King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 248

December 1999

Relative Continuity of Direct Sums of M-Injective Modules

Javed Ahsan, Liu Zhongkui

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

RELATIVE CONTINUITY OF DIRECT SUMS OF M-INJECTIVE MODULES

Javed Ahsan

Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia

Liu Zhongkui

Department of Mathematics
Northwest Normal University
Lanzhou, Gansu 730070, P.R. China

Abstract

Let M be a left R-module and $\mathcal{K}$ be an M-natural class with some additional conditions. It is proved that every direct sum of M-injective left R-modules in $\mathcal{K}$ is $\mathcal{KS}$-continuous ($\mathcal{KS}$-quasi-continuous) if and only if every direct sum of M-injective left R-modules in $\mathcal{K}$ is M-injective.

AMS Subject Classification: 16 P 40.

Let R be a ring with identity. It is well known that R is left noetherian if and only if every direct sum of injective left R-modules is injective. Based on this, many characterizations of left noetherian rings by generalized injectivity of some left R-modules have been obtained. For example, it was shown that R is left noetherian if and only if every direct sum of injective left R-modules is continuous (or quasi-continuous) (cf. [5]). On the other hand, Albu, Nastasescu, Golan, Goldman, Stenstrom, Teply, Enochs, Ahsan and others have studied the situations when all direct sums of non-singular injective left R-modules are injective, when all direct sums of τ-torsionfree injective left R-modules are injective for a hereditary torsion theory τ, and when all direct sums of τ-torsion injective left R-modules are injective for a stable hereditary torsion theory τ. These results are well presented in Golan's book "Torsion Theories", and have been generalized in [12] by considering when all direct sums of M-injective left R-modules in an M-natural class $\mathcal{K}$ are M-injective. In this note we consider when all direct sums of M-injective left R-modules in an M-natural class $\mathcal{K}$ are $\mathcal{KS}$-continuous (or $\mathcal{KS}$-quasi-continuous). We will show that for an M-natural class $\mathcal{K}$, all direct sums of M-injective left R-modules in $\mathcal{K}$ are $\mathcal{KS}$-continuous (or $\mathcal{KS}$-quasi-continuous) if and only if all direct sums of M-injective left R-modules in $\mathcal{K}$ are M-injective.

Throughout this note we write $A \leq_e B$ $(A|B)$ to denote that A is an essential submodule (a direct summand) of B.

Let M be a left R-module. We say that a left R-module N is subgenerated by M, or that M is a subgenerator for N, if N is isomorphic to a submodule of an M-generated module. Following [11], we denote by $\sigma[M]$ the full subcategory of R-Mod whose objects are all R-modules generated by M. By [11, 17.9], every module N in $\sigma[M]$ has an injective hull $I(N)$ in $\sigma[M]$, which is also called an M-injective hull of N. It is known that the M-injective hulls of a left R-module in $\sigma[M]$ are unique up to isomorphism. In the following, we always denote by $I(N)$ the M-injective hull of N for any left R-module $N \in \sigma[M]$.

According to [2], a subclass $\mathcal{K}$ in $\sigma[M]$ which is closed under submodules, direct sums, isomorphic copies, and M-injective hulls is called an M-natural class. There exist a large number of examples of M-natural classes. Among them are $\sigma[M]$ and all natural classes in the sense of [9]. In particular, hereditary torsionfree classes, stable hereditary torsion classes, and saturated classes in the sense of Dauns (cf. [1]) are examples of M-natural classes.

For an M-natural class $\mathcal{K}$ and a left R-module N, we denote by $H_{\mathcal{K}}(N)$ the set $\{L \leq N | N/L \in \mathcal{K}\}$.

Let M, N be left R-modules. Define the family

$$\mathcal{A}(N,M) = \Big\{A \subseteq M | \exists\, X \subseteq N,\ \exists\, f \in \mathrm{Hom}(X,M),\ f(X) \leq_e A\Big\}.$$

Consider the properties

$\mathcal{A}(N,M)$-(C_1) : For all $A \in \mathcal{A}(N,M)$, $\exists\, A^*|M$, such that $A \leq_e A^*$.

