Relative Continuity of Direct Sums of M-Injective Modules

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DIRECT SUMS OF $M$-INJECTIVE MODULES

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Abstract
Let $M$ be a left $R$-module and $\mathcal{K}$ be an $M$-natural class with some additional conditions. It is proved that every direct sum of $M$-injective left $R$-modules in $\mathcal{K}$ is $\mathcal{KS}$-continuous (or $\mathcal{KS}$-quasi-continuous) if and only if every direct sum of $M$-injective left $R$-modules in $\mathcal{K}$ is $M$-injective.

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Let $R$ be a ring with identity. It is well known that $R$ is left noetherian if and only if every direct sum of injective left $R$-modules is injective. Based on this, many characterizations of left noetherian rings by generalized injectivity of some left $R$-modules have been obtained. For example, it was shown that $R$ is left noetherian if and only if every direct sum of injective left $R$-modules is continuous (or quasi-continuous) (cf. [5]). On the other hand, Albu, Nastasescu, Golan, Goldman, Stenstrom, Teply, Enochs, Ahsan and others have studied the situations when all direct sums of non-singular injective left $R$-modules are injective, when all direct sums of $\tau$-torsionfree injective left $R$-modules are injective for a hereditary torsion theory $\tau$, and when all direct sums of $\tau$-torsion injective left $R$-modules are injective for a stable hereditary torsion theory $\tau$. These results are well presented in Golan's book "Torsion Theories", and have been generalized in [12] by considering when all direct sums of $M$-injective left $R$-modules in an $M$-natural class $\mathcal{K}$ are $M$-injective. In this note we consider when all direct sums of $M$-injective left $R$-modules in an $M$-natural class $\mathcal{K}$ are $\mathcal{KS}$-continuous (or $\mathcal{KS}$-quasi-continuous). We will
show that for an \( M \)-natural class \( \mathcal{K} \), all direct sums of \( M \)-injective left \( R \)-modules in \( \mathcal{K} \) are \( \mathcal{K} \)-continuous (or \( \mathcal{K} \)-quasi-continuous) if and only if all direct sums of \( M \)-injective left \( R \)-modules in \( \mathcal{K} \) are \( M \)-injective.

Throughout this note we write \( A \leq_e B \ (A|B) \) to denote that \( A \) is an essential submodule (a direct summand) of \( B \).

Let \( M \) be a left \( R \)-module. We say that a left \( R \)-module \( N \) is subgenerated by \( M \), or that \( M \) is a subgenerator for \( N \), if \( N \) is isomorphic to a submodule of an \( M \)-generated module. Following [11], we denote by \( \sigma[M] \) the full subcategory of \( R \)-Mod whose objects are all \( R \)-modules generated by \( M \). By [11, 17.9], every module \( N \) in \( \sigma[M] \) has an injective hull \( I(N) \) in \( \sigma[M] \), which is also called an \( M \)-injective hull of \( N \). It is known that the \( M \)-injective hulls of a left \( R \)-module in \( \sigma[M] \) are unique up to isomorphism. In the following, we always denote by \( I(N) \) the \( M \)-injective hull of \( N \) for any left \( R \)-module \( N \in \sigma[M] \).

According to [2], a subclass \( \mathcal{K} \) in \( \sigma[M] \) which is closed under submodules, direct sums, isomorphic copies, and \( M \)-injective hulls is called an \( M \)-natural class. There exist a large number of examples of \( M \)-natural classes. Among them are \( \sigma[M] \) and all natural classes in the sense of [9]. In particular, hereditary torsionfree classes, stable hereditary torsion classes, and saturated classes in the sense of Dauns (cf. [1]) are examples of \( M \)-natural classes.

