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Coincidence Point Results in Locally Convex Spaces

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Abstract

In this paper we prove the existence of coincidence points in Hausdorff locally convex spaces for nonself single-valued and multivalued maps satisfying a non-expansive type condition. The results herein contain the work of many authors including recent results by Latif and Tweddle and Martinez-Yanez.

1. Introduction

Geometric fixed point theory in functional analysis for multivalued mappings has been extensively developed. One of its developments has led to substantial weakenings in the assumption that values of the mapping be subsets of its domain. We shall continue this approach in our work.

The longstanding Horn's conjecture about coincidence points in [4] reads: Let Y be a compact convex subset of a Banach space X and $f, g : Y \to Y$ commuting continuous maps. Then f and g have a coincidence point.

Contributions in this area of investigations have been made by a number of authors; see for example [1, 2, 8, 9, 10, 11, 14, 18] and references therein. Recently coincidence and common fixed point results have been obtained by Daffer and Kaneko [2], Kaneko [6] and Latif and Tweddle [9] for self and nonself f-contraction and f-nonexpansive single-valued as well as multivalued maps in the set up of metric spaces and Banach

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spaces. We shall generalize these results to the case of Hausdorff locally convex topological vector spaces for mappings without commutativity condition. In particular we prove a coincidence point result for f-contraction maps satisfying the inwardness condition in a Hausdorff locally convex space, which contains results of Massa [11] and Martinez-Yanez [12].

As applications of our result, we establish some theorems concerning coincidence points of f-nonexpansive maps, which in turn generalize and strengthen the results due to Chang and Yen [1], Itoh and Takahashi [5], Lami Dozo [8], Latif and Tweddle [9], Singh and Chen [15], Su and Sehgal [16], Taylor [18] and Zhang [19].

We now fix our terminology. Throughout this paper, X will denote a Hausdorff locally convex topological vector space, P the family of continuous seminorms generating the topology of X and $K(X)$ the family of nonempty compact subsets of X. For each $p \in P$ and $A, B \in K(X)$, we define

$$D_p(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}[p(a-b)],\ \sup_{b\in B}\inf_{a\in A}[p(a-b)]\right\}.$$

Although p is only a seminorm, D_p is a Hausdorff metric on $K(X)$ [7].

Let M be a nonempty subset of X. A mapping $T : M \to K(X)$ is called a multivalued contraction if there exists a constant k_p, $0 \le k_p < 1$ such that for each $x, y \in M$ and for each $p \in P$, we have

$$D_p\Big(T(x), T(y)\Big) \le k_p \quad p(x-y).$$

The map T is called nonexpansive if for each $x, y \in M$ and $p \in P$,

$$D_p\Big(T(x), T(y)\Big) \le p(x-y).$$

Let $f : M \to X$ be a single-valued map. Then $T : M \to K(X)$ is called an f-contraction if there exists k_p, $0 \le k_p < 1$ such that for each $x, y \in M$ and for each

$p \in P$, we have

$$D_p\Big(T(x), T(y)\Big) \leq k_p \qquad p\Big(f(x) - f(y)\Big).$$

If we have the Lipschitz constant $k_p = 1$ for all $p \in P$, then T is called an f-nonexpansive mapping. A point x in M is said to be a coincidence point of f and T if $f(x) \in T(x)$. We denote by $C(f \cap T)$, the set of coincidence points of f and T.

A subset M of X is said to be starshaped if there exists a $q \in M$, called the star-centre of M, such that for any $x \in M$ and $0 \leq \alpha \leq 1$, $\alpha q + (1-\alpha)x \in M$.

For any $x, y \in X$, we set

$$(x, y] = \{(1-\alpha)x + \alpha y : 0 < \alpha \leq 1\}$$

$$[x, y] = \{(1-\alpha)x + \alpha y : 0 \leq \alpha \leq 1\}.$$

A mapping $T : M \to K(X)$ is said to satisfy the boundary condition (α) if for all $x \in M$ and all $y \in T(x)$, $(x, y] \cap M \neq \phi$ (cf. [11] and [15]). As noted by Massa [11], the boundary condition (α) may be restated as:

$$\text{for all } \ x \in M, \ T(x) \subset I_M(x) = \{z : z = x + \alpha(y - x), \ \ y \in M, \ \alpha \geq 1\}.$$

In [19], Zhang called such a mapping T an inward mapping. If for all $x \in M$, $T(x) \subset \operatorname{cl}\Big(I_M(x)\Big)$, then T is called weakly inward where cl stands for closure.

A variant of Theorem 2 due to Massa [11] stated below will be needed.

THEOREM A. *Let M be a closed subset of a Hausdorff locally convex space X and $T : M \to K(X)$ a contraction satisfying*

$$(x, y] \cap M \neq \phi \ \textit{ for all } \ x \in M \ \textit{ and } \ y \in T(x).$$

Then T has a fixed point.

