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Nonlocal Multipoint Boundary Value Problems

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Abstract. In this work, we investigate the existence of solutions of the multipoint boundary value problem $-u'' = f(t, u, u')$, $0 < t < 1$, $u(0) = \sum_{i=1}^{m} a_i u(\xi_i)$ and $u(1) = \sum_{j=1}^{n} b_j u(\eta_j)$, where the boundary conditions are nonlocal. We rely on the upper and lower solutions method to provide a constructive method for obtaining at least one solution.

Keywords. multipoint boundary value problems,upper and lower solutions,

Mathematics Subject Classification :34B10-34B15

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1.INTRODUCTION

The purpose of this paper is to establish the existence of solutions for a class of second order ordinary differential equations subject to nonlocal boundary conditions. More specifically, we consider the following nonlinear multipoint boundary value problem

$$\begin{cases} -u'' = f(t,u,u') \quad 0<t<1 \\ u(0) = \sum_{i=1}^{m} a_i u(\xi_i) \\ u(1) = \sum_{j=1}^{n} b_j u(\eta_j) \end{cases} \qquad (P)$$

where $f : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous function, $\xi_i \in (0,1)$ and $a_i \in \mathbb{R}_+$, $i = 1,2,\ldots,m$ and, $\eta_j \in (0,1)$ and $b_j \in \mathbb{R}_+$, $j = 1,2,\ldots,n$.

Mathematical models leading to the above so called "*nonlocal*" boundary value problems, were investigated by Bitsadze and Samarskii (see [19]). Recently, several papers have been devoted to the study of problem (P) assuming that $0 < \xi_1 < \xi_2 < \ldots < \xi_m$, $0 < \eta_1 < \eta_2 < \ldots < \eta_n$ with $a_i \equiv 0$ for all $i = 1,\ldots,m$ (see for instance [12], [15] and [16]) and $a_i \neq 0$ for all $i = 1,\ldots,m-2$ (see [17]). In general, the analysis is done by reducing the problem to a three-point problem, with boundary conditions $x(0) = 0$, $x(1) = ax(\eta)$ where $a \in \mathbb{R}$ and $\eta \in (0,1)$. The case $a = 1$ has been investigated in [4], [17], [21].

In almost all the above papers, the main assumption is that f is allowed to grow linearly (see [15], [16], [17]). In [12], the authors assume $f = g+h$ where g satisfies a sign condition and h is allowed a nonlinear growth. In [4], the authors used an integral monotonicity condition which generalizes the usual sign condition. All these conditions have proven sufficient in order to obtain a priori bounds on solutions. Then degree theoretical methods are used to prove existence.

In the present work, we consider the case of the general boundary conditions. We do not assume that the ξ_i and η_j are ordered. Also, our assumptions on the nonlinearity f are less restrictive; and, in fact, these are natural assumptions for the setting of the upper and lower solutions method on which, we shall rely to study our problem. However, unlike the usual upper and lower solutions method where one modifies accordingly the given problem (see for instance [1], [2],

[5], [8] and the references therein), we work directly with problem (P) and obtain its solution as a limit of a subsequence of a sequence of functions with bounded C^1−norm . As a byproduct, we will provide a constructive method to get at least one solution of problem (P).This method can,also, be used to analyze problems with Caratheodory nonlinearities.We should point out that our approach is different from the monotone iterative method used in [9] to analyze a different problem with Newman boundary conditions.Also, our results remain independent, since we do not assume any Lipschitz condition on the nonlinearity, and we do not rely on anti-maximum principles. As a matter of fact, we believe , this is the first time that the method of upper and lower solutions is being used in the context of nonlocal boundary value problems.

Finally, by a solution of problem (P), we mean a function $u \in C^2((0,1);\ \mathbb{R}) \cap C^0([0,1];\ \mathbb{R})$ satisfying the differential equation for all $t \in (0,1)$ and the boundary conditions.

The paper is organized as follows. In section 2, we present some notations and definitions that will be used throughout the paper. In section 3, we state and prove our main result. Section 4 is devoted to an application.

