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CONVERGENCE OF A CLASS OF STOCHASTIC PROCESSES WITH INDEPENDENT INCREMENTS AND ITS APPLICATIONS

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ABSTRACT

We consider a class of stochastic processes generated by independent and identically distributed Bernoulli random variables. For such processes we give conditions for convergence to some known processes such as Poisson process, Brownian motion and Binomial process. As applications we consider a population of n individuals. Each of these individuals generates a discrete time branching stochastic process. We study the number of ancestors $S(n,t)$ whose offspring at time t exceeds level $\theta(t)$, where $\theta(t)$ is some positive valued function. Using general theorems we prove that $S(n,t)$ may be approximated as $t \to \infty$ and $n \to \infty$ by some stochastic processes with independent increments.

Key words: population, ancestor, branching process, Poisson process, Brownian motion, Binomial process, exceedance, Skorohod topology.

AMS Subject Classification: Primary 60J80; Secondary 60G70, 60F05

1. INTRODUCTION

We consider a population containing n individuals of the same type at time zero. Each of these individuals (ancestors) initiates a discrete time branching population process. Let $\theta(t), t \in \mathbb{N}_0 = \{0, 1, \ldots\}$ be a positive valued function and $S(n,t)$ be the number of ancestors having more than $\theta(t)$ descendants at time t.

Branching processes started by the initial ancestors may be considered as population processes describing population growth in different regions of an area R. Then it is easy to see that $S(n,t)$ is the number of regions of R whose population at time t exceeds level $\theta(t)$. Process $S(n,t)$ can be associated with a problem on the number of vertexes of rooted random trees as well (see [1], for example). In fact each realization of the scheme under the consideration can be interpreted as a forest containing n rooted trees. Consequently a realization of $S(n,t)$ is the number of trees in the forest having more than $\theta(t)$ vertexes of the level t.

We note here the rise of interest in recent years to problems concerning extrema in branching stochastic processes. For example the recent publications in this direction have been devoted to the asymptotic behaviour of the expectation of the maxima of branching processes ([2], [3]), to the limit distribution for the maximum family size ([4], [5]) and to other problems. Limit distributions for the index of the first process in a sequence of branching processes exceeding some fixed or increasing levels were obtained in [6]. Thus the study of $S(n,t)$ can be considered as a contribution to this program of investigation of the extrema in population processes.

It follows from well-known properties of branching processes (see [7], for example) that if n is fixed and the process is critical or subcritical, then $S(n,t)$ in the long run equals to zero with probability 1, for any level function $\theta(t)$.

What happens if the size of the initial population is large? In other words what is the asymptotic behaviour of $S(n,t)$ if the number of initial ancestors increases? To answer these questions we consider family of stochastic process $Y(x,t) = S([m(t)x],t)$, where $x \in [0,\infty)$ and $m(t) \to \infty$ as $t \to \infty$. We approximate $Y(x,t)$ by some known processes with independent increments. Behaviour of the parameter $m(t)$ and the form of limit processes naturally depend on criticality of the initial branching process. It turns out that, if the process is supercritical, then $Y(x,t)$ may be approximated by either a "binomial process" (process with independent and binomially distributed increments) or by the Brownian motion depending on the behaviour of $m(t)$. If the process is subcritical or critical, then the approximating process is either a Poisson process or the Brownian motion.

Now we give a rigorous definition of the process $S(n,t)$. Let $\mathcal{A}_t^i$ be the random population at time t generated by i-th initial ancestor, $i = 1, 2, \ldots, n$. For any positive valued function $\theta(t)$ functional $S(n,t) = S(n,t)[\theta]$ can be defined as following

$$S(n,t) = \#\{i : \text{ card } \mathcal{A}_t^i > \theta(t)\}.$$

Let $X_i(t) = \text{card } \mathcal{A}_t^i$ be i-th branching process and $X(t)$ be a branching process such that $X(t) \overset{d}{=} X_i(t)$ for all $i \geq 1$. We denote $\{P_k, k \geq 0\}$ the offspring distribution of $X(t)$ and put

$$\begin{aligned} f(s) &= \sum_{k=0}^{\infty} P_k s^k, \quad R(x,t) = P\{X(t) > x\}, Q(t) = R(0,t), \\ A &= \sum_{k=1}^{\infty} k P_k, \quad \sigma^2 = \sum_{k=1}^{\infty} k(k-1)P_k. \end{aligned}$$

Let $D = D[0,1]$ be the space of functions $y(x)$ defined on $[0,1]$ that are right continuous and have left-hand limits at any point $x \in [0,1]$. It is clear that realizations of the $Y(x,t) = S([m(t)x],t)), x \in [0,1]$, are elements of D for any $t \in N_0$. We equip D with the Skorohod metric ([8], see also [9]). The topology of weak convergence of measures on Borel sets in D induced by this metric is called the Skorohod topology. In this paper we approximate the process $Y(x,t)$ with respect to Skorohod topology on $D[0,1]$. In the case of Poisson limiting process we prove convergence in Skorohod topology on $D[0,\infty)$.