The mapping T from M into 2^X (the family of all nonempty subsets of X) is said to be demiclosed if for every net $\{x_\alpha\}$ in M and any $y_\alpha \in T(x_\alpha)$ such that $x_\alpha \xrightarrow{\omega} x$

and $y_\alpha \to y$, we have $x \in M$ and $y \in T(x)$ where $\to$ and $\xrightarrow{\omega}$ denote strong and weak convergence, respectively. We say X satisfies Opial's condition if for each $x \in X$ and every net $\{x_\alpha\}$ converging weakly to x, we have

$$\liminf p(x_\alpha - y) > \liminf p(x_\alpha - x), \quad \text{for } \; y \neq x \; \text{ and } \; p \in P.$$

The Hilbert spaces and Banach spaces having a weakly continuous duality mapping satisfy Opial's condition [8].

2. Results

We shall follow the arguments used by Latif and Tweddle [9] to prove the following.

THEOREM 2.1. *Let M be a nonempty subset of a Hausdorff locally convex space X. Let $f : M \to X$ be any map with its range G closed and $T : M \to K(X)$ an f-contraction map such that $T(x) \subset I_G(z)$ for all $x \in f^{-1}(z)$. Then $C(f \cap T) \neq \phi$.*

Proof. Define $J : G \to K(X)$ by $J(z) = Tf^{-1}(z)$ for all $z \in G$. For each $z \in G$ and $x, y \in f^{-1}(z)$, the f-contractiveness of T implies that

$$D_p(Tx, Ty) \leq k_p\, p\Big(f(x) - f(y)\Big) = 0.$$

Hence $J(z) = T(a)$ for all $a \in f^{-1}(z)$. Now we show that J is a contraction. For any $\omega, z \in G$, we have $D_p\Big(J(\omega), J(z)\Big) = D_p\Big(T(x), T(y)\Big)$ for any $x \in f^{-1}(\omega), y \in f^{-1}(z)$ and $p \in P$. But T is an f-contraction so there exists $k_p \in (0,1)$ such that for all $p \in P$, we have

$$\begin{aligned} D_p\Big(J(\omega), J(z)\Big) &= D_p\Big(T(x), T(y)\Big) \\ &\leq k_p\, p\Big(f(x) - f(y)\Big) = k_p\, p(\omega - z) \end{aligned}$$

which implies that J is a contraction. Also note that for all $z \in G, J(z) \subset I_G(z)$; that is, J is an inward mapping. It follows from Theorem A (cf. proof of Theorem

1[15]), that there exists $z_0 \in G$ such that $z_0 \in J(z_0)$. Since $J(z_0) = T(x_0)$ for any $x_0 \in f^{-1}(z_0)$, so $f(x_0) \in T(x_0)$. $\square$

The application of Theorem 2.1 yields the following coincidence point result for f-nonexpansive maps.

THEOREM 2.2. *Let M be a nonempty subset of a Hausdorff locally convex space X and $f : M \to X$ with its range G closed and starshaped. Let $T : M \to K(X)$ be an f-nonexpansive map which satisfies the following conditions:*

(i) $T(x) \subset I_G(z)$ for all $x \in f^{-1}(z)$.

(ii) $T(M)$ bounded and $(f - T)M$ closed.

Then $C(f \cap T) \neq \phi$.

Proof. Let q be a star-centre of G. Then $I_G(z)$ is also starshaped with respect to q for each $z \in G$ (see Theorem 1.2 [19]). For each n, define $T_n : M \to K(X)$ by

$$T_n(x) = k_n T(x) + (1 - k_n)q$$

where $\{k_n\}$ is any sequence with $k_n \to 1$ as $n \to \infty$ and $0 < k_n < 1$.

We have for all $p \in P$,

$$D_p\Big(T_n(x), T_n(y)\Big) \leq k_n\, p\Big(f(x) - f(y)\Big).$$

This implies that each T_n is an f-contraction. The condition (i) and starshapedness of $I_G(z)$ imply that $T_n(x) \subset I_G(z)$ for all $x \in f^{-1}(z)$. By Theorem 2.1, there exists $x_n \in M$ such that $f(x_n) \in T_n(x_n) = k_n T(x_n) + (1 - k_n)q$. So there is some $u_n \in T(x_n)$ such that

$$f(x_n) = k_n u_n + (1 - k_n)q.$$

Thus $f(x_n) - u_n = (k_n - 1)\, u_n + (1 - k_n)q \to 0$ as $n \to \infty$, by the boundedness of $T(M)$.