2.DEFINITIONS AND NOTATIONS

Let I denote the closed real interval $[0,1]$ and $\overset{\circ}{I}$ its interior. For $k = 0, 1, 2, \ldots$, let $C^k(I)$ denote the space of real-valued functions which are k−times continuously differentiable on I. For $u \in C^k(I)$, we define its norm by

$$|u|_k = \max\left(\|u\|_0, \|u'\|_0, \ldots, \left\|u^{(k)}\right\|_0\right)$$

where $\|v\|_0 = \sup\{|v(t)|;\ t \in I\}$. The space of measurable real-valued functions whose p−th power of the absolute value is *Lebesgue* integrable over I is denoted by $L^p(I)$. We denote the norm in $L^p(I)$ by $\|u\|_p$. Also, we shall refer to the *Sobolev* space $W^{2,p}(I)$, which may be defined by

$$W^{2,p}(I) := \{u \in L^p(I);\ u', u'' \in L^p(I)\}$$

with its usual norm

$$\|u\|_{W^{2,p}(I)}=\left(\sum_{k=0}^{2}\left\|u^{(k)}\right\|_{L^2}^2\right)^{\frac{1}{2}}$$

Definition 1 *A function $\beta \in C^2\left(\overset{\circ}{I}\right)\cap C^0\left(I\right)$ is called an upper solution for problem (P) if*

$$\begin{aligned} -\left(\beta\left(t\right)\right)'' &\geq f\left(t,\beta\left(t\right),\beta'(t)\right) \quad \text{for all } t\in\overset{\circ}{I} \\ \beta\left(0\right) &\geq \sum_{i=1}^{m} a_i\beta\left(\xi_i\right) \\ \beta\left(1\right) &\geq \sum_{j=1}^{n} b_j\beta\left(\eta_j\right) \end{aligned} \tag{1}$$

Definition 2 *A function $\alpha \in C^2(\overset{\circ}{I})\cap C^0\left(I\right)$ is called a lower solution for problem (P) if*

$$\begin{aligned} -\left(\alpha\left(t\right)\right)'' &\leq f\left(t,\alpha\left(t\right),\alpha'(t)\right) \quad \text{for all } t\in\overset{\circ}{I} \\ \alpha\left(0\right) &\leq \sum_{i=1}^{m} a_i\alpha\left(\xi_i\right) \\ \alpha\left(1\right) &\leq \sum_{j=1}^{n} b_j\alpha\left(\eta_j\right) \end{aligned} \tag{2}$$

3.MAIN RESULT

In this section, we state and prove our main result.
Let $\alpha \in C^2(\overset{\circ}{I})\cap C^0\left(I\right)$ and $\beta \in C^2\left(\overset{\circ}{I}\right)\cap C^0\left(I\right)$ be such that $\alpha \leq \beta$.
On the nonlinearity f, we shall impose the following conditions:

(H1) $f: I\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous and such that there exists a continuous function $\theta: I\longrightarrow \mathbb{R}$ with the property that

$$f\left(t,u,z\right)-f\left(t,v,z\right)\geq -\theta\left(t\right)\left(u-v\right) \quad \text{for all } t\in I\ ,\alpha\leq v\leq u\leq\beta \text{ and all } z\in\mathbb{R}.$$

(H2) There exists $K>0$ such that $|f(t,u,z)|\leq K\left(1+|z|^2\right)$ for all $t\in I$, u such that $\alpha\leq u\leq\beta$ and all $z\in\mathbb{R}$

Theorem 3 *Assume* **(H1)** **and** **(H2)** *are satisfied. Suppose that problem* (P) *has a lower solution* α *and an upper solution* β *such that* $\alpha \leq \beta$*. Then problem* (P) *has a solution* u *such that* $\alpha \leq u \leq \beta$.

Proof. The proof of this theorem is based on the following lemmas.

Lemma 4 *Let* u *be a solution of* (P) *such that* $\alpha \leq u \leq \beta$*. Then there exists a positive constant* C_0*, (independent of* u*) such that*

$$\|u'\|_0 \leq C_0.$$

Proof. Since $\alpha \leq u \leq \beta$, i.e. $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in I$, we have

$$\alpha(1) - \beta(0) \leq u(1) - u(0) \leq \beta(1) - \alpha(0)$$

Let $\omega := \max\{|\alpha(1) - \beta(0)|, |\beta(1) - \alpha(0)|\}$. Then $|u(1) - u(0)| \leq \omega$.

Choose $C_0 > \omega$ so that

$$K\left[\max_{t\in I} \beta(t) - \min_{t\in I} \alpha(t)\right] < \int_{\omega}^{C_0} \frac{s\,ds}{1+s^2}$$

It is clear that C_0 depends only on α, β and K.

