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ON $(\aleph, U)$-COHERENCE OF MODULES AND RINGS

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1. Introduction

A ring R is called left coherent if every finitely generated left ideal is finitely presented. As generalizations of left coherent rings, the concepts of left $\aleph$-coherent rings and left $(\aleph, U)$-coherent rings were introduced and investigated for any infinite cardinal number $\aleph$ and any flat right R-module U by Loustaunau in [12] and by Oyonarte and Torrecillas in [13], respectively. Let U be a flat right R-module. A ring R is said to be left $\aleph$-coherent $\Big($resp. left $(\aleph, U)$-coherent$\Big)$ if every finitely generated left ideal is $\aleph$-finitely presented $\Big($resp. $(\aleph, U)$-finitely presented$\Big)$. It was proved in [12] and [13] that a ring R is left $\aleph$-coherent $\Big($resp. left $(\aleph, U)$-coherent$\Big)$ if and only if the $\aleph$-product $\prod\limits_{i\in I}^{\aleph} R \Big($resp. $\prod\limits_{i\in I}^{\aleph} U\Big)$ is flat as a right R-module for any index set I. In this paper we consider $(\aleph, U)$-coherence of left R-modules. A left R-module M is said to be $(\aleph, U)$-coherent if every finitely generated submodule of every finitely generated M-projective module in $\sigma[M]$ is $(\aleph, U)$-finitely presented in $\sigma[M]$. It is proved under some additional conditions that a left R-module M is $(\aleph, U)$-coherent if and only if $\prod\limits_{i\in I}^{\aleph} U$ is M-flat as a right R-module.

It is well known that R is left coherent if and only if every finitely presented left R-module M has a finite 2-presentation in the sense of Bourbaki [1], that is, there exists an exact sequence $0 \longrightarrow K_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ such that F_0 and F_1 are finitely generated free left R-modules and K_2 is finitely generated (see, for example, [6] or [8]). In section 3, using the concepts of finite n-presentations, as defined in [1], $\aleph$-finite n-presentations and $(\aleph, U)$-finite n-presentations, where U is a flat right R-module, we define a dimension, called $(\aleph, U)$-coherent dimension, for a left R-module M. We show under some additional conditions that a left R-module M is $(\aleph, U)$-coherent if and only if the $(\aleph, U)$-coherent dimension of M is equal to zero. Thus the $(\aleph, U)$-coherent dimension can be used to measure how far a left R-module is from being left $(\aleph, U)$-coherent. We also give some characterizations of left $(\aleph, U)$-coherent dimension of rings and show that the left $\aleph$-coherent dimension of ring R introduced in [11] is the supremum of $(\aleph, U)$-coherent dimensions of R for all flat right R-modules U.

Throughout, R denotes an associative ring with identity. For any left R-module M we denote by M^+ the character module $Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M.

Let M be a left R-module. We say that a left R-module N is subgenerated by M, or that M is a subgenerator for N, if N is isomorphic to a submodule of an M-generated module. Following [18], we denote by $\sigma[M]$ the full subcategory of R-Mod whose objects are left R-modules subgenerated by M.

2. $(\aleph, U)$-coherent Modules

Let $\aleph$ be an infinite cardinal number and M a left R-module. Following Loustaunau [12], M is said to be $\aleph$-finitely generated, denoted $\aleph$-fg, if every subset X of M, with $|X| < \aleph$, is contained in a finitely generated submodule of M. For example, every left R-module is $\aleph_0$-fg, and every finitely generated left R-module is $\aleph$-fg for all $\aleph > \aleph_0$. If $\aleph > |M|$ and M is $\aleph$-fg, then M is finitely generated.

Let U be a flat right R-module. According to [14], a left R-module M is called $(\aleph, U)$-finitely generated, denoted $(\aleph, U)$-fg, if for any subset $S \subseteq U \bigotimes_R M$ with $|S| < \aleph$ there exists a finitely generated submodule $N \leq M$ such that $S \subseteq U \bigotimes_R N$. Clearly every $\aleph$-finitely generated module is $(\aleph, U)$-finitely generated for any flat R-module U.

The following result appeared in [13].