$\mathcal{A}(N,M)$-(C_2) : For all $A \in \mathcal{A}(N,M)$, if $X|M$ is such that $A \cong X$, then $A|M$.

$\mathcal{A}(N,M)$-(C_3): For all $A \in \mathcal{A}(N,M)$ and $X|M$, if $A|M$ and $A \cap X = 0$, then $A \oplus X|M$.

According to [7], M is said to be N-extending, N-quasi-continuous or N-continuous, respectively, if M satisfies $\mathcal{A}(N,M)$-(C_1), $\mathcal{A}(N,M)$-(C_1) and $\mathcal{A}(N,M)$-(C_3), $\mathcal{A}(N,M)$-(C_1) and $\mathcal{A}(N,M)$-(C_2).

LEMMA 1 ([7, Proposition 2.4]). *A left R-module M is (quasi-) continuous* (cf. [2]) *if and only if M is M-(quasi-) continuous if and only if M is N-(quasi-) continuous for every left R-module N.*

Given an M-natural class $\mathcal{K}$, a left R-module N is called $\mathcal{K}$-cocritical if $N \in \mathcal{K}$ and $N/P \notin \mathcal{K}$ for any $0 \neq P \subset N$.

DEFINITION 2. Let $\mathcal{K}$ be an M-natural class. A left R-module M is said to be $\mathcal{KS}$-extending, $\mathcal{KS}$-quasi-continuous or $\mathcal{KS}$-continuous, respectively, if for any direct sum $C = \bigoplus_{i \in I} C_i$ of $\mathcal{K}$-cocritical modules $C_i (i \in I)$, M is C-extending, C-quasi-continuous or C-continuous.

Clearly (quasi-) continuous modules are $\mathcal{KS}$-(quasi-) continuous. The following example shows that the converse is not true.

EXAMPLE 3 (cf. [6]). Let R be a left noetherian V-ring which is not artinian semisimple (see, for example, [3]). Then, by [7, Corollary 3.7], every left R-module is N-continuous for every semisimple left R-module N. Thus every left R-module is $\mathcal{KS}$-continuous, where $\mathcal{K} = R$-Mod. If all left R-modules are quasi-continuous, then for every left R-module M, $M \oplus E(M)$ is quasi-continuous, and so M is injective by [8, Lemma C], where $E(M)$ denotes the injective hull of M. Thus R is artinian semisimple, a contradiction. Hence there exists a left R-module M which is not quasi-continuous.

LEMMA 4. *Any direct summand of a $\mathcal{KS}$-continuous ($\mathcal{KS}$-quasi-continuous) left R-module is $\mathcal{KS}$-continuous ($\mathcal{KS}$-quasi-continuous).*

Proof. It follows from the fact that condition $\mathcal{A}(N,M) - (C_i)$, $(i = 1,2,3)$ is inherited by direct summands of M ([7, Proposition 2.4]).

LEMMA 5 ([7]). *If M is N-(quasi-) continuous and $A \in \mathcal{A}(N, M)$ is a direct summand of M, then A is indeed (quasi-) continuous.*

Let c be any cardinal. A left R-module M is called c-limited provided every direct sum of non-zero submodules of M contains at most c direct summands (cf. [10]).

We say an M-natural class $\mathcal{K}$ satisfies $(*)$ (cf. [12]), if for any cyclic submodule N of M, and every ascending chain $N_1 \leq N_2 \leq \cdots$ with each $N_i \in H_{\mathcal{K}}(N)$, the union $\bigcup_i N_i$ belongs to $H\mathcal{K}(N)$.

THEOREM 6. *The following conditions are equivalent for an M-natural class $\mathcal{K}$ with $(*)$.*

(1) *$H\mathcal{K}(A)$ has ACC for any cyclic (or finitely generated) submodule A of M.*

(2) *Every direct sum of M-injective left R-modules in $\mathcal{K}$ is M-injective.*

(3) *Every direct sum of M-injective left R-modules in $\mathcal{K}$ is $\mathcal{K}\mathcal{S}$-continuous.*

(4) *Every direct sum of M-injective left R-modules in $\mathcal{K}$ is $\mathcal{K}\mathcal{S}$-quasi-continuous.*

(5) *There exists a cardinal c such that every direct sum of M-injective left R-modules in $\mathcal{K}$ is the direct sum of a c-limited module and a $\mathcal{K}\mathcal{S}$-continuous module.*

(6) *There exists a cardinal c such that every direct sum of M-injective left R-modules in $\mathcal{K}$ is the direct sum of a c-limited module and a $\mathcal{K}\mathcal{S}$-quasi-continuous module.*

Proof. (1) $\Leftrightarrow$ (2). Follows from [12, Theorem 2.4].

