theorem 1 alongwith (4.2). Hence by Remark 2, we note that q^* can not be a best uniform approximant to $f_{H,\beta}$ from the class π_m^* This completes the proof. ∎

Remark 4. *An immediate consequence of Theorem 3 is that $H_{s-1}(x, S_\beta) + q^*(x)$ is a best uniform approximant to $f(x)$ from the class $\Phi_{m,\beta}$.*

5. UNIQUENESS

We retain the setting of the previous sections in order to establish the uniqueness of the solution of the Problem (P*). More precisely, we prove the following theorem:

Theorem 2. *There is exactly one best uniform approximant p^* to $f_{H,\beta}$ from π_m^*.*

Proof. If q^* is another best uniform approximant to $f_{H,\beta}$ from π_m^* then there exist N points $\alpha_i \in crit(e_{f_{H,\beta},q^*}), i = 1, 2...., N$, satisfying the conditions (a), (b) and (c) of Theorem 1. Using the properties of best approximant to $f_{H,\beta}$ we observe that $|f_H(\alpha_i) - p^*(\alpha_i)| \leq \left\| e_{f_{H,\beta},p^*} \right\|_\infty = \left\| e_{f_{H,\beta},q^*} \right\|_\infty = |f_{H,\beta}(\alpha_i) - q^*(\alpha_i)|$ for each $i = 1, 2...., N$, and consequently,

$$\left| E_{f_{H,\beta},p^*}(\alpha_i) \right| \leq \left| E_{f_{H,\beta},q^*}(\alpha_i) \right| \tag{5.1}$$

We set $D(x) := R_{p^*}(x) - R_{q^*}(x)$. Then $D \in \pi_{m-s}$ and

$$D(\alpha_i) = E_{f_{H,\beta},q^*}(\alpha_i) - E_{f_{H,\beta},p^*}(\alpha_i), \; i = 1, 2, ..., N. \tag{5.2}$$

Note that if $D(\alpha_i) \neq 0$ for any $i = 1, 2...., N$ then (5.2) provides us $sgn(D(\alpha_i)) = sgn(E_{f_{H,\beta},q^*}(\alpha_i))$. Thus, the polynomial D has either a double zero at α_i, or it has a zero in (α_i, α_{i+1}) implying that $D \equiv 0$. Hence $R_{p^*}(x) = R_{q^*}(x)$, and consequently, $p^* = q^*$. ∎

6. CONVERGENCE

In this section we discuss the convergence of the sequence of best uniform approximants $\{q_k^*\}_{k=s-1}^\infty$ to $f_{H,\beta}$ with the conditions that f is "sufficiently differentiable" and the set S_β (cf Problem (P)) is replaced by

$$S_f = \left\{ f^{(j)}(u_i) : j \in N_i, i \in I_k. \right\}. \tag{6.1}$$

In this case, we shall write $f_{H,f}$ and $\Phi_{m,f}$ instead of $f_{H,\beta}$ and $\Phi_{m,\beta}$(cf (2.2)).

Theorem 3. *Assume that $f \in C^{n^*}[a,b]$ with $n^* = (\max_{i\in I_k} n_i) - 1$ and that the set S_β (cf (1.2)) is replaced by S_f . If $q_m^* \in \pi_m^*$ is the best approximant to $f_{H,f}$ in the sense of Theorem 1 then*

$$\lim_{m\to\infty} \|q_m^* - f_{H,f}\|_\infty = 0$$

Consequently, the sequence $\{q_m^* + H_{s-1}(.,S_f)\}_{m=s-1}^{\infty}$ will converge uniformly to f.

The crux of the proof of this theorem is in an extension of a result based on the Weierstrass approximation theorem [11, p.160]. We state it in the next Lemma without proof which is a routine exercise:

Lemma 2. *For any $f \in C^r[a,b]$, and for a given $\varepsilon > 0$, there exists a polynomial p such that*

$$\left\|f^{(j)} - p^{(j)}\right\|_\infty < \varepsilon$$

for all $j = 0, 1, 2, ..., r$.

Proof of Theorem 3: In the notations of (2.2) and (6.1), we can write

$$f_{H,f}(x) := f(x) - H_{s-1}(x, S_f). \tag{6.2}$$

The polynomial $H_{s-1}(x,S_f)$ due to its interpolation properties may be expressed as

$$H_{s-1}(x,S_f) = \sum_{l\in I_k} \sum_{n\in N_l} f^{(n)}(u_l) L_{n,l}(x) \tag{6.3}$$

where $L_{n,l}(x)$ are the fundamental polynomials of degree $s-1$ satisfying the conditions

$$L_{n,l}^{(j)}(u_i) = \begin{cases} 1 & \text{for } l = i, n = j \\ 0 & \text{otherwise} \end{cases} .$$

For a given $\varepsilon > 0$, we can fix a polynomial p of degree $r > s$ such that (cf Lemma 2)

$$\left\|f^{(j)} - p^{(j)}\right\|_\infty < \frac{\varepsilon}{2\tau},\ j = 0, 1, ..., n^*, \tag{6.4}$$

where $\tau = \max\{1, \lambda\}$ with $\lambda = \max_{x\in[a,b]} \sum_{l\in I_k} \sum_{n\in N_l} |L_{n,l}(x)|$. From (6.1), (6.3) and (6.4) it follows that

$$\max_{x\in[a,b]} |H_{s-1}(x,S_f) - H_{s-1}(x,S_p)| < \frac{\varepsilon}{2}. \tag{6.5}$$

We set $q(x) := p(x) - H_{s-1}(x,S_p)$. Then $q \in \pi_r$ and $q^{(j)}(u_i) = 0$ for $j \in N_i$ and $i \in I_k$. Hence, W as defined in (2.1) is a factor of the polynomial q. This shows that $q \in \pi_r^*$. Using (6.2), (6.3) and (6.5) it can be seen that

$$\|f_{H,f} - q\|_\infty < \varepsilon. \tag{6.6}$$

Now consider the best uniform approximant q_r^* to f_H from π_r^* (cf Theorem 1) and note that

$$\varepsilon > \|f_{H,f} - q\|_\infty \geq \|f_{H,f} - q_r^*\|_\infty \geq \|f_{H,f} - q_m^*\|_\infty$$

for all $m \geq r$. The last inequality follows from the relation $\pi_r^* \subseteq \pi_m^*$. This proves the desired result. ■

Acknowledgment: The author acknowledges with thanks the research facilities availed at King Fahd University of Petroleum and Minerals, Saudi Arabia and the University of Alberta, Canada during the preparation of this paper.

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