LEMMA 2.1. *Let $M_1, \dots, M_k$ be left R-modules. Then $\bigoplus_{i=1}^{k} M_i$ is $(\aleph, U)$-fg if and only if every M_i is $(\aleph, U)$-fg.*

Suppose that I is a set and $\{M_i | i \in I\}$ is a family of right R-modules. Let $x = (x_i)_{i\in I} \in \prod_{i\in I} M_i$. We define the support of x as $\mathrm{supp}(x) = \{i \in I | x_i \neq 0\}$. For an infinite cardinal number $\aleph$, define the $\aleph$-product of the M_i's as

$$\prod_{i\in I}^{\aleph} M_i = \left\{ x \in \prod_{i\in I} M_i \mid |\mathrm{supp}(x)| < \aleph \right\}.$$

Clearly one may view the direct sum and the direct product of a family of modules as two special cases of the same object, namely, the $\aleph$-product of the family of modules. $\aleph$-products of some families of modules have been studied by Loustaunau [12], Dauns [3], [4], Teply [16], [17] and Oyonarte and Torrecillas [13–15]. It may be noted that the concept of $\aleph$-products of modules, as described by Dauns in [4], is a special case of the more general notion of *filter sums of modules* (see, for example, Franzen [5] and Laradji [9, 10].

LEMMA 2.2. *Let N be a left R-module. Then the following statements are equivalent:*

(1) *N is $(\aleph, U)$-finitely generated.*

(2) *The canonical map $\phi_N : \left(\prod_{i\in I}^{\aleph} U\right) \bigotimes_R N \to \prod_{i\in I}^{\aleph} \left(U \bigotimes_R N\right)$ defined by $\phi_N\left((u_i)_{i\in I} \otimes x\right) = (u_i \otimes x)_{i\in I}$ is an epimorphism of abelian groups for all index sets I.*

Let M be a left R-module and $N \in \sigma[M]$ be finitely generated. Then N is said to be $(\aleph, U)$-finitely presented in $\sigma[M]$, denoted by $(\aleph, U)$-fp, if in any exact sequence $0 \longrightarrow K \to P \longrightarrow N \longrightarrow 0$ in $\sigma[M]$ with P finitely generated and M-projective, K is $(\aleph, U)$-fg. By [18, Chapter 5], it is easy to see that $(\aleph, U)$-fp left R-modules in $\sigma[M]$ need not be $(\aleph, U)$-fp in R-Mod.

PROPOSITION 2.3. *Let M be a finitely generated left R-module such that M is a projective generator in $\sigma[M]$. Then a finitely generated module $N \in \sigma[M]$ is $(\aleph, U)$-finitely presented in $\sigma[M]$ if and only if there exists an exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow N \longrightarrow 0$ in $\sigma[M]$ with P finitely generated and M-projective and K $(\aleph, U)$-fg.*

Proof. Since M is a projective generator in $\sigma[M]$ and N is finitely generated, there exists an exact sequence $0 \longrightarrow K \longrightarrow M^n \longrightarrow N \longrightarrow 0$ in $\sigma[M]$. Now the result follows from Lemma 2.1 and Schanuel's Lemma (see [18, 50.2]).

Let U be a right R-module. A left R-module N is called an $(\aleph, U)$-Mittag-Leffler module if, for every index set I, the canonical map $\phi_N : \left(\prod_{i\in I}^{\aleph} U\right) \bigotimes_R N \longrightarrow \prod_{i\in I}^{\aleph}\left(U \bigotimes_R N\right)$ defined by $\phi_N\Big((u_i)_{i\in I} \otimes x\Big) = (u_i \otimes x)_{i\in I}$ is monic.

LEMMA 2.4. *Let M be an $(\aleph, U)$-Mittag-Leffler left R-module. If M is a generator of $\sigma[M]$, then every finitely generated M-projective module $N \in \sigma[M]$ is an $(\aleph, U)$-Mittag-Leffler module.*

Proof. Since M is a generator of $\sigma[M]$ and N is finitely generated, there exists an exact sequence

$$0 \longrightarrow K \longrightarrow M^n \longrightarrow N \longrightarrow 0$$

in $\sigma[M]$. By [18], N is projective in $\sigma[M]$. Thus the sequence splits. Hence $M^n \cong N \oplus K$. Now the following commutative exact diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & \left(\prod_{i\in I}^{\aleph} U\right) \otimes N & \longrightarrow & \left(\prod_{i\in I}^{\aleph} U\right) \otimes M^n \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \prod_{i\in I}^{\aleph} (U \otimes N) & \longrightarrow & \prod_{i\in I}^{\aleph} (U \otimes M^n)
\end{array}$$

shows that the canonical map $\left(\prod_{i\in I}^{\aleph} U\right) \bigotimes_R N \longrightarrow \prod_{i\in I}^{\aleph}\left(\bigotimes_R N\right)$ is a monomorphism. This means that N is an $(\aleph, U)$-Mittag-Leffler module.