(2) $\Leftrightarrow$ (3). Suppose that $N = \bigoplus_{i \in N} N_i$ is the direct sum of M-injective left R-modules $N_i \in \mathcal{K}$, $i \in I$. Then N is M-injective by (2). On the other hand, N is in $\mathcal{K}$, and so $N \in \sigma[M]$. Thus N is quasi-injective. Now clearly N is $\mathcal{K}\mathcal{S}$-continuous by Lemma 1.

(3) $\Leftrightarrow$ (4). Clear.

(4) $\Leftrightarrow$ (1). By [12, Theorem 2.5], it is sufficient to show that every direct sum of M-injective hulls of $\mathcal{K}$-cocritical left R-modules is M-injective.

Let C_i, $i \in I$ be $\mathcal{K}$-cocritical left R-modules. Then $C_i \in \mathcal{K}$, $i \in I$. Set

$$N = \Big(\bigoplus_{i\in I} I(C_i)\Big) \oplus I\Big(\bigoplus_{i\in I} I(C_i)\Big),$$
$$L = N \oplus I(N).$$

Then clearly L is a direct sum of M-injective left R-modules. Since $\mathcal{K}$ is closed under direct sums and M-injective hulls, it follows that L is a direct sum of M-injective left R-modules in $\mathcal{K}$. Thus L is $\mathcal{KS}$-quasi-continuous. Denote

$$S = \Big(\bigoplus_{i\in I} C_i\Big) \oplus \Big(\bigoplus_{i\in I} C_i\Big).$$

Then L is S-quasi-continuous. For the submodule $A = N \oplus 0$ of L, define an R-homomorphism $f : S \to L$ as the induced R-homomorphism $S = \Big(\bigoplus_{i\in I} C_i\Big) \oplus \Big(\bigoplus_{i\in I} C_i\Big) \to \Big(\bigoplus_{i\in I} I(C_i)\Big) \oplus I\Big(\bigoplus_{i\in I} I(C_i)\Big) \oplus 0$ (by the natural maps $C_i \to I(C_i)$ and $\bigoplus_{i\in I} C_i \to I\Big(\bigoplus_{i\in I} I(C_i)\Big)$.
Since $C_i \leq_e I(C_i)$, we have

$$\bigoplus_{i\in I} C_i \leq_e \bigoplus_{i\in I} I(C_i) \leq_e I\Big(\bigoplus_{i\in I} I(C_i)\Big).$$

Thus

$$\begin{aligned} f(S) &= \Big(\big(\bigoplus_{i\in I} C_i\big) \oplus \big(\bigoplus_{i\in I} C_I\big)\Big) \oplus 0 \\ &\leq_e \Big(\big(\bigoplus_{i\in I} I(C_i)\big) \oplus I\big(\bigoplus_{i\in I} I(C_i)\big)\Big) \oplus 0 = A. \end{aligned}$$

This means that $A \in \mathcal{A}(S, L)$. By Lemma 5, it follows that A is quasi-continuous. Thus N is quasi-continuous. By [8, Lemma C], $\bigoplus_{i\in I} I(C_i)$ is $I\Big(\bigoplus_{i\in I} I(C_i)\Big)$-injective. Hence $\bigoplus_{i\in I} I(C_i)$ is M-injective.

The implications (3) $\Rightarrow$ (5) $\Rightarrow$ (6) are clear.

(6) $\Rightarrow$ (4). Note that, by Lemma 4, any direct summand of a $\mathcal{KS}$-quasi-continuous left R-module is $\mathcal{KS}$-quasi-continuous. By analogy with the proof of [12, Theorem 2.6], the proof can be completed.