We want to prove that $|u'(t)| \leq C_0$ for all $t \in I$. Suppose, on the contrary, that this is not true. Then there exists $\sigma \in I$ such that $|u'(\sigma)| > C_0$.

By the mean value theorem, there exists $\rho \in I$ such that $u'(\rho) = u(1) - u(0)$. Hence $|u'(\rho)| \leq \omega$. Thus, we have

$$|u'(\rho)| \leq \omega < C_0, \text{ and } |u'(\sigma)| > C_0.$$

Since u is C^1 on I, there exists an interval $[\tau_1, \tau_2] \subset I$ such that one of the following situations, holds

(i). $u'(\tau_1) = \omega$, $u'(\tau_2) = C_0$, and $\omega < u'(t) < C_0$ for all $t \in [\tau_1, \tau_2]$,

(ii). $u'(\tau_1) = C_0$, $u'(\tau_2) = \omega$, and $\omega < u'(t) < C_0$ for all $t \in [\tau_1, \tau_2]$,

(iii). $u'(\tau_1) = -\omega$, $u'(\tau_2) = -C_0$, and $-C_0 < u'(t) < -\omega$ for all $t \in [\tau_1, \tau_2]$,

$(iv)\,.u'(\tau_1) = -C_0, u'(\tau_2) = -\omega$, and $-C_0 < u'(t) < -\omega$ for all $t \in [\tau_1, \tau_2]$.

We consider only the first case, since the other cases can be handled similarly.

Multiplying the equation of (P) by u' and using **(H2)** we will obtain

$$u''(t)u'(t) \leq K\left(1 + |u'(t)|^2\right)u'(t) \text{ for all } t \in [\tau_1, \tau_2]$$

This implies that

$$\int_{\tau_1}^{\tau_2} \frac{u''(t)\,u'(t)\,dt}{1 + (u'(t))^2} \leq K\int_{\tau_1}^{\tau_2} u'(t)\,dt = K[u(\tau_2) - u(\tau_1)]$$

which yields

$$\int_{u'(\tau_1)}^{u'(\tau_2)} \frac{s\,ds}{1+s^2} \leq K\left(\max_{t\in I}\beta(t) - \min_{t\in I}\alpha(t)\right).$$

This leads to

$$\int_{\omega}^{C_0} \frac{s\,ds}{1+s^2} \leq K\left(\max_{t\in I}\beta(t) - \min_{t\in I}\alpha(t)\right)$$

This, obviously, contradicts the choice of C_0.

Hence, we have

$$\|u'\|_0 \leq C_0 := C_0(K, \alpha, \beta).$$

Lemma 5 *Let $M_0 > 0$ and $\varphi : I \times \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. Then for each $\gamma, \delta \in \mathbb{R}$, the problem*

$$\begin{cases} -u''(t) + M_0 u(t) = \varphi(t, u'(t)) & 0 < t < 1 \\ u(0) = \gamma, \quad u(1) = \delta \end{cases}$$

has a unique solution $u \in C^2(\overset{\circ}{I}) \cap C^0(I)$.

Proof.

Existence: For $0 \leq \lambda \leq 1$ consider the one parameter family of problems

$$\begin{cases} -u''(t) + M_0 u(t) = \lambda\varphi(t, u'(t)) & 0 < t < 1 \\ u(0) = \gamma \\ u(1) = \delta \end{cases} \qquad (Q_\lambda)$$

For $\lambda = 0$ the problem (Q_0) has a unique solution given by

$$u(t) = \left(\sinh\sqrt{M_0}\right)^{-1}\left[\delta\sinh\sqrt{M_0}\, t + \gamma\sinh\sqrt{M_0}\ (1-t)\right]$$

Let $0 < \lambda \leq 1$, and let u be a possible solution of (Q_λ). Then, there exists a positive constant R, independent of λ, such that $\|u\|_2 \leq R$.