As $(f-T)M$ is closed and $f(x_n)-u_n \in (f-T)M$, we get that $0 \in (f-T)M$. Hence there is a point $x_0 \in M$ such that $f(x_0) \in T(x_0)$. $\square$

THEOREM 2.3. *Let M be a nonempty weakly compact subset of a Hausdorff locally convex space X and $f: M \to X$ a weakly continuous map with its range G starshaped. Let $T: M \to K(X)$ be an f-nonexpansive map which satisfies the following conditions:*

(i) $T(x) \subset I_G(z)$ for all $x \in f^{-1}(z)$.

(ii) $f-T$ is demiclosed.

Then $C(f \cap T) \neq \phi$.

Proof. Note that G is weakly compact and hence it is a closed and bounded subset of X. If q is a star-centre of G, then $I_G(z)$ is also starshaped with the star-centre q. As in the proof of Theorem 2.2, we get a sequence $\{x_n\}$ in M and $u_n \in T(x_n)$ such that $f(x_n)-u_n = \frac{1-k_n}{k_n}\Big(q-f(x_n)\Big)$. Since M is weakly compact, we may assume that $x_n \xrightarrow{\omega} x \in M$ and hence $f(x_n) \xrightarrow{\omega} f(x)$. It follows that for each $p \in P, \sup\limits_n p\Big(f(x)-f(x_n)\Big) < +\infty$. Thus for each $p \in P$, $p\Big(f(x_n)-u_n\Big) = \frac{1-k_n}{k_n}\, p\Big(q-f(x_n)\Big) \to 0$ as $n \to \infty$. Hence by the demiclosedness of $f-T$, we get that $0 \in (f-T)(x_0)$. Hence $f(x_0) \in T(x_0)$. $\square$

We shall need the following variant of Nadler's result [13], as stated in Lemma [15].

LEMMA 2.4. *If $A, B \in K(X)$, then for each $a \in A$, there is a $b \in B$ such that $p(a-b) \leq D_p(A,B)$ for all $p \in P$.*

An analogue of Lemma 3.1 due to Latif and Tweddle [9] is established below.

LEMMA 2.5. *Let M be a nonempty weakly compact subset of Hausdorff locally convex space X satisfying Opial's condition. Let $f: M \to X$ be a weakly continuous map and $T: M \to K(X)$ an f-nonexpansive multivalued map. Then $f-T$ is demiclosed.*

Proof. Let $\{x_\alpha\}$ be a net in M and $y_\alpha \in (f-T)(x_\alpha)$ be such that $x_\alpha \xrightarrow{\omega} x$ and

$y_\alpha \to y$. Obviously $x \in M$ and $f(x_\alpha) \xrightarrow{\omega} f(x)$. Since $y_\alpha \in f(x_\alpha) - T(x_\alpha)$; therefore we have $y_\alpha = f(x_\alpha) - u_\alpha$, for some $u_\alpha \in T(x_\alpha)$. As $T(x)$ is compact so by Lemma 2.4, there is a $u_\alpha \in T(x)$ such that for all $p \in P$,

$$p(u_\alpha - v_\alpha) \leq D_p\Big(T(x_\alpha), T(x)\Big).$$

The f-nonexpansiveness of T gives for each $p \in P$,

$$D_p\Big(T(x_\alpha), T(x)\Big) \leq p\Big(f(x_\alpha) - f(x)\Big).$$

Thus

$$p(u_\alpha - v_\alpha) \leq p\Big(f(x_\alpha) - f(x)\Big) \text{ for all } p \in P.$$

Passing to the limit with respect to α, we obtain

$$\begin{aligned} \liminf p\Big(f(x_\alpha) - f(x)\Big) &\geq \liminf p(u_\alpha - v_\alpha) \\ &= \liminf p\Big(f(x_\alpha) - y_\alpha - v_\alpha\Big), \text{ for all } p \in P. \end{aligned} \tag{$*$}$$

By compactness of $T(x)$, for a convenient subnet still denoted by $\{v_\alpha\}$, we have $v_\alpha \to v \in T(x)$. Consequently $(*)$ yields

$$\liminf p\Big(f(x_\alpha) - f(x)\Big) \geq \liminf p\Big(f(x_\alpha) - y - v\Big) \text{ for all } p \in P.$$

Since X satisfies Opial's condition and $f(x_\alpha) \xrightarrow{\omega} f(x)$ so $f(x) = y + v$. Thus $y = f(x) - v \in f(x) - T(x)$, which proves that $f - T$ is demiclosed.

The above Lemma leads to the following results for Opial spaces.

COROLLARY 2.6. *Let M be a nonempty weakly compact subset of a Hausdorff locally convex space X satisfying Opial's condition and $f : M \to X$ a weakly continuous map with its range G starshaped. Let $T : M \to K(X)$ be an f-nonexpansive map such that $T(x) \subset I_G(z)$ for all $x \in f^{-1}(z)$. Then $C(f \cap T) \neq \phi$.*

COROLLARY 2.7. Let M be a nonempty weakly compact starshaped subset of a Hausdorff locally convex space X satisfying Opial's condition. Suppose that $T : M \to X$ is a nonexpansive map such that $T(x) \in I_M(x)$ for all $x \in M$. Then T has a fixed point.

