Initial Value Problems for Integro-differential Inclusions

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Abstract. This paper is devoted to the study of the existence of solutions of initial value problems for scalar integro-differential inclusions. We shall rely on the topological transversality theorem to prove our main result.

Key Words: Initial value problem, integro-differential inclusions, a priori bounds on solutions, topological transversality theorem.

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1. Introduction

In this paper, we are concerned with the study of the existence of solutions of initial value problems for integro-differential inclusions of the form

\[ x'(t) \in F \left(t, x(t), \int_0^t K(t, s, x(s))ds \right) \quad t \in (0, T] \]

\[ x(0) = 0. \]  \hspace{1cm} (1)

Here \( F : J \times R^2 \to R \) is a set-valued map, and \( K : D \times R \to R \) is continuous, where \( D = \{(t, s) \in R^2; 0 \leq s \leq t < +\infty\}, J = [0, T]. \) Our objective is to provide sufficient conditions on the set-valued map \( F \) and the function \( K \) that insure the existence of solutions of (1). Our method of proof is based on topological transversality arguments. We note that such problems have application in the theory of closed-loop control problems. For, consider the scalar state equation \( \dot{x}(t) = f(t, x(t), u(t)) \) subject to the constraint \( u(t) \in U(t, x(t)), \) where \( U(t, x(t)) \) is a nonempty compact subset of \( R. \) Suppose the controls are generated by a closed-loop law in the form \( u(t) = (Kx)(t), \) where \( K \) is a nonlinear Volterra integral operator; i.e., \( u(t) = \int_0^t K(t, s, x(s))ds. \) Then

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$U(t, x(t)) := \left\{ \int_0^t K(t, s, x(s))ds; \; x(t) \in R \right\}$.

Let $F \left( t, x(t), \int_0^t K(t, s, x(s))ds \right) = f(t, x(t), U(t, x(t)))$. Thus, we see that we have a problem of the form (1). Also, we should point out that when $F$ is single-valued, problem (1) has been examined in [3] and [7].

2. Preliminaries

In this section we introduce notations, definitions and results that will be used in the remainder of the paper.

2.1 Function spaces (see [2]).

Let $J$ be a compact interval in $R$. $C(J)$ is the Banach space of continuous real-valued functions defined on $J$, with the norm $\|x\|_0 = \sup\{|x(t)|; \; t \in J\}$ for $x \in C(J)$; $C^k(J)$ is the Banach space of $k$-times continuously differentiable functions. $L^p(J) = \left\{ x: J \to R \text{ measurable}; \; \int_J |x(t)|^p dt < +\infty \right\}$ and for $x \in L^p(J)$ define $\|x\|_{L^p} = \left( \int_J |x(t)|^p dt \right)^{1/p}$. The Sobolev spaces $W^{k,p}(J)$ are defined as follows

$$W^{1,p}(J) = \{ x: J \to R; x \text{ absolutely continuous and } x' \in L^p(J), \; 1 \leq p \leq \infty \}$$

and

$$W^{k,p}(J) = \{ x \in W^{k-1,p}(J); \; x' \in W^{k-1,p}(J) \} \quad k \geq 2.$$ 

Note that the embeddings $j : W^{k,p}(J) \to C^{k-1}(J)$ are completely continuous, $J$ being compact.

2.2 Set-valued Maps

Let $X$ and $Y$ be Banach spaces. A set-valued map $G : X \to 2^Y$ is said to be compact if $G(X) = \overline{\{ G(x); x \in X \}}$ is compact. $G$ has convex (closed, compact) values if $G(x)$ is convex (closed, compact) for every $x \in X$. $G$ is bounded on bounded subsets of $X$ if $G(B)$ is bounded in $Y$ for every bounded subsets $B$ of $X$. A set-valued map $G$ is upper semicontinuous (usc for short) at $z_0 \in X$ if for every open set $O$ containing $Gz_0$, there exists a neighborhood $M$ of $z_0$ such that $G(M) \subset O$. $G$ is usc on $X$ if it is usc
at every point of $X$. If $G$ is nonempty and compact-valued then $G$ is usc if and only if $G$ has a closed graph. The set of all bounded closed convex and nonempty subsets of $X$ is denoted by $bcc(X)$. A set-valued map $G : J \rightarrow bcc(X)$ is measurable if for each $x \in X$, the function $t \mapsto \text{dist}(x, G(t))$ is measurable on $J$. If $X \subset Y$, $G$ has a fixed point if there exists $x \in X$ such that $x \in Gx$. For more details on set-valued maps we refer to [4].

