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Abdelkader Boucherif

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

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Abdelkader Boucherif
Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
P.O. Box 5046, Dhahran 31261, Saudi Arabia
e-mail: aboucher@kfupm.edu.sa

Abstract. This paper is devoted to the study of the existence of solutions of initial value problems for scalar integro-differential inclusions. We shall rely on the topological transversality theorem to prove our main result.
Key Words: Initial value problem, integro-differential inclusions, a priori bounds on solutions, topological transversality theorem..

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1. Introduction

In this paper, we are concerned with the study of the existence of solutions of initial value problems for integro-differential inclusions of the form

$$\begin{aligned} x'(t) &\in F\left(t, x(t), \int_0^t K(t,s,x(s))ds\right) \qquad t \in (0,T] \\ x(0) &= 0. \end{aligned} \tag{1}$$

Here $F : J \times R^2 \to R$ is a set-valued map, and $K : D \times R \to R$ is continuous, where $D = \{(t,s) \in R^2; 0 \leq s \leq t < +\infty\}, J = [0,T]$. Our objective is to provide sufficient conditions on the set-valued map F and the function K that insure the existence of solutions of (1). Our method of proof is based on topological transversality arguments. We note that such problems have application in the theory of closed-loop control problems. For, consider the scalar state equation $\dot{x}(t) = f(t, x(t), u(t))$ subject to the constraint $u(t) \in U(t, x(t))$, where $U(t, x(t))$ is a nonempty compact subset of R. Suppose the controls are generated by a closed-loop law in the form $u(t) = (Kx)(t)$, where K is a nonlinear Volterra integral operator; i.e., $u(t) = \int_0^t K(t,s,x(s))ds$. Then

$U(t,x(t)) := \left\{\int_0^t K(t,s,x(s))ds;\ x(t) \in R\right\}$.

Let $F\left(t, x(t), \int_0^t K(t,s,x(s))ds\right) = f(t,x(t),U(t,x(t)))$. Thus, we see that we have a problem of the form (1). Also, we should point out that when F is single-valued, problem (1) has been examined in [3] and [7].

2. Preliminaries

In this section we introduce notations, definitions and results that will be used in the remainder of the paper.

2.1 Function spaces (see [2]).

Let J be a compact interval in R. $C(J)$ is the Banach space of continuous real-valued functions defined on J, with the norm $\|x\|_0 = \sup\{|x(t)|;\ t \in J\}$ for $x \in C(J)$; $C^k(J)$ is the Banach space of k-times continuously differentiable functions. $L^p(J) = \left\{x : J \to R \text{ measurable}; \int_J |x(t)|^p dt < +\infty\right\}$ and for $x \in L^p(J)$ define $\|x\|_{L^p} = \left(\int_J |x(t)|^p dt\right)^{1/p}$. The Sobolev spaces $W^{k,p}(J)$ are defined as follows

$$W^{1,p}(J) = \{x : J \to R; x \text{ absolutely continuous and } x' \in L^p(J), \quad 1 \le p \le \infty\}$$

and

$$W^{k,p}(J) = \left\{x \in W^{k-1,p}(J);\quad x' \in W^{k-1,p}(J)\right\} \qquad k \ge 2.$$

Note that the embeddings $j : W^{k,p}(J) \to C^{k-1}(J)$ are completely continuous, J being compact.

2.2 Set-valued Maps

Let X and Y be Banach spaces. A set-valued map $G : X \to 2^Y$ is said to be compact if $G(X) = \overline{\cup\{G(x); x \in X\}}$ is compact. G has convex (closed, compact) values if $G(x)$ is convex (closed, compact) for every $x \in X$. G is bounded on bounded subsets of X if $G(B)$ is bounded in Y for every bounded subsets B of X. A set-valued map G is upper semicontinuous (usc for short) at $z_0 \in X$ if for every open set O containing Gz_0, there exists a neighborhood $\mathcal{M}$ of z_0 such that $G(\mathcal{M}) \subset O$. G is usc on X if it is usc at every point of X. If G is nonempty and compact-valued then G is usc if and only if G has a closed graph. The set of all bounded closed convex and nonempty subsets of X is denoted by $bcc(X)$. A set-valued map $G : J \to bcc(X)$ is measurable if for each $x \in X$, the function $t \mapsto \text{dist}(x, G(t))$ is measurable on J. If $X \subset Y$, G has a fixed point if there exists $x \in X$ such that $x \in Gx$. For more details on set-valued maps we refer to [4].