A generalization of a result in [5] is contained in the following.

COROLLARY 2.8. *Let M be a nonempty weakly compact subset of a Hausdorff locally convex space X satisfying Opial's condition and $f : M \to X$ a weakly continuous map with its range G starshaped. Let $T : M \to K(X)$ be an f-nonexpansive map such that for each $z \in \partial G$ (boundary of G), $T(x) \subset G$ for all $x \in f^{-1}(z)$. Then $C(f \cap T) \neq \phi$.*

Proof. For all $z \in G$, $G \subset I_G(z)$ and $I_G(z) = X$ if z is an interior point of G so we obtain that $T(x) \subset I_G(z)$ for all $x \in f^{-1}(z)$. The result follows from Corollary 2.6.

Next we use Theorem 2.2 to obtain the following result.

THEOREM 2.9. *Let M be a nonempty subset of a Hausdorff locally convex space X satisfying Opial's condition. Suppose $f : M \to X$ is a map with its range G weakly closed and starshaped. Let $T : M \to K(X)$ be an f-nonexpansive map which satisfies the following conditions:*

(i) $T(x) \subset I_G(z)$ for all $x \in f^{-1}(z)$.

(ii) $T(M) \subseteq B$ for some weakly compact subset B of X.

Then $C(f \cap T) \neq \phi$.

Proof. If we can show that $(f - T)M$ is closed, then the conclusion would follow from Theorem 2.2. Let y be a limit point of $(f - T)M$. Then there is a net $\{y_\alpha\}$ with $y_\alpha \in (f - T)M$ such that $y_\alpha \to y$. Since $y_\alpha \in (f - T)M$ so there is $\{x_\alpha\}$ in M such that $y_\alpha \in (f - T)(x_\alpha)$. Thus $y_\alpha = f(x_\alpha) - u_\alpha$ for some $u_\alpha \in T(x_\alpha)$. This implies that $f(x_\alpha) - y_\alpha = u_\alpha \in T(x_\alpha) \subseteq B$ so there is a $b \in B$ and a subset $\Big\{f(x_\alpha) - y_\alpha\Big\}$ (say) which converges weakly to b in B. Since $y_\alpha \xrightarrow{\omega} y$, it follows that $f(x_\alpha) \xrightarrow{\omega} y + b = z$. As G is weakly closed so $z \in G$ and hence $z = f(a)$ for some a in M. For each α, $f(x_\alpha) - y_\alpha \in T(x_\alpha)$ implies by Lemma 2.4 and f-nonexpansiveness of T that there is a $z_\alpha \in T(a)$ such that

$$p\Big(f(x_\alpha) - y_\alpha - z_\alpha\Big) \le p\Big(f(x_\alpha) - f(a)\Big).$$

Since $T(a)$ is compact so there is subnet still denoted by $\{z_\alpha\}$ such that $z_\alpha \to u \in T(a)$ and $y_\alpha + z_\alpha \to y + u$. Thus

$$\liminf p\Big(f(x_\alpha) - y - u\Big) \le \liminf p\Big(f(x_\alpha) - f(a)\Big).$$

By Opial's condition of X and $f(x_\alpha) \xrightarrow{\omega} f(a)$, we obtain $y + u = f(a)$. That is $y = f(a) - u \in (f - T)M$ as required. □

REMARKS 2.10. Let I denote the identity map on M.

(i) If $f = I$ and T is a single-valued map in Theorem 2.1, then a fixed point result of Martinez-Yanez [12] is obtained. Moreover, Theorem 2 of Massa [11] is a special case of Theorem 2.1.

(ii) In case $f = I$ in Theorem 2.2, we get Theorem 1 [15] and if in addition T is a single-valued self map on M, then Dotson's fixed point theorem [3] and Tarafdar's Theorem 1.2 [17] follow from it as immediate corollaries.

(iii) For $f = I$, Theorem 2.3 provides an analogue of Theorem 3.8 [19] for an inward map on a locally convex space and if in addition, T is a single-valued self map on M, then Corollary 2.6 (i) [18] follows from it.

(iv) Take $f = I$ on a convex set M, X a Banach space and $T : M \to K(M)$. Then Corollary 2.6 reduces to Theorem 3.2 [8].

(v) Corollary 2.7 generalizes Theorem 11 and Corollaries 12 and 15 due to Chang and Yen [1].

(vi) If $f = I$ and $T : M \to X$, then Corollary 2.8 implies conclusion of Theorem 3 [16] for weakly compact sets of an Opial space.

(vii) Theorem 2.9 extends Theorem 2 [15].

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