For, let $w(t) = (\delta - \gamma)t + \gamma$ .It can be easily shown that any solution u of (Q_λ) can be written as $v(t) + w(t)$, where v is a solution of the integral equation

$$v(t) = \lambda\int_0^1 G(t,s)\varphi(s, v'(s) + (\delta-\gamma))\,ds - M_0\int_0^1 G(t,s)w(s)ds$$

where $G(t,s)$ is the Green's function for the homogeneous problem corresponding to (Q_0). Also, we have

$$v'(t) = \lambda\int_0^1 \frac{\partial G}{\partial t}(t,s)\varphi(s, v'(s) + (\delta-\gamma))\,ds - M_0\int_0^1 \frac{\partial G}{\partial t}(t,s)w(s)ds$$

Since G is uniformly continuous on $I \times I$ and I is compact ,we have $|G(t,s)| \leq G_0$ for all $(t,s) \in I \times I$ .Also, by assumption $|\varphi(s,z)| \leq \varphi_0$ for all $(s,z) \in I \times \mathbb{R}$.

This implies that $|v(t)| \leq G_0\varphi_0 + M_0 G_0 \max(|\gamma|, |\delta|) := \theta_0$, a constant.

Similarly, $|v'(t)| \leq [\varphi_0 + M_0\max(|\gamma|, |\delta|)] \left\|\frac{\partial G}{\partial t}\right\|_{L^1} := \theta_1$, a constant.

Therefore , there is a positive constant R , depending only on θ_0 and θ_1 such that $\|u\|_2 \leq R$. As a consequence of the topological transversality theorem of Granas (see for instance [10] , [3]) the problem (Q_λ) , for $\lambda = 1$, has a solution . This proves the existence of solutions.

Uniqueness: Suppose that problem (Q_λ) has two solutions u_1 and u_2.Then we have

$$\begin{cases} -(u_1-u_2)''(t)+M_0(u_1-u_2)(t)=\lambda\varphi(t,u_1'(t))-\lambda\varphi(t,u_2'(t)) & 0<t<1 \\ (u_1-u_2)(0)=0, \quad (u_1-u_2)(1)=0 \end{cases}$$

Let $v(t)=(u_1-u_2)(t)$ and $v(\tau_0)=\max\{v(t);t\in I\}$.
The cases $\tau_0=0$ and $\tau_0=1$ are obvious.
If $0<\tau_0<1$, then $v'(\tau_0)=0$ and

$$-v''(\tau_0)+M_0v(\tau_0)=0$$

that is

$$M_0v(\tau_0)=v''(\tau_0)\leq 0$$

which implies that

$$v(\tau_0)\leq 0$$

and $u_1(t)\leq u_2(t)$ for all $t\in I$. Similarly, we obtain $u_2(t)\leq u_1(t)$ for all $t\in I$.

Proof. of Theorem 3: It follows from **(H1)** that there exists a constant $M>0$ such that the function

$$s\longmapsto f(t,s,z)+Ms$$

is increasing for all $t\in I$ and all $z\in\mathbb{R}$.

Let C_0 be the constant from lemma 4.Consider $N_0>Max\left(C_0,\|\alpha'\|_0,\|\beta'\|_0\right)$. Define a function $h:\mathbb{R}\to\mathbb{R}$ by $h(z)=\max(-N_0,\ \min\ (z,N_0))$. Then h is a continuous and bounded function. In fact, we have $h(z)=z$ for all z such that $|z|\leq N_0$; and $|h(z)|\leq N_0$ for all $z\in\mathbb{R}$.

Define a sequence of functions $(u_k)_{k\in\mathbb{N}}$ in the following way

$$\begin{cases} u_0=\beta \\ -(u_{k+1})''(t)+Mu_{k+1}(t)=f\left(t,u_k(t),h\left(u_{k+1}'(t)\right)\right)+Mu_k(t) & t\in\overset{\circ}{I} \\ u_{k+1}(0)=\sum\limits_{i=1}^{m}a_iu_k(\xi_i) \\ u_{k+1}(1)=\sum\limits_{j=1}^{n}b_ju_k(\eta_j) \end{cases}$$

For each k, the function $\varphi(t,z) = f(t, u_k(t), h(z)) + Mu_k(t)$ is bounded for all $t \in I$ and all $z \in \mathbb{R}$.Then lemma 5, shows that the above problem admits a unique solution $u_{k+1} \in C^2(\overset{\circ}{I}) \cap C^0(I)$.Thus, the sequence $(u_k)_{k\in \mathrm{N}}$ is well defined.

Claim 1 $\alpha \leq u_k \leq \beta$ *for all* $k \in \mathbb{N}$.