We denote by $\mathcal{S}^2$ the class of all semisimple left R-modules in $\sigma[M]$.

COROLLARY 7. *The following conditions are equivalent for a left R-module M.*

(1) *M is a locally noetherian module (that is, every finitely generated submodule of M is noetherian).*

(2) *Every direct sum of M-injective left R-modules in $\sigma[M]$ is M-injective.*

(3) *Every direct sum of M-injective left R-modules in $\sigma[M]$ is S^2-continuous.*

(4) *Every direct sum of M-injective left R-modules in $\sigma[M]$ is S^2-quasi-continuous.*

(5) *There exists a cardinal c such that every direct sum of M-injective left R-modules in $\sigma[M]$ is the direct sum of a c-limited module and an S^2-continuous module.*

(6) *There exists a cardinal c such that every direct sum of M-injective left R-modules in $\sigma[M]$ is the direct sum of a c-limited module and an S^2-quasi-continuous module.*

COROLLARY 8. *Let S^2 be the class of all semisimple left R-modules. Then the following conditions are equivalent:*

(1) *R is a noetherian ring.*

(2) *Every direct sum of injective left R-modules is S^2-continuous (S^2-quasi-continuous).*

(3) *There exists a cardinal c such that every direct sum of injective left R-modules is the direct sum of a c-limited module and an S^2-continuous (S^2-quasi-continuous) module.*

Given a stable hereditary torsion theory τ on R-Mod, many equivalent conditions were presented in [9] and [12] to characterize the ring which has ACC on τ-dense left ideals. Here we have

COROLLARY 9. *Let τ be a stable hereditary torsion theory on R-Mod and $\mathcal{TS}$ be the class of all τ-torsion semisimple left R-modules. Then the following conditions are equivalent:*

(1) *R has ACC on τ-dense left ideals.*

(2) *Every direct sum of τ-torsion injective left R-modules is injective.*

(3) *Every direct sum of τ-torsion injective left R-modules is $\mathcal{TS}$-continuous.*

(4) *Every direct sum of τ-torsion injective left R-modules is $\mathcal{TS}$-quasi-continuous.*

(5) *There exists a cardinal c such that every direct sum of τ-torsion injective R-modules is the direct sum of a c-limited module and a $\mathcal{TS}$-continuous module.*

(6) *There exists a cardinal c such that every direct sum of τ-torsion injective R-modules is the direct sum of a c-limited module and a $\mathcal{TS}$-quasi-continuous module.*

References

[1] Dauns, J., Classes of modules, Forum Math., 3(1991), 327–338.

[2] Dung, N.V., Huynh, D.V., Smith, P.F. and Wisbauer, R., Extending Modules, Pitman Research Notes in Math. Ser. 313, Longman Sci. and Tech. 1994.

[3] Cozzens, J.H. and Johnson, J.L., An application of differential algebra to ring theory, Proc. Amer. Math. Soc., 31(1972), 354–356.

[4] Golan, J.S., Torsion Theories, Longman Sci. and Tech. 1986.

[5] Liu Zhongkui, Characterizations of rings by their modules, Comm. Algebra, 21(10)(1993), 3663–3671.

[6] Liu Zhongkui, Characterizations of V-modules by their relative quasi-continuity, submitted.

[7] Lopez-Permouth, S.R.., Oshiro, K. and Tariq Rizvi, S., On the relative (quasi-)continuity of modules, Comm. Algebra, 26(1998), 3497–3510.

[8] Osofsky, B.L. and Smith, P.F., Cyclic modules whose quotients have all complement submodules direct summands, J. Algebra, 139(1991), 342–354.

[9] Page, S.S. and Zhou Yiqiang, Direct sums of injective modules and chain conditions, Canadian J. of Math. 46(3)(1994), 634–647.

[10] Van Huynh, D. and Smith, P.F., Some rings characterized by their modules, Comm. Algebra, 18(1990), 1971–1988.

[11] Wisbauer, R., Foundations of Module and Ring Theory, Gordon and Breach Science Publishers, 1991.

[12] Zhou Yiqiang, Direct sums of M-injective modules and module classes, Comm. Algebra, 23(30) (1995), 927–940.