PROPOSITION 2.5. *Let U be a flat right R-module and M a finitely generated $(\aleph, U)$-Mittag-Leffler left R-module such that M is a projective generator in $\sigma[M]$. Suppose that $N \in \sigma[M]$ is finitely generated. Then the following conditions are equivalent:*

(1) *N is $(\aleph, U)$-finitely presented in $\sigma[M]$.*

(2) $\phi_N : \left(\prod_{i\in I}^{\aleph} U\right) \bigotimes_R N \to \prod_{i\in I}^{\aleph}\left(U \bigotimes_R N\right)$ *is an isomorphism of abelian groups for all index sets I.*

Proof. Since M is a projective generator in $\sigma[M]$ and $N \in \sigma[M]$ is finitely generated, there exists an exact sequence of left R-modules in $\sigma[M]$

$$0 \longrightarrow K \longrightarrow P \longrightarrow N \longrightarrow 0$$

with P finitely generated and projective in $\sigma[M]$. Consider the following commutative exact diagram:

$$\begin{array}{ccccccc}
\left(\prod_{i\in I}^{\aleph} U\right) \otimes K & \longrightarrow & \left(\prod_{i\in I}^{\aleph} U\right) \otimes P & \longrightarrow & \left(\prod_{i\in I}^{\aleph} U\right) \otimes N & \longrightarrow & 0 \\
\phi_K \downarrow & & \phi_P \downarrow & & \phi_N \downarrow & & \\
\prod_{i\in I}^{\aleph} (U \otimes K) & \xrightarrow{f} & \prod_{i\in I}^{\aleph} (U \otimes P) & \longrightarrow & \prod_{i\in I}^{\aleph} (U \otimes N) & \longrightarrow & 0
\end{array}$$

where f is a monomorphism. By Lemmas 2.2 and 3.4, ϕ_P is an isomorphism and ϕ_N is an epimorphism. Thus ϕ_N is an isomorphism if and only if ϕ_N is a monomorphism if and only if ϕ_K is an epimorphism if and only if K is $(\aleph, U)$-finitely generated if and only if N is $(\aleph, U)$-finitely presented in $\sigma[M]$.

COROLLARY 2.6. *Let U be a flat right R-module and let M be a finitely generated $(\aleph, U)$-Mittag-Leffler left R-module such that M is a projective generator in $\sigma[M]$. If $0 \longrightarrow K \longrightarrow N \longrightarrow L \longrightarrow 0$ is an exact sequence in $\sigma[M]$ with K and L $(\aleph, U)$-fp in $\sigma[M]$, then N is $(\aleph, U)$-fp in $\sigma[M]$.*

The following corollary is a generalization of [2, Lemma 5].

COROLLARY 2.7. *Let U and M be as in Corollary 2.6 and let $N \in \sigma[M]$ be finitely generated. If there is an exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow N \longrightarrow 0$ in $\sigma[M]$ where K is $(\aleph, U)$-fg and P is $(\aleph, U)$-fp, then N is $(\aleph, U)$-fp.*

Proof. Since P is $(\aleph, U)$-fp, there exits, by Proposition 2.3, an exact sequence

$$0 \longrightarrow A \longrightarrow Q \xrightarrow{f} P \longrightarrow 0$$

in $\sigma[M]$ in which Q is finitely generated M-projective and A is $(\aleph, U)$-fg. Assume $K \leq P$ and let $g = f|_{f^{-1}(K)}$. We have a commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & f^{-1}(K) & \longrightarrow & Q & \longrightarrow & N & \longrightarrow & 0 \\
\downarrow & & g\downarrow & & f\downarrow & & 1\downarrow & & \downarrow \\
0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

where g is onto, and both K and $\text{Ker}(g) = A$ are $(\aleph, U)$-fg. Thus $f^{-1}(K)$ is $(\aleph, U)$-fg, and hence N is $(\aleph, U)$-fp.

A left R-module M is said to be $(\aleph, U)$-coherent if every finitely generated submodule of every finitely generated M-projective module in $\sigma[M]$ is $(\aleph, U)$-fp. For example, every left R-module is $(\aleph_0, U)$-coherent for any flat right R-module U. If R is left coherent, then ${}_R R$ is left $(\aleph, U)$-coherent for all infinite cardinal number $\aleph$ and all flat right R-module U. If $\aleph > \aleph'$, then every $(\aleph, U)$-coherent module is $(\aleph', U)$-coherent.