Proof. Suppose, by induction, that $\alpha \leq u_j \leq \beta$ for all $j = 0, 1, \ldots, k$.
It follows from the definition of the upper solution β that

$$\begin{cases} -(u_{k+1}-\beta)''(t) + M(u_{k+1}-\beta)(t) \leq \\ f\left(t, u_k(t), h\left(u'_{k+1}(t)\right)\right) + Mu_k(t) - f(t, \beta(t), h(\beta'(t))) - M\beta(t) \quad \forall t \in \overset{\circ}{I} \\ (u_{k+1}-\beta)(0) \leq \sum_{i=1}^{m} a_i (u_k - \beta)(\xi_i) \leq 0 \\ (u_{k+1}-\beta)(1) \leq \sum_{j=1}^{n} b_j (u_k - \beta)(\eta_j) \leq 0 \end{cases}$$

Let $(u_{k+1}-\beta)(t_0) = \max\{(u_{k+1}-\beta)(t); t \in I\}$. If $0 < t_0 < 1$, we have $u'_{k+1}(t_0) = \beta'(t_0)$ and

$$\begin{aligned} &-(u_{k+1}-\beta)''(t_0) + M(u_{k+1}-\beta)(t_0) \leq \\ &f(t_0, u_k(t_0), h(\beta'(t_0))) + Mu_k(t_0) - f(t_0, \beta(t_0), h(\beta'(t_0))) - M\beta(t_0) \leq 0 \end{aligned}$$

Thus $M(u_{k+1}-\beta)(t_0) \leq (u_{k+1}-\beta)''(t_0) \leq 0$.This implies that

$$u_k(t) \leq \beta(t) \text{ for all } t \in I, \text{ and all } k \in \mathbb{N}$$

Similar arguments imply that

$$u_k(t) \geq \alpha(t) \text{ for all } t \in I, \text{ and all } k \in \mathbb{N}$$

Claim 2 *There exists a positive constant* C_3*, independent of* k*, such that*

$$\|u_k\|_{W^{2,p}\left(\overset{\circ}{I}\right)} \leq C_3 \quad \textit{for all } k \in \mathbb{N}.$$

Proof. Let

$$\rho(f) := \max\left\{|f(t,u,z)|\,;\ t \in I,\ \alpha \le u \le \beta \text{ and } |z| \le N_0\right\}$$

and

$$\ell_0 := \rho(f) + M(\|\beta\|_0 + \|\alpha\|_0).$$

Then ℓ_0 does not depend on $k \in \mathbb{N}$ and

$$\left|f\left(t, u_k(t), h\left(u'_{k+1}(t)\right)\right) + Mu_k(t)\right| \le \ell_0$$

for all $t \in \overset{\circ}{I}$.

Now, theorem 9.15 & lemma 9.17 in [14] imply that there exists a positive constant C_3, independent of $k \in \mathbb{N}$, such that

$$\|u_k\|_{W^{2,p}\left(\overset{\circ}{I}\right)} \le C_3 \quad \text{for all } k \in \mathbb{N}.$$

This proves the claim.

It follows from the compactness of the injection $W^{2,p}(\overset{\circ}{I}) \subset C^1(I)$ that there exists a subsequence $\left(u_{k_j}\right)_{k_j \in \mathrm{N}}$ of $(u_k)_{k \in \mathrm{N}}$ that converges in $C^1(I)$. Let $u := \lim\limits_{k_j \to +\infty} u_{k_j}$.

We have

$$u'_{k_j+1}(t) - u'_{k_j+1}(0) = \int_0^t u''_{k_j+1}(s)\,ds$$

$$= \int_0^t [-Mu_{k_j+1}(s) + f\left(t, u_{k_j}(s), h\left(u'_{k_j+1}(s)\right)\right) + Mu_{k_j}(s)]ds$$

It can be easily seen that

$$\left|f\left(t, u_{k_j}(s), h\left(u'_{k_j+1}(s)\right)\right) + M(u_{k_j}(s) - u_{k_j+1}(s))\right| \le 2\ell_0$$

Now, we use the Lebesgue dominated convergence theorem to prove that

$$u'(t) - u'(0) = -\int_0^t f(t, u(s), h(u'(s)))\,ds$$

which shows that $\quad -u''(t) = f(t, u(t), h(u'(t)))$.