2.3 Topological Transversality Theory for Set-valued Maps (see [5]).

Let X be a Banach space, C a convex subset of X and U an open subset of C. $K_{\partial U}(\overline{U}, 2^C)$ shall denote the set of all set-valued maps $G : \overline{U} \to 2^C$ which are compact, usc with closed convex values and have no fixed points on ∂U (i.e., $u \not\in Gu$ for all $u \in \partial U$). A compact homotopy is a set-valued map $H : [0,1] \times \overline{U} \to 2^C$ which is compact, usc with closed convex values. If $u \not\in H(\lambda, u)$ for every $\lambda \in [0,1], u \in \partial U$, H is said to be fixed point free on ∂U. Two set-valued maps $F, G \in K_{\partial U}(\overline{U}, 2^C)$ are called homotopic in $K_{\partial U}(\overline{U}, 2^C)$ if there exists a compact homotopy $H : [0,1] \times \overline{U} \to 2^C$ which is fixed point free on ∂U and such that $H(0, \cdot) = F$ and $H(1, \cdot) = G$. $G \in K_{\partial U}(\overline{U}, 2^C)$ is called essential if every $F \in K_{\partial U}(\overline{U}, 2^C)$ such that $G|_{\partial U} = F|_{\partial U}$, has a fixed point. Otherwise G is called inessential.

Theorem 1 (Topological transversality theorem). *Let F, G be two homotopic set-valued maps in $K_{\partial U}(\overline{U}, 2^C)$. Then F is essential if and only if G is essential.*

Theorem 2. *Let $G : \overline{U} \to 2^C$ be the constant set-valued map $G(u) \equiv u_0$. Then, if $u_0 \in U$, G is essential*

Theorem 3 (Nonlinear Alternative). *Let U be an open subset of a convex set C, with $0 \in U$. Let $H : [0,1] \times \overline{U} \to 2^C$ be a compact homotopy such that $H_0 \equiv 0$. Then, either*

(i) *$H(1, \cdot)$ has a fixed point in $\overline{U}$, or*

(ii) *there exists $u \in \partial U$ such that $u \in H(\lambda, u)$ for some $u \in (0,1)$.*

3. Main Result

In this section, we state and prove our main result. We assume that

(H1) $F : J \times R^2 \to bcc(R),\ (t, x, y) \mapsto F(t, x, y)$ is

(i) measurable in t, for each $x, y \in R$

(ii) usc with respect to $(x, y) \in R^2$ for a.e. $t \in J$

(H2) $|F(t, x(t), v)| \leq f(t)|x(t)| + |v| + h(t)$ and $|K(t, s, x(s))| \leq f(t)g(s)\psi(|x(s)|)$ where f, g, h are continuous nonnegative real-valued functions on J, and $\psi : R_+ \to R_+$ is continuous nondecreasing such that $\int_1^{+\infty} \frac{d\sigma}{\psi(\sigma)} = +\infty$.

Our main result reads as follows.

Theorem 4. *If the assumptions (H1) and (H2) are satisfied, then the initial value problem (1) has at least one solution.*

Remark. Condition (H2) has been used in [3] and [7] in the case of integro-differential equation.

Proof. This proof will be given in several steps, and uses some ideas from [5].

Step 1. Consider the set-valued operator $\Phi : C(J) \to L^2(J)$ defined by

$$(\Phi x)(t) = F\left(t, x(t), \int_0^t K(t, s, x(s))ds\right).$$

Φ is well defined, usc, with convex values and sends bounded subsets of $C(J)$ into bounded subsets of $L^2(J)$. In fact, we have

$$\Phi x := \left\{u : J \to R \text{ measurable; } u(t) \in F\left(t, x(t), \int_0^t K(t, s, x(s))ds\right) \text{ a.e. } t \in J\right\}.$$

Let $z \in C(J)$. If $u \in \Phi z$ then

$$\begin{aligned} |u(t)| &\leq f(t)\left[|z(t)| + \int_0^t g(s)\psi(|z(s)|)ds\right] + h(t) \\ &\leq \max_{t \in J} f(t)\left[\|z\|_0 + T \max_{s \in J} g(s) \max_{x \leq \|z\|_0} \psi(x)\right] + \|h\|_0. \end{aligned}$$

Hence $\|u\|_{L^2} \leq C_0$ for some constant C_0. This shows that Φ is well defined. It is clear that Φ is convex valued.

Now, let B be a bounded subset of $C(J)$. Then, there exists $k > 0$ such that $\|u\|_0 \leq k$ for $u \in B$. So, for $w \in \Phi u$ we have $\|w\|_{L^2} \leq C_1$, where $C_1 = T \sup_{x \leq k} \theta(x)$,

$$\theta(x) = \sup_{t \in J} \left\{ f(t) \left[x(t) + \int_0^t g(s)\psi(|x(s)|)ds \right] + h(t) \right\}.$$

Also, we can argue as in [5, p. 16] to show that Φ is usc.