In part 2 we provide main theorems on convergence of $S(n,t)$. Proofs of these theorems are based on two preliminary results which are given in part 3. Part 4 contains proofs of main theorems.

2. MAIN THEOREMS AND COROLLARIES

First we consider the critical case, i.e., the case of $A = 1$, $0 < \sigma^2 < \infty$. We assume that there exits the following

$$\lim_{t\to\infty} \frac{\theta(t)}{t} = \theta \in [0,\infty] \tag{1}$$

and consider $Y(x,t) = S([tx],t)$, i.e., $m(t) = t$.

Theorem 1. *If $A = 1$, $0 < \sigma^2 < \infty$ and (1) is satisfied, then $Y(x,t) \xrightarrow{\mathcal{D}} Y(x)$ as $t \to \infty$, where $\mathcal{D}$ means convergence in the Skorohod topology on $D[0,\infty)$ and $Y(x), x \in [0,\infty)$ is the Poisson process with $EY(x) = 2x\exp\{-2\theta/\sigma^2\}/\sigma^2$ for $\theta \in [0,\infty)$ and it is a "zero process" (i.e., $Y(x) \equiv 0$ with probability 1 for all $x \in [0,\infty)$) for $\theta = \infty$.*

Theorem 1 gives an approximation of $S(n,t)$ for the case when $n = o(t)$ or $n \asymp t$ as $t \to \infty$. Now we consider the case when $n/t \to \infty$, $t \to \infty$. More precisely we put $m(t) = a(t)t$, where $a(t) \to \infty$. We define the stochastic process $W_t^{(1)}(x)$ as follows

$$W_t^{(1)}(x) = \frac{S([ta(t)x],t) - [ta(t)x]R(\theta(t),t)}{\sqrt{a(t)}},$$

where $R(\theta(t),t) = P\{X(t) > \theta(t)\}$, $x \in [0,1]$.

Theorem 2. *If $A = 1$, $0 < \sigma^2 < \infty$ and (1) is satisfied, then $W_t^{(1)}(x) \xrightarrow{\mathcal{D}} W^{(1)}(x)$ as $t \to \infty$, where $\mathcal{D}$ means convergence in the Skorohod topology on $D[0,1]$ and $W^{(1)}(x)$ is the Brownian motion with zero shift and with the diffusion parameter $2\sigma^{-2}\exp\{-2\theta/\sigma^2\}$ for $\theta \in [0,\infty)$ and it is a zero process for $\theta = \infty$.*

Example 1. Suppose that the initial ancestors are labeled by $1, 2, \ldots$ Let $\nu(t)$ be the number of the first ancestor having more than $\theta(t)$ descendants at time t, i.e.

$$\nu(t) = \min\{k : X_k(t) > \theta(t)\}.$$

Then it follows from Theorem 1 that

$$\lim_{t\to\infty} P\{\nu(t) \le tx\} = \lim_{t\to\infty} P\{Y(xt,t) \ge 1\} = P\{Y(x) \ge 1\}.$$

Thus we have the following limit theorem for $\nu(t)$.

Corollary 1. *If $A = 1$, $0 < \sigma^2 < \infty$ and condition (1) is satisfied, then $\nu(t)/t \to \nu$ as $t \to \infty$ in distribution and*

$$P\{\nu > x\} = \exp\{-\frac{2x}{\sigma^2}e^{-2\theta/\sigma^2}\}.$$

Now we consider the case of supercritical processes. It is known [7] that if $A > 1$, $EX(1)\ln X(1) < \infty$, then $X(t)A^{-t}$ converges with probability one to a random variable W and the Laplace transform $\varphi(\lambda)$ of W satisfies the following equation

$$\varphi(\lambda) = f\left(\varphi\left(\frac{\lambda}{A}\right)\right).$$

It is also known that the distribution function $\pi(x)$ of W is absolute continuous for $x > 0$ and has an atom of the mass q at $x = 0$. Here q is the extinction probability.

We assume that there exists

$$\lim_{t\to\infty} \theta(t)A^{-t} = \theta \in [0,\infty] \tag{2}$$

and

$$\sum_{k=2}^{\infty} kP_k \ln k < \infty \tag{3}$$

and consider "discrete time" process $S(n,t), n = 0,1,\dots$ for $t \in \mathbb{N}_0$. Note that here n is the time parameter.