2.3 Topological Transversality Theory for Set-valued Maps (see [5]).

Let $X$ be a Banach space, $C$ a convex subset of $X$ and $U$ an open subset of $C$. $K_{BU}(\overline{U}, 2^C)$ shall denote the set of all set-valued maps $G : \overline{U} \rightarrow 2^C$ which are compact, usc with closed convex values and have no fixed points on $\partial U$ (i.e., $u / \in \text{G}u$ for all $u \in \partial U$). A compact homotopy is a set-valued map $H : [0,1] \times \overline{U} \rightarrow 2^C$ which is compact, usc with closed convex values. If $u / \in H(\lambda, u)$ for every $\lambda \in [0,1], u \in \partial U$, $H$ is said to be fixed point free on $\partial U$. Two set-valued maps $F, G \in K_{BU}(\overline{U}, 2^C)$ are called homotopic in $K_{BU}(\overline{U}, 2^C)$ if there exists a compact homotopy $H : [0,1] \times \overline{U} \rightarrow 2^C$ which is fixed point free on $\partial U$ and such that $H(0, \cdot) = F$ and $H(1, \cdot) = G$. $G \in K_{BU}(\overline{U}, 2^C)$ is called essential if every $F \in K_{BU}(\overline{U}, 2^C)$ such that $G|_{\partial U} = F|_{\partial U}$, has a fixed point. Otherwise $G$ is called inessential.

**Theorem 1** (Topological transversality theorem). Let $F, G$ be two homotopic set-valued maps in $K_{BU}(\overline{U}, 2^C)$. Then $F$ is essential if and only if $G$ is essential.

**Theorem 2.** Let $G : \overline{U} \rightarrow 2^C$ be the constant set-valued map $G(u) \equiv u_0$. Then, if $u_0 \in U$, $G$ is essential.

**Theorem 3** (Nonlinear Alternative). Let $U$ be an open subset of a convex set $C$, with $0 \in U$. Let $H : [0,1] \times \overline{U} \rightarrow 2^C$ be a compact homotopy such that $H_0 \equiv 0$. Then, either

(i) $H(1, \cdot)$ has a fixed point in $\overline{U}$, or

(ii) there exists $u \in \partial U$ such that $u \in H(\lambda, u)$ for some $u \in (0,1)$.

3. Main Result
In this section, we state and prove our main result. We assume that

(H1) \( F : J \times R^2 \rightarrow bcc(R), (t, x, y) \mapsto F(t, x, y) \) is

(i) measurable in \( t \), for each \( x, y \in R \)

(ii) usc with respect to \( (x, y) \in R^2 \) for a.e. \( t \in J \)

(H2) \(|F(t, x(t), v)| \leq f(t)|x(t)| + |v| + h(t) \) and \(|K(t, s, x(s))| \leq f(t)g(s)\psi(|x(s)|)\) where

\( f, g, h \) are continuous nonnegative real-valued functions on \( J \), and \( \psi : R_+ \rightarrow R_+ \)

is continuous nondecreasing such that \( \int_1^{+\infty} \frac{d\sigma}{\psi(\sigma)} = +\infty \).

Our main result reads as follows.

**Theorem 4.** If the assumptions (H1) and (H2) are satisfied, then the initial value problem (1) has at least one solution.

**Remark.** Condition (H2) has been used in [3] and [7] in the case of integro-differential equation.

**Proof.** This proof will be given in several steps, and uses some ideas from [5].

**Step 1.** Consider the set-valued operator \( \Phi : C(J) \rightarrow L^2(J) \) defined by

\[
(\Phi x)(t) = F\left(t, x(t), \int_0^t K(t, s, x(s))ds\right).
\]

\( \Phi \) is well defined, usc, with convex values and sends bounded subsets of \( C(J) \) into bounded subsets of \( L^2(J) \). In fact, we have

\[
\Phi x := \left\{ u : J \rightarrow R \text{ measurable; } u(t) \in F\left(t, x(t), \int_0^t K(t, s, x(s))ds\right) \text{ a.e. } t \in J \right\}.
\]

Let \( z \in C(J) \). If \( u \in \Phi z \) then

\[
|u(t)| \leq f(t)\left[|z(t)| + \int_0^t g(s)\psi(|z(s)|)ds\right] + h(t)
\]

\[
\leq \max_{t \in J} f(t)\left[\|z\|_0 + T\max_{s \in J} g(s)\max_{x \leq \|z\|_0} \psi(x)\right] + \|h\|_0.
\]
Hence \( \|u\|_{L^2} \leq C_0 \) for some constant \( C_0 \). This shows that \( \Phi \) is well defined. It is clear that \( \Phi \) is convex valued.

Now, let \( B \) be a bounded subset of \( C(J) \). Then, there exists \( k > 0 \) such that \( \|u\|_0 \leq k \) for \( u \in B \). So, for \( w \in \Phi u \) we have \( \|w\|_{L^2} \leq C_1 \), where \( C_1 = T \sup_{x \leq k} \theta(x) \),

\[
\theta(x) = \sup_{t \in J} \left\{ f(t) \left[ x(t) + \int_0^t g(s)\psi(|x(s)|)ds \right] + h(t) \right\}.
\]

Also, we can argue as in [5, p. 16] to show that \( \Phi \) is usc.