Step 2. Let $x \in H_0^1(J)$ be a possible solution of (1). Then there exists a positive constant M_0, not depending on x, such that

$$|x(t)| \leq M_0 \text{ for } t \in J.$$

For, it follows from the differential inclusion that

$$x(t)x'(t) \in x(t)F\left(t, x(t), \int_0^t K(t, s, x(s))ds\right).$$

Assumption (H2) yields

$$x(t)x'(t) \leq |x(t)x'(t)| \leq f(t)|x(t)| \left[|x(t)| + \int_0^t g(s)\psi(|x(s)|)ds \right] + h(t)|x(t)|$$

which implies that

$$x(t)^2 \leq 2 \int_0^t \left[f(s)|x(s)| \left(|x(s)| + \int_0^s g(\tau)\psi(|x(\tau)|)d\tau \right) \right] ds + 2 \int_0^t h(s)|x(s)|ds.$$

Then (see [7, Theorem 3] or [3])

$$x(t) \leq \int_0^t h(s)ds + \int_0^t f(s)E^{-1} \left[E\left(\int_0^s h(\tau)d\tau \right) + \int_0^s (f(\tau) + g(\tau))d\tau \right] ds$$

where $E(r) = \displaystyle\int_0^r \frac{d\sigma}{\sigma + \psi(\sigma)}$ and E^{-1} is the inverse of E. This shows that $|x(t)| \leq M_0$ for $t \in J$ where

$$M_0 = \|h\|_{L^1} + \sup_{t \in J} \left\{ \int_0^t f(s)E^{-1} \left[E\left(\int_0^s h(\tau)d\tau \right) + \int_0^s (f(\tau) + g(\tau))\, d\tau \right] ds \right\}.$$

Step 3. For $0 \le \lambda \le 1$ consider the one-parameter family of problems

$$x'(t) \in \lambda F\left(t, x(t), \int_0^t K(t,s,x(s))ds\right) \quad t \in J, \ \ x(0) = 0. \tag{1$_\lambda$}$$

It follows from Step 2 that if x is a solution of $(1)_\lambda$ for some $\lambda \in [0,1]$, then

$$|x(t)| \le M_0 \text{ for } t \in J$$

and M_0 does not depend on λ.

Define $\Phi_\lambda : C(J) \to L^2(J)$ by

$$(\Phi_\lambda x)(t) = \lambda F\left(t, x(t), \int_0^t K(t,s,x)(s))ds\right).$$

Step 1 shows that Φ_λ is usc, has convex values and sends bounded subsets of $C(J)$ into bounded subsets of $L^2(J)$. Let $j : H_0^1(J) \to C(J)$ be the completely continuous embedding. The operator $L : H_0^1(J) \to L^2(J)$, defined by $(Lx)(t) = x'(t)$ has a bounded inverse. We denote by L^{-1} this inverse. Let $B_{M_0+1} := \{x \in C(J); \|x\|_0 \le M_0 + 1\}$. Define a set-valued map $H : [0,1] \times B_{M_0+1} \to C(J)$ by

$$H(\lambda, x) = (j \circ L^{-1} \circ \Phi_\lambda)(x).$$

We can easily show that the fixed points of $H(\lambda, \cdot)$ are solutions of $(1)_\lambda$. Moreover, H is a compact homotopy between $H(0,\cdot) \equiv 0$ and $H(1,\cdot)$. In fact, H is compact since Φ_λ is bounded on bounded subsets and j is completely continuous. Also, H is usc with closed convex values. Since solutions of $(1)_\lambda$ satisfy $\|x\|_0 \le M_0 < M_0 + 1$ we see that $H(\lambda,\cdot)$ has no fixed points on ∂B_{M_0+1}.

Now, $H(0,\cdot)$ is essential by Theorem 2. Hence H_1 is essential. This implies that $j \circ L^{-1} \circ \Phi$ has a fixed point. Therefore problem (1) has a solution .

This completes the proof of the main result.

Remark. If we want to consider a nonzero initial condition, $x(0) = x_0$, then we let $y(t) = x(t) - x_0$, and hence y will be a solution of the following problem:

$$\begin{cases} y'(t) \in F\left(t, y(t) + x_0, \int_0^t K\left(t, s, y(s) + x_0\right) ds\right) \\ y(0) = 0. \end{cases}$$

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