For an \( M \)-natural class \( \mathcal{K} \) and a left \( R \)-module \( N \), we denote by \( H_{\mathcal{K}}(N) \) the set 
\[
\{ L \leq N|N|L \in \mathcal{K} \}\.
\]

Let \( M, N \) be left \( R \)-modules. Define the family 
\[
\mathcal{A}(N, M) = \{ A \subseteq M | \exists X \subseteq N, \exists f \in \text{Hom}(X, M), f(X) \leq_e A \}.
\]

Consider the properties

\( \mathcal{A}(N, M) -(C_1) \) : For all \( A \in \mathcal{A}(N, M) \), \( \exists A^*|M \), such that \( A \leq_e A^* \).

\( \mathcal{A}(N, M) -(C_2) \) : For all \( A \in \mathcal{A}(N, M) \), if \( X|M \) is such that \( A \cong X \), then \( A|M \).
For all \( A \in \mathcal{A}(N, M) \) and \( X|M \), if \( A|M \) and \( A \cap X = 0 \), then \( A \oplus X|M \).

According to [7], \( M \) is said to be \( N \)-extending, \( N \)-quasi-continuous or \( N \)-continuous, respectively, if \( M \) satisfies \( \mathcal{A}(N, M)-(C_1) \), \( \mathcal{A}(N, M)-(C_1) \) and \( \mathcal{A}(N, M)-(C_2) \), \( \mathcal{A}(N, M)-(C_1) \) and \( \mathcal{A}(N, M)-(C_2) \).

**Lemma 1** ([7, Proposition 2.4]). A left \( R \)-module \( M \) is (quasi-) continuous (cf. [2]) if and only if \( M \) is \( M \)-(quasi-) continuous if and only if \( M \) is \( N \)-(quasi-) continuous for every left \( R \)-module \( N \).

Given an \( M \)-natural class \( \mathcal{K} \), a left \( R \)-module \( N \) is called \( \mathcal{K} \)-cocritical if \( N \in \mathcal{K} \) and \( N/P \notin \mathcal{K} \) for any \( 0 \neq P \subset N \).

**Definition 2.** Let \( \mathcal{K} \) be an \( M \)-natural class. A left \( R \)-module \( M \) is said to be \( \mathcal{K}S \)-extending, \( \mathcal{K}S \)-quasi-continuous or \( \mathcal{K}S \)-continuous, respectively, if for any direct sum \( C = \bigoplus_{i \in I} C_i \) of \( \mathcal{K} \)-cocritical modules \( C_i (i \in I) \), \( M \) is \( C \)-extending, \( C \)-quasi-continuous or \( C \)-continuous.

Clearly (quasi-) continuous modules are \( \mathcal{K}S \)-(quasi-) continuous. The following example shows that the converse is not true.

**Example 3** (cf. [6]). Let \( R \) be a left noetherian \( V \)-ring which is not artinian semisimple (see, for example, [3]). Then, by [7, Corollary 3.7], every left \( R \)-module is \( N \)-continuous for every semisimple left \( R \)-module \( N \). Thus every left \( R \)-module is \( \mathcal{K}S \)-continuous, where \( \mathcal{K} = R \)-Mod. If all left \( R \)-modules are quasi-continuous, then for every left \( R \)-module \( M \), \( M \oplus E(M) \) is quasi-continuous, and so \( M \) is injective by [8, Lemma C], where \( E(M) \) denotes the injective hull of \( M \). Thus \( R \) is artinian semisimple, a contradiction. Hence there exists a left \( R \)-module \( M \) which is not quasi-continuous.

**Lemma 4.** Any direct summand of a \( \mathcal{K}S \)-continuous (\( \mathcal{K}S \)-quasi-continuous) left \( R \)-module is \( \mathcal{K}S \)-continuous (\( \mathcal{K}S \)-quasi-continuous).

**Proof.** It follows from the fact that condition \( \mathcal{A}(N, M) - (C_i) \), \( (i = 1, 2, 3) \) is inherited
by direct summands of $M$ ([7, Proposition 2.4]).

**Lemma 5** ([7]). If $M$ is $N$-(quasi-) continuous and $A \in \mathcal{A}(N, M)$ is a direct summand of $M$, then $A$ is indeed (quasi-) continuous.