Also, passing to the limit in the boundary conditions, we see that

$$u(0) = \sum_{i=1}^{m} a_i u(\xi_i) \text{ and } u(1) = \sum_{j=1}^{n} b_j u(\eta_j).$$

Consequently, we see that the problem

$$\begin{cases} -u''(t) = f(t, u(t), h(u'(t))) & t \in \overset{\circ}{I} \\ u(0) = \sum_{i=1}^{m} a_i u(\xi_i) \\ u(1) = \sum_{j=1}^{n} b_j u(\eta_j) \end{cases}$$

has a solution u such that $\|u'\|_0 \leq N_0$ (see lemma 4 and definition of h).Since $h(z) = z$ for all z such that $|z| \leq N_0$ it follows that u is also a solution of our problem (P).

4.APPLICATION

In this section, we shall apply our main result to prove existence of solutions of problem (P) in a special case.

Assume that f does not depend on u' and satisfies, in addition to **(H1)**, the following conditions

(H3) $f(t, u) = f(t, -u) \quad$ for all $(t, u) \in I \times \mathbb{R}$,

(H4) there exists a continuously differentiable function $g : \mathbb{R} \longrightarrow \mathbb{R}_+^*$ such that $\quad |f(t, u)| \leq g(u) \quad$ for all $(t, u) \in I \times \mathbb{R}$,

and $\limsup_{u \to +\infty} \dfrac{2G(u)}{u^2} \leq \lambda_1$, where $G(u) = \int_0^u g(s)\, ds$ and λ_1 is the first eigenvalue of the operator $u \to -u''$ on $W_0^{1,2}(]-2, 2[)$.

Assume that for all $i = 1, 2, \ldots, m$ $a_i \geq 0, \sum_{i=1}^{m} a_i \leq 1$ and for all $j = 1, 2, \ldots, n$ $b_j = 0$. In this case, we shall refer to problem (P) as problem $(P)_0$.

Remark. As we shall see below these conditions will enable us to prove the existence of nontrivial (i.e. nonconstant) upper and lower solutions of problem $(P)_0$.

Theorem 6 *If the assumptions* **(H1)**, **(H3)** *and* **(H4)** *are satisfied, then problem* $(P)_0$ *has at least one solution.*

Proof. Consider the following two-point boundary value problem

$$\begin{cases} -v'' = g(v) & \text{on }]-2, 2[\\ v(-2) = 0 \\ v(2) = 0 \end{cases} \tag{3}$$

which has a solution $v \in \mathcal{C}^2((-2,2); \mathbb{R}) \cap \mathcal{C}^0([-2,2]; \mathbb{R})$ (see for instance [11]).By the maximum principle ([7],p.144) the solution v is strictly positive in$]-2,2[$.Using theorem 1 in [13], we see that u is even and strictly decreasing in $]0,2[$.Let v^* be the restriction of v to the interval $[0,1]$.

Claim 3 v^* *is an upper solution for problem* $(P)_0$.

Proof. The definition of v^* and condition **(H4)** imply that

$$\begin{array}{l} -(v^*)''(t) = g(v^*(t)) \geq f(t, v^*(t)) \quad t \in I \\ v^*(0) \geq \sum_{i=1}^{m} a_i v^*(\xi_i) \\ v^*(1) \geq \sum_{j=1}^{n} 0 v^*(\eta_j) = 0 \end{array}$$

since $v^*(\xi_i) \leq v^*(0)$ for all $i = 1, 2, \ldots, m$ (recall that v^* is decreasing and $\sum_{i=1}^{m} a_i \leq 1$).

This ends the proof of claim 3.

Claim 4 *$-v^*$ is a lower solution for problem $(P)_0$.*

Proof. We have that

$$-(v^*)''(t) = g(v^*(t)) \quad t \in I.$$

Hence, by **(H4)**

$$(v^*)''(t) = -g(v^*(t)) \leq f(t, v^*(t)) \quad t \in I.$$

This shows that

$$-(-v^*)''(t) \leq f(t, v^*(t)) \quad t \in I.$$

Now, **(H3)** yields

$$-(-v^*)''(t) \leq f(t, -v^*(t)) \quad t \in I.$$

Also

$$(-v^*)(0) \leq \sum_{i=1}^{m} a_i(-v^*)(\xi_i) \text{ and } (-v^*)(1) \leq \sum_{j=1}^{n} 0\,(-v^*)(\eta_j).$$

This completes the proof of claim 4.

Now, we see that all the assumptions of theorem 3 are satisfied. Hence problem $(P)_0$ has a solution.

The proof of theorem 6 is complete.

Remark. Assumption (H2) is not needed here because f does not depend on u'.

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