THEOREM 2.8. *Let U be a flat right R-module and M a finitely generated $(\aleph, U)$-Mittag-Leffler left R-module such that M is a projective generator in $\sigma[M]$. Then the following conditions are equivalent:*

(1) $\prod_{i\in I}^{\aleph} U$ *is M-flat.*

(2) *Every finitely generated submodule of an $(\aleph, U)$-finitely presented module in $\sigma[M]$ is $(\aleph, U)$-finitely presented in $\sigma[M]$.*

(3) *M is $(\aleph, U)$-coherent.*

(4) *Every finitely generated submodule of an M-projective module in $\sigma[M]$ is $(\aleph, U)$-finitely presented in $\sigma[M]$.*

(5) *Every finitely generated submodule of M is $(\aleph, U)$-finitely presented in $\sigma[M]$.*

Proof. (1) $\Rightarrow$ (2). Let $N \in \sigma[M]$ be $(\aleph, U)$-finitely presented and L a finitely generated submodule of N. Since $\prod_{i\in I}^{\aleph} U$ is M-flat, it follows from [18, 17.4] that $\prod_{i\in I}^{\aleph} U$ is flat in $\sigma[M]$. Thus we have the following commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \left(\prod_{i\in I}^{\aleph} U\right) \otimes L & \longrightarrow & \left(\prod_{i\in I}^{\aleph} U\right) \otimes N \\ \downarrow & & \phi_L \downarrow & & \phi_N \downarrow \\ 0 & \longrightarrow & \prod_{i\in I}^{\aleph} (U \otimes L) & \longrightarrow & \prod_{i\in I}^{\aleph} (U \otimes N) \end{array}$$

Since N is $(\aleph, u)$-finitely presented in $\sigma[M]$, by Proposition 2.5, ϕ_N is an isomorphism. Thus ϕ_L is a monomorphism. On the other hand, by Lemma 2.2, ϕ_L is an epimorphism since L is finitely generated. Hence ϕ_L is an isomorphism, which implies that L is $(\aleph, U)$-finitely presented in $\sigma[M]$ by Proposition 2.5.

(2) $\Rightarrow$ (3). It follows from the fact that every finitely generated M-projective module in $\sigma[M]$ is $(\aleph, U)$-finitely presented in $\sigma[M]$.

(3) ⇒ (4). Let $P \in \sigma[M]$ be projective in $\sigma[M]$ and L a finitely generated submodule of P. Since M is a generator in $\sigma[M]$ and P is M-projective, it is easy to see that there exists a natural number k such that L is isomorphic to a submodule M^k. Now the result follows from (3).

(4) ⇒ (5). Clear.

(5) ⇒ (1). By [18, 12.15], it is enough to show that for any finitely generated submodule N of M, $\left(\prod_{i\in}^{\aleph} U\right) \otimes N \to \left(\prod_{i\in I}^{\aleph} U\right) \otimes M$ is a monomorphism.

Consider the following commutative diagram.

$$\begin{array}{ccc} \left(\prod_{i\in I}^{\aleph} U\right) \otimes N & \xrightarrow{f} & \left(\prod_{i\in I}^{\aleph} U\right) \otimes M \\ \phi_N \downarrow & & \phi_M \downarrow \\ \prod_{i\in I}^{\aleph} (U \otimes N) & \xrightarrow{g} & \prod_{i\in I}^{\aleph} (U \otimes M) \end{array}$$

where g is a monomorphism. Since M is an $(\aleph, U)$-mittag-Leffler module, ϕ_M is a monomorphism. Moreover N is $(\aleph, U)$-finitely presented in $\sigma[M]$. Thus ϕ_N is an isomorphism by Proposition 2.5. Hence f is a monomorphism. We conclude that $\prod_{i\in I}^{\aleph} U$ is M-flat.

Corollary 2.9. *Let U be a flat right R-module. Then the following conditions are equivalent:*

(1) $\prod_{i\in I}^{\aleph} U$ *is flat.*

(2) *Every finitely generated submodule of an $(\aleph, U)$-finitely presented module is $(\aleph, U)$-finitely presented.*

(3) *R is left $(\aleph, U)$-coherent.*

(4) *Every finitely generated submodule of a projective module is $(\aleph, U)$-finitely presented.*

(5) *Every finitely generated left ideal of R is $(\aleph, U)$-finitely presented.*

Recall that a ring R is said to be $\aleph$-left coherent if every finitely generated left ideal is $\aleph$-fp.

COROLLARY 2.10. *A ring R is $\aleph$-left coherent if and only if $_RR$ is $(\aleph, U)$-coherent for any flat right R-module U.*

Proof. Follows from Theorem 2.8 and from [12, Theorem 1.6].

3. $(\aleph, U)$-coherent Dimension of Modules

Let M be a left R-module and $N \in \sigma[M]$. According to [1], we will say that N is n-finitely presented (n-$\aleph$-finitely presented, n-$\aleph$-U-finitely presented) in $\sigma[M]$, denoted n-FP $\Big($resp. $(n, \aleph)$-FP, $(n, \aleph, U)$-FP$\Big)$, if there exists an exact sequence

$$0 \longrightarrow K_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where $P_0, \ldots, P_{n-1}$ are finitely generated M-projective modules in $\sigma[M]$ and $K_n \in \sigma[M]$ is finitely generated $\Big($resp. $\aleph$-fg, $(\aleph, U)$-fg$\Big)$. In this case, we also say that M has a finite n-presentation $\Big($resp. an $\aleph$-finite n-presentation, an $(\aleph, U)$-finite n-presentation$\Big)$.