**Step 2.** Let \( x \in H^1_0(J) \) be a possible solution of (1). Then there exists a positive constant \( M_0 \), not depending on \( x \), such that

\[
|x(t)| \leq M_0 \text{ for } t \in J.
\]

For, it follows from the differential inclusion that

\[
x(t)x'(t) \in x(t)F \left( t, x(t), \int_0^t K(t, s, x(s))ds \right).
\]

Assumption (H2) yields

\[
x(t)x'(t) \leq |x(t)x'(t)| \leq f(t)|x(t)| \left[ |x(t)| + \int_0^t g(s)\psi(|x(s)|)ds \right] + h(t)|x(t)|
\]

which implies that

\[
x(t)^2 \leq 2 \int_0^t \left[ f(s)|x(s)| \left( |x(s)| + \int_0^s g(\tau)\psi(|x(\tau)|)d\tau \right) \right] ds + 2 \int_0^t h(s)|x(s)|ds.
\]

Then (see [7, Theorem 3] or [3])

\[
x(t) \leq \int_0^t h(s)ds + \int_0^t f(s)E^{-1} \left[ E \left( \int_0^s h(\tau)d\tau \right) + \int_0^s (f(\tau) + g(\tau))d\tau \right] ds
\]

where \( E(r) = \int_0^r \frac{d\sigma}{\sigma + \psi(\sigma)} \) and \( E^{-1} \) is the inverse of \( E \). This shows that \( |x(t)| \leq M_0 \)

for \( t \in J \) where

\[
M_0 = \|h\|_{L^1} + \sup_{t \in J} \left\{ \int_0^t f(s)E^{-1} \left[ E \left( \int_0^s h(\tau)d\tau \right) + \int_0^s (f(\tau) + g(\tau))d\tau \right] ds \right\}.
\]
Step 3. For $0 \leq \lambda \leq 1$ consider the one-parameter family of problems

$$x'(t) \in \lambda F \left( t, x(t), \int_0^t K(t, s, x(s))ds \right) \quad t \in J, \; x(0) = 0. \quad (1)_\lambda$$

It follows from Step 2 that if $x$ is a solution of $(1)_\lambda$ for some $\lambda \in [0, 1]$, then

$$|x(t)| \leq M_0 \text{ for } t \in J$$

and $M_0$ does not depend on $\lambda$.

Define $\Phi_\lambda : C(J) \to L^2(J)$ by

$$(\Phi_\lambda x)(t) = \lambda F \left( t, x(t), \int_0^t K(t, s, x(s))ds \right).$$

Step 1 shows that $\Phi_\lambda$ is usc, has convex values and sends bounded subsets of $C(J)$ into bounded subsets of $L^2(J)$. Let $j : H^1_0(J) \to C(J)$ be the completely continuous embedding. The operator $L : H^1_0(J) \to L^2(J)$, defined by $(Lx)(t) = x'(t)$ has a bounded inverse. We denote by $L^{-1}$ this inverse. Let $B_{M_0+1} := \{x \in C(J); \|x\|_0 \leq M_0 + 1\}$.

Define a set-valued map $H : [0, 1] \times B_{M_0+1} \to C(J)$ by

$$H(\lambda, x) = (j \circ L^{-1} \circ \Phi_\lambda)(x).$$

We can easily show that the fixed points of $H(\lambda, \cdot)$ are solutions of $(1)_\lambda$. Moreover, $H$ is a compact homotopy between $H(0, \cdot) \equiv 0$ and $H(1, \cdot)$. In fact, $H$ is compact since $\Phi_\lambda$ is bounded on bounded subsets and $j$ is completely continuous. Also, $H$ is usc with closed convex values. Since solutions of $(1)_\lambda$ satisfy $\|x\|_0 \leq M_0 < M_0 + 1$ we see that $H(\lambda, \cdot)$ has no fixed points on $\partial B_{M_0+1}$.

Now, $H(0, \cdot)$ is essential by Theorem 2. Hence $H_1$ is essential. This implies that $j \circ L^{-1} \circ \Phi$ has a fixed point. Therefore problem $(1)$ has a solution.

This completes the proof of the main result.

Remark. If we want to consider a nonzero initial condition, $x(0) = x_0$, then we let $y(t) = x(t) - x_0$, and hence $y$ will be a solution of the following problem:

$$\begin{align*}
\left\{ \begin{array}{l}
y'(t) \in F \left( t, y(t) + x_0, \int_0^t K(t, s, y(s) + x_0)ds \right) \\
y(0) = 0.
\end{array} \right.
\end{align*}$$

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References