Let $c$ be any cardinal. A left $R$-module $M$ is called $c$-limited provided every direct sum of non-zero submodules of $M$ contains at most $c$ direct summands (cf. [10]).

We say an $M$-natural class $\mathcal{K}$ satisfies ($\ast$) (cf. [12]), if for any cyclic submodule $N$ of $M$, and every ascending chain $N_1 \leq N_2 \leq \cdots$ with each $N_i \in H_{\mathcal{K}}(N)$, the union $\bigcup N_i$ belongs to $H_{\mathcal{K}}(N)$.

**Theorem 6.** The following conditions are equivalent for an $M$-natural class $\mathcal{K}$ with ($\ast$).

1. $H_{\mathcal{K}}(A)$ has ACC for any cyclic (or finitely generated) submodule $A$ of $M$.

2. Every direct sum of $M$-injective left $R$-modules in $\mathcal{K}$ is $M$-injective.

3. Every direct sum of $M$-injective left $R$-modules in $\mathcal{K}$ is $\mathcal{KS}$-continuous.

4. Every direct sum of $M$-injective left $R$-modules in $\mathcal{K}$ is $\mathcal{KS}$-quasi-continuous.

5. There exists a cardinal $c$ such that every direct sum of $M$-injective left $R$-modules in $\mathcal{K}$ is the direct sum of a $c$-limited module and a $\mathcal{KS}$-continuous module.

6. There exists a cardinal $c$ such that every direct sum of $M$-injective left $R$-modules in $\mathcal{K}$ is the direct sum of a $c$-limited module and a $\mathcal{KS}$-quasi-continuous module.

**Proof.** (1) $\iff$ (2). Follows from [12, Theorem 2.4].

(2) $\iff$ (3). Suppose that $N = \bigoplus_{i \in I} N_i$ is the direct sum of $M$-injective left $R$-modules $N_i \in \mathcal{K}$, $i \in I$. Then $N$ is $M$-injective by (2). On the other hand, $N$ is in $\mathcal{K}$, and so $N \in \sigma[M]$. Thus $N$ is quasi-injective. Now clearly $N$ is $\mathcal{KS}$-continuous by Lemma 1.

(3) $\iff$ (4). Clear.

(4) $\iff$ (1). By [12, Theorem 2.5], it is sufficient to show that every direct sum of $M$-injective hulls of $\mathcal{K}$-cocritical left $R$-modules is $M$-injective.
Let \( C_i, \ i \in I \) be \( K \)-cocritical left \( R \)-modules. Then \( C_i \in K, \ i \in I \). Set

\[
N = \left( \bigoplus_{i \in I} I(C_i) \right) \oplus I\left( \bigoplus_{i \in I} I(C_i) \right), \quad L = N \oplus I(N).
\]

Then clearly \( L \) is a direct sum of \( M \)-injective left \( R \)-modules. Since \( K \) is closed under direct sums and \( M \)-injective hulls, it follows that \( L \) is a direct sum of \( M \)-injective left \( R \)-modules in \( K \). Thus \( L \) is \( KS \)-quasi-continuous. Denote

\[
S = \left( \bigoplus_{i \in I} C_i \right) \oplus \left( \bigoplus_{i \in I} C_i \right).
\]

Then \( L \) is \( S \)-quasi-continuous. For the submodule \( A = N \oplus 0 \) of \( L \), define an \( R \)-homomorphism \( f : S \to L \) as the induced \( R \)-homomorphism \( S = \left( \bigoplus_{i \in I} C_i \right) \oplus \left( \bigoplus_{i \in I} C_i \right) \to \left( \bigoplus_{i \in I} I(C_i) \right) \oplus I\left( \bigoplus_{i \in I} I(C_i) \right) \oplus 0 \) (by the natural maps \( C_i \to I(C_i) \) and \( \bigoplus_{i \in I} C_i \to I\left( \bigoplus_{i \in I} I(C_i) \right) \)). Since \( C_i \leq e I(C_i) \), we have

\[
\bigoplus_{i \in I} C_i \leq e \bigoplus_{i \in I} I(C_i) \leq e I\left( \bigoplus_{i \in I} I(C_i) \right).
\]