By [18, 25.2] and Schanuel's Lemma, it is easy to see that if M is a finitely generated left R-module such that M is a projective generator in $\sigma[M]$, then a finitely generated left R-module $N \in \sigma[M]$ is finitely presented in $\sigma[M]$ if and only if N is 1-FP, and that N is $(\aleph, U)$-fp if and only if N is $(1, \aleph, U)$-FP. Clearly we have the following implications:

$$(n+1)\text{-FP} \Rightarrow (n+1, \aleph)\text{-FP} \Rightarrow (n+1, \aleph, U)\text{-FP} \Rightarrow n\text{-FP},$$

but not conversely.

From [8, Theorem 3.3] it is clear that a ring R is left coherent if and only if every 1-FP left R-module is 2-FP. Generalizing this result, we give the following definition.

DEFINITION 3.1. Let U be a flat right R-module and M a left R-module. We define the left $(\aleph, U)$-coherent dimension of M, denoted by $(\aleph, U)$-lc.dim M, as

$$\inf \{n \mid \text{every } (n+1)\text{-FP left} R\text{-module in } \sigma[M] \text{ is } (n+2, \aleph, U)\text{-FP}\}.$$

If no such n exists, we say that $(\aleph, U)$-lc.dim $M = \infty$.

REMARK 3.2. Let $U = R_R$ and M be a finitely generated left R-module such that M is a generator in $\sigma[M]$. Take $\aleph > |M|^{\aleph_0}$. For every $(\aleph, R)$-fg left R-module K with $K \leq P$, where $P \in \sigma[M]$ is finitely generated M-projective, we have $|K| \leq |P| \leq |M|^{\aleph_0} < \aleph$. This implies that K is finitely generated. Thus when $U = R_R$ and $\aleph > |M|^{\aleph_0}$, Definition 3.1 gives a concept of left coherent dimension, denoted by lc.dim M, for any finitely generated left R-module M where M is a generator in $\sigma[M]$, that is,

$$\text{lc.dim } M = \inf\{n \mid \text{ every } (n+1)\text{-FP left } R\text{-module in } \sigma[M] \text{ is } (n+2)\text{-FP}\}.$$

If no such n exists, we say that lc.dim $M = \infty$.

REMARK 3.3. If $U = R_R$ and $M = {}_RR$, then Definition 3.1 gives the concept of left $\aleph$-coherent dimension of R, denoted by $\aleph$-lc.dim R, that is,

$$\aleph\text{-lc.dim } R = \inf\{n \mid \text{ every}(n+1)\text{-FP left } R\text{-module is } (n+2, \aleph)\text{-FP}\}.$$

If no such n exists, we say that $\aleph$-lc.dim $R = \infty$. Left $\aleph$-coherent dimension of R has been investigated in [11].

PROPOSITION 3.4. *Let M be a finitely generated $(\aleph, U)$-Mittag-Leffler left R-module such that M is a projective generator in $\sigma[M]$. Then M is $(\aleph, U)$-coherent if and only if $(\aleph, U)$-lc.dim$M = 0$.*

Proof. Suppose that M is $(\aleph, U)$-coherent. Then, every finitely generated submodule of every finitely generated M-projective left R-module in $\sigma[M]$ is $(\aleph, U)$-fp. Let $N \in \sigma[M]$ be 1-FP. Then there exists an exact sequence

$$0 \longrightarrow K_1 \longrightarrow P_0 \longrightarrow N \longrightarrow 0$$

such that $P_0 \in \sigma[M]$ is finitely generated and M-projective and K_1 is finitely generated. Thus K_1 is $(\aleph, U)$-fp. This means that there exists an exact sequence

$$0 \longrightarrow K_2 \longrightarrow P_1 \longrightarrow K_1 \longrightarrow 0$$

such that $P_1 \in \sigma[M]$ is finitely generated M-projective and K_2 is $(\aleph, U)$-fg. Now it is clear that N is $(2, \aleph, U)$-FP.

Conversely suppose that $(\aleph, U)$-lc.dim $M = 0$. Then every 1-FP left R-module is $(2, \aleph, U)$-FP. Let L be a finitely generated submodule of a finitely generated M-projective module $P \in \sigma[M]$. Then P/L is 1-FP. Thus it is $(2, \aleph, U)$-FP. So there exists an exact sequence

$$0 \longrightarrow K_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P/L \longrightarrow 0$$

such that $P_0, P_1 \in \sigma[M]$ are finitely generated and M-projective and K_2 is $(\aleph, U)$-fg. Since M is a projective generator in $\sigma[M]$ and L is finitely generated, we can take an exact sequence $0 \longrightarrow H \longrightarrow M^n \longrightarrow L \longrightarrow 0$ in $\sigma[M]$. Then by Schanuel's Lemma (see [18, 50.2]), $H \oplus P_1 \oplus P \simeq K_2 \oplus M^n \oplus P_0$. Now by Lemma 2.1, it follows that H is $(\aleph, U)$-fg, which implies that L is $(\aleph, U)$-fp. Thus M is $(\aleph, U)$-coherent.

Because of this proposition, we may regard our $(\aleph, U)$-coherent dimension as a measure of how far a left R-module M is from being $(\aleph, U)$-coherent. When $U = R_R$ and $M = {}_RR$, then left $\aleph$-coherent dimension of R may be regarded as a measure of how far a ring R is from being left $\aleph$-coherent.

LEMMA 3.5. *If $(\aleph, U)$-lc.dim $M = m$, then for any $n \geq m$, every $(n+1)$-FP left R-module in $\sigma[M]$ is $(n+2, \aleph, U)$-FP.*

Proof. Suppose that $N \in \sigma[M]$ is $(m+2)$-FP. Then there exists an exact sequence

$$0 \longrightarrow K_{m+2} \longrightarrow P_{m+1} \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_0 \longrightarrow N \longrightarrow 0$$

where $P_{m+1}, P_m, \ldots, P_0 \in \sigma[M]$ are finitely generated and M-projective and $K_{m+2} \in \sigma[M]$ is finitely generated. Denote $K_1 = \mathrm{Ker}(P_0 \longrightarrow N)$. Then K_1 is $(m+1)$-FP. Since $(\aleph, U)$-lc.dim $M = m$, it follows that K_1 is $(m+2, \aleph, U)$-FP in $\sigma[M]$. Thus there exists an exact sequence

$$0 \longrightarrow H_{m+2} \longrightarrow G_{m+1} \longrightarrow G_m \longrightarrow \cdots \longrightarrow G_0 \longrightarrow K_1 \longrightarrow 0$$

where $G_{m+1}, G_m, \ldots, G_0 \in \sigma[M]$ are finitely generated and M-projective and $H_{m+2} \in \sigma[M]$ is $(\aleph, U)$-finitely generated. Therefore N is $(m+3, \aleph, U)$-FP.

Now the result follows by induction.

LEMMA 3.6. *Let X be a right R-module, and M a left R-module. Then the following conditions are equivalent:*

(1) $Ext_R^n(X, M^+) = 0$.

(2) $Tor_n^R(X, M) = 0$.

(4) $Ext_R^n(M, X^+) = 0$.

We are now ready to give our characterization of $(\aleph, U)$-coherent dimension of rings.

THEOREM 3.7. *Let $\aleph$ be an infinite cardinal number and U a flat right R-module. For an integer $m \geq 0$, the following conditions are equivalent.*

(1) $(\aleph, U)$-*lc.dim* $R \leq m$.

(2) $Tor_{m+1}^R\left(\prod_I^{\aleph} U, M\right) = 0$ *for each $(m+1)$-FP left R-module M and for every set I.*

(3) *For each set I, if $Tor_{m+1}^R\left(\prod_I^{\aleph} U, N\right) = 0$ for all $(m+2, \aleph, U)$-FP left R-modules N, then $Tor_{m+1}^R\left(\prod_I^{\aleph} U, M\right) = 0$ for all $(m+1)$-FP left R-modules M.*

(4) *If X is a right R-module such that $Ext_R^{m+1}(X, N^+) = 0$ for all $(m+2, \aleph, U)$-FP left R-modules N, then $Ext_R^{m+1}(X, M^+) = 0$ for all $(m+1)$-FP left R-modules M.*

(5) *If X is a right R-module such that $Ext_R^{m+1}(N, X^+) = 0$ for all $(m+2, \aleph, U)$-FP left R-modules N, then $Ext_R^{m+1}(M, X^+) = 0$ for all $(m+1)$-FP left R-modules M.*