Thus

\[
f(S) = \left( \bigoplus_{i \in I} C_i \oplus \bigoplus_{i \in I} C_i \right) \oplus 0 \\
\leq e \left( \bigoplus_{i \in I} I(C_i) \oplus I\left( \bigoplus_{i \in I} I(C_i) \right) \right) \oplus 0 = A.
\]

This means that \( A \in A(S, L) \). By Lemma 5, it follows that \( A \) is quasi-continuous. Thus \( N \) is quasi-continuous. By \([8, \text{Lemma C}]\), \( \bigoplus_{i \in I} I(C_i) \) is \( I\left( \bigoplus_{i \in I} I(C_i) \right) \)-injective. Hence \( \bigoplus_{i \in I} I(C_i) \) is \( M \)-injective.

The implications (3) \( \Rightarrow \) (5) \( \Rightarrow \) (6) are clear.

(6) \( \Rightarrow \) (4). Note that, by Lemma 4, any direct summand of a \( KS \)-quasi-continuous left \( R \)-module is \( KS \)-quasi-continuous. By analogy with the proof of \([12, \text{Theorem 2.6}]\), the proof can be completed.

We denote by \( S^2 \) the class of all semisimple left \( R \)-modules in \( \sigma[M] \).

**Corollary 7**. The following conditions are equivalent for a left \( R \)-module \( M \).
(1) $M$ is a locally noetherian module (that is, every finitely generated submodule of $M$ is noetherian).

(2) Every direct sum of $M$-injective left $R$-modules in $\sigma[M]$ is $M$-injective.

(3) Every direct sum of $M$-injective left $R$-modules in $\sigma[M]$ is $S^2$-continuous.

(4) Every direct sum of $M$-injective left $R$-modules in $\sigma[M]$ is $S^2$-quasi-continuous.

(5) There exists a cardinal $c$ such that every direct sum of $M$-injective left $R$-modules in $\sigma[M]$ is the direct sum of a $c$-limited module and an $S^2$-continuous module.

(6) There exists a cardinal $c$ such that every direct sum of $M$-injective left $R$-modules in $\sigma[M]$ is the direct sum of a $c$-limited module and an $S^2$-quasi-continuous module.

**Corollary 8.** Let $S^2$ be the class of all semisimple left $R$-modules. Then the following conditions are equivalent:

(1) $R$ is a noetherian ring.

(2) Every direct sum of injective left $R$-modules is $S^2$-continuous ($S^2$-quasi-continuous).

(3) There exists a cardinal $c$ such that every direct sum of injective left $R$-modules is the direct sum of a $c$-limited module and an $S^2$-continuous ($S^2$-quasi-continuous) module.

Given a stable hereditary torsion theory $\tau$ on $R$-Mod, many equivalent conditions were presented in [9] and [12] to characterize the ring which has $ACC$ on $\tau$-dense left ideals. Here we have

**Corollary 9.** Let $\tau$ be a stable hereditary torsion theory on $R$-Mod and $TS$ be the class of all $\tau$-torsion semisimple left $R$-modules. Then the following conditions are equivalent:

(1) $R$ has $ACC$ on $\tau$-dense left ideals.

(2) Every direct sum of $\tau$-torsion injective left $R$-modules is injective.
(3) Every direct sum of $\tau$-torsion injective left $R$-modules is $TS$-continuous.

(4) Every direct sum of $\tau$-torsion injective left $R$-modules is $TS$-quasi-continuous.

(5) There exists a cardinal $c$ such that every direct sum of $\tau$-torsion injective $R$-modules is the direct sum of a $c$-limited module and a $TS$-continuous module.

(6) There exists a cardinal $c$ such that every direct sum of $\tau$-torsion injective $R$-modules is the direct sum of a $c$-limited module and a $TS$-quasi-continuous module.

References


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