Proof. (1) $\Rightarrow$ (2). Suppose that M is $(m+1)$-FP. Then M is $(m+2, \aleph, U)$-FP by Lemma 3.5. Thus there exists an exact sequence

$$0 \longrightarrow K_{m+2} \longrightarrow P_{m+1} \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where $P_{m+1}, P_m, \ldots, P_0$ are finitely generated projective left R-modules and K_{m+2} is $(\aleph, U)$-finitely generated. Denote $K_{m+1} = \mathrm{Ker}(P_m \to P_{m=1})$ and $K_m = \mathrm{Ker}(P_{m-1} \to P_{m-2})$. If $m = 0$ then take $K_m = P_{m-1} = M$. Consider the following exact sequence:

$$0 \longrightarrow K_{m+1} \longrightarrow P_m \longrightarrow K_m \longrightarrow 0.$$

We obtain a commutative diagram

$$\begin{array}{ccc}
\mathrm{Tor}_1^R\left(\prod_I^{\aleph} U, K_m\right) & \xrightarrow{f} & \left(\prod_I^{\aleph} U\right) \otimes K_{m+1} \\
\alpha \big\downarrow & & \beta \big\downarrow \\
0 & \longrightarrow & \prod_I^{\aleph} (U \otimes K_{m+1})
\end{array}$$

where f is a monomorphism. When K_{m+2} is $(\aleph, U)$-fg, K_{m+1} is $(\aleph, U)$-fp and hence β is an isomorphism by Proposition 2.5. Thus α is an isomorphism and hence $\mathrm{Tor}_1^R\Big(\prod_I^{\aleph} U, K_m\Big) = 0$. Now it is easy to see that $\mathrm{Tor}_{m+1}^R\Big(\prod_I^{\aleph} U, M\Big) = 0$.

(2) $\Rightarrow$ (3). Clear.

(3) $\Rightarrow$ (1). Let N be an $(m+2, \aleph, U)$-FP left R-module. By analogy with the proof of (1) $\Rightarrow$ (2), we can obtain $\mathrm{Tor}_{m+1}^R\Big(\prod_I^{\aleph} U, N\Big) = 0$, for every set I. Thus, by (3), it follows that $\mathrm{Tor}_{m+1}^R\Big(\prod_I^{\aleph} U, M\Big) = 0$ for all $(m+1)$-FP left R-modules M. To complete the proof it is enough to show that every $(m+1)$-FP left R-module $(m+2, \aleph, U)$-FP.

Let M be an $(m+1)$-FP left R-module. If $m = 0$, then the result follows from Proposition 3.4, Theorem 2.8 and from the fact that R is left $(\aleph, U)$-coherent if and only if every $\aleph$-product of any family of copies of U is flat as a right R-module (by Corollary 2.9) since every left R-module is a direct limit of finitely generated left R-modules. Now suppose that $m \geq 1$. Then there exists an exact sequence

$$0 \longrightarrow K_{m+1} \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where $P_m, \ldots, P_0$ are finitely generated projective left R-modules and K_{m+1} is finitely generated. Denote $K_m = \mathrm{Ker}(P_{m-1} \to P_{m-2})$ $\Big($if $m = 1$ then set $K_m = \mathrm{Ker}(P_0 \to M)\Big)$. Then $\mathrm{Tor}_I^R\Big(\prod_I^{\aleph} U, K_m\Big) \cong \mathrm{Tor}_{m+1}^R\Big(\prod_I^{\aleph} U, M\Big) = 0$. Thus we obtain a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Big(\prod_I^{\aleph} U\Big) \otimes K_{m+1} & \longrightarrow & \Big(\prod_I^{\aleph} U\Big) \otimes P_m & \longrightarrow & \Big(\prod_I^{\aleph} U\Big) \otimes K_m \\
\big\downarrow & & \gamma\big\downarrow & & \beta\big\downarrow & & \alpha\big\downarrow \\
0 & \longrightarrow & \prod_I^{\aleph} (U \otimes K_{m+1}) & \longrightarrow & \prod_I^{\aleph} (U \otimes P_m) & \longrightarrow & \prod_I^{\aleph} (U \otimes K_m)
\end{array}$$

with exact rows, where α and β are isomorphisms by Proposition 2.5 since P_m and K_m are finitely presented. Thus γ is an isomorphism and, hence K_{m+1} is $(\aleph, U)$-fp by Proposition 2.5. Now the result follows.

(1) $\Rightarrow$ (4). Follows from Lemma 3.5.

(4) ⇒ (2). Let N be an $(m+2, \aleph, U)$-FP left R-module. By analogy with the proof of (1) ⇒ (2), we can obtain $\mathrm{Tor}^R_{m+1}\Big(\prod_I^{\aleph} R, N\Big) = 0$, for every set I. Thus $\mathrm{Ext}_R^{m+1}\Big(\prod_I^{\aleph} U, N^+\Big) = 0$ by Lemma 3.6. From (4), it follows that $\mathrm{Ext}_R^{m+1}\Big(\prod_I^{\aleph} U, M^+\Big) = 0$ for every $(m+1)$-FP left R-module M. Now we have $\mathrm{Tor}^R_{m+1}\Big(\prod_I^{\aleph} U, M\Big) = 0$ for every set I and every $(m+1)$-FP left R-module M by Lemma 3.6.

(4) ⇒ (5). Follows from Lemma 3.6.

We use w.gl.dim R to denote the weak global dimension of ring R. As a direct consequence of Definition 3.1 and Theorem 3.7 we have

COROLLARY 3.8. *$(\aleph, U)$-lc.dim $R \leq \aleph$-lc.dim $R \leq$ lc.dim $R \leq$ w.gl.dim R.*

From [8], it follows that lc.dim R can be much smaller than w.gl.dim R and that $\aleph$-lc.dim R can be much smaller than lc.dim R. The following example shows that $(\aleph, U)$-lc.dim R can be much smaller than $\aleph$-lc.dim R.

EXAMPLE 3.9. Let ω_1 be the first uncountable ordinal number and let $B = \mathbb{Z}_2[x_\mu | \mu \leq \omega_1]$ be the commutative polynomial ring with relations $x_\alpha = x_\alpha x_\beta$ for $\alpha < \beta \leq \omega_1$ and $x_\alpha^2 = x_\alpha$ for $\alpha < \omega_1$. By [12], B is $\aleph_1$-coherent but not coherent. Moreover it is immediate to prove that the ring $A = \mathbb{Z}_2[x_\mu | \mu \leq \omega_1]$ is not $\aleph_1$-coherent. Take $M = B$ and

$$U = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}, \quad R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}.$$

Then, by [13], the matrix ring R is $(\aleph_1, U)$-coherent but not $\aleph_1$-coherent. Thus $(\aleph_1, U)$-lc.dim $R = 0$ but $\aleph_1$-lc.dim $R \neq 0$.

The following proposition is clear.

PROPOSITION 3.10. *$\aleph$-lc.dim $R = Sup_U\Big((\aleph, U)$-lc.dim $R\Big)$.*

According to [6], a left (right) R-module X is called 2-FP-injective (2-FP flat) if $\mathrm{Ext}^1_R(M, X) = 0$ $\big(\mathrm{Tor}^R_1(X, M) = 0\big)$ for each 2-FP let R-module M. We will say that a left (right) R-module X is called $(2, \aleph, U)$-FP-injective $\Big((2, \aleph, U)$-FP -flat$\Big)$ if $\mathrm{Ext}^1_R(M, X) = 0$ $\Big(\mathrm{Tor}^R_1(X, M) = 0\Big)$ for each $(2, , \aleph, U)$-FP left R-module M. A left R-module X is called FP-injective if $\mathrm{Ext}^1_R(M, X) = 0$ for each finitely presented left R-module M. As an immediate consequence of Theorem 3.7 when $m = 0$, we have the following result, some parts of which are well known.

COROLLARY 3.11. *Let U be a flat right R-module. Then the following statements are equivalent:*

(1) *R is left $(\aleph, U)$-coherent.*

(2) *For every set I, $\prod_I^{\aleph} U$ is a flat right R-module*

(3) *For every set I, if the right R-module $\prod_I^{\aleph} U$ is $(2, \aleph, U)$-FP-flat, then it is flat.*

(4) *For every set I, if $\left(\prod_I^{\aleph} U\right)^+$ is $(2, \aleph, U)$-FP-injective, then it is FP-injective.*

Proof. The result follows from [12, Theorem 1.6], and from Theorem 3.7, Proposition 3.4 and Lemma 3.6, bearing in mind that each left R-module is a direct limit of finitely presented modules, and that the functor $\mathrm{Tor}_1^R(X, -)$ preserves direct limits.

For special $\aleph$ and special U (for example, $\aleph > |R|^{\aleph_0}$ and $U = R_R$), we have

COROLLARY 3.12 (see [8]). *For an integer $m \geq 0$, the following conditions on a ring R are equivalent:*

(1) *$lc.dim R \leq m$.*

(2) *$Tor_{m+1}^R\left(\prod_I R, M\right) = 0$ for each $(m+1)$-FP left R-module M and for every set I.*

(3) *For each set I, if $Tor_{m+1}^R\left(\prod_I R, N\right) = 0$ for all $(m+2)$-FP left R-modules N, then $Tor_{m+1}^R\left(\prod_I R, M\right) = 0$ for all $(m+1)$-FP left R-modules M.*

(4) *If X is a right R-module such that $Ext_R^{m+1}(X, N^+) = 0$ for all $(m+2)$-FP left R-modules N, then $Ext_R^{m+1}(X, M^+) = 0$ for all $(m+1)$-FP left R-modules M.*

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