Theorem 3. *If $A > 1$ and conditions (2) and (3) are satisfied, then $S(n,t) \xrightarrow{w} \xi(n),\quad n \in \mathbb{N}_0$ as $t \to \infty$, where w means convergence in the weak sence and $\xi(n)$ is a stochastic process with independent and binomially distributed increments such that*

$$P\{\xi(n_i) - \xi(n_{i-1}) = k\} = \binom{n_i - n_{i-1}}{k} [1 - \pi(\theta)]^k \pi(\theta)^{n_i - n_{i-1} - k}$$

for any $0 \le n_{i-1} < n_i < \infty,\ n_i \in \mathbb{N}_0,$ for $\theta \in [0,\infty)$ and it is a zero process for $\theta = \infty$.

Example 2. Let the offspring distribution be the positive geometric, i.e. $P_k = \alpha(1-\alpha)^{k-1}, k \ge 1$ and $P_0 = 0$. In this case the offspring generating function has the form $f(s) = \alpha s(1-\beta s)^{-1}, \beta = 1-\alpha$ and $A = \alpha^{-1}$ and the equation for the Laplace transform is:

$$\varphi\left(\frac{\lambda}{\alpha}\right) = \frac{\alpha\varphi(\lambda)}{1-\beta\varphi(\lambda)}.$$

Now it is not difficult to check that the Laplace transform $\varphi(\lambda) = \alpha(\alpha+\lambda)^{-1}$ satisfies the above equation. Hence the limit distribution $\pi(x)$ is exponential with the density function $\alpha e^{-\alpha x}$ and Theorem 3 gives the following result.

Corollary 2. *If conditions of Theorem 3 are satified and the offspring distribution is the positive geometric with the parameter* $0 < \alpha < 1$, *then for* $\theta \in [0, \infty)$ *the limit process* $\xi(n)$ *in Theorem 3 is binomial such that*

$$P\{\xi(n_i) - \xi(n_{i-1}) = k\} = \binom{n_i - n_{i-1}}{k} e^{-\alpha\theta k}\left[1 - e^{-\alpha\theta}\right]^{n_i - n_{i-1} - k}.$$

Example 3. If we consider the process $\nu(t)$ defined in Example 1, then it follows from Theorem 3 that

$$\lim_{t\to\infty} P\{\nu(t) \le n\} = \lim_{t\to\infty} P\{S(n,t) \ge 1\} = P\{\xi(n) \ge 1\}.$$

Consequently we obtain the following limit theorem for $\nu(t)$.

Corollary 3. *If* $A > 1$ *and conditions (2) and (3) are satisfied, then for any fixed* n

$$\lim_{t\to\infty} P\{\nu(t) \le n\} = 1 - \{\pi(\theta)\}^n.$$

In particular, if the offspring distribution is as in Example 2, then the limit distribution of $\nu(t)$ is positive geometric with the parameter $e^{-\alpha\theta}$.

Theorem 3 shows that stochastic process $S(n,t)$ for fixed $n \in \mathbb{N}_0$ can be approximated as $t \to \infty$ by a binomial process. Now we consider the case when $n \to \infty$. Let $a(t)$ be a positive function such that $a(t) \to \infty$ as $t \to \infty$. We consider the following stochastic process

$$W_t^{(2)}(x) = \frac{S([a(t)x], t) - [a(t)x]R(\theta(t), t)}{\sqrt{a(t)}},$$

where $x \in [0, 1]$.

Theorem 4. *If* $A > 1$ *and conditions (2) and (3) are satisfied, then* $W_t^{(2)}(x) \xrightarrow{\mathcal{D}} W^{(2)}(x)$ *as* $t \to \infty$, *where* $W^{(2)}(x), x \in [0, 1]$, *is the Brownian motion with zero shift and with diffusion parameter* $\pi(\theta)(1 - \pi(\theta))$ *for* $\theta \in [0, \infty)$ *and it is a zero* *process for* $\theta = \infty$.

Example 4. If, as in Example 2, the offspring distribution is the positive geometric of the parameter $0 < \alpha < 1$, then it is not difficult to see that the Brownian motion in Theorem 4 has the diffusion parameter $e^{-\alpha\theta}\left(1 - e^{-\alpha\theta}\right)$.

Let now $A < 1$, i.e., the initial process is subcritical. In this case we use the following limit theorem for subcritical processes (See [10], p. 29). If $A < 1$, there exist

$$\lim_{t\to\infty} P\{X(t) = j | X(t) > 0\} = P_j^*; \quad j \geq 1, \tag{4}$$

and the generating function $F^*(s)$ of P_j^*, $j \geq 1$ satisfies the equation

$$1 - F^*(s)) = A(1 - F^*(s)). \tag{5}$$

It is also known that, if $A \leq 1$, then $Q(t) = R(0,t) \to 0$ as $t \to \infty$. If $A < 1$ and in addition $EX(1) \ln X(1) < \infty$, then we have the following asymptotics for $Q(t)$ (see [10], p. 56)

$$Q(t) \sim KA^t, 0 < K = \prod_{m=0}^{\infty} B(P\{X(m) = 0\}) < \infty, \tag{6}$$

where $B(s) = (1 - f(s))/(A(1 - s))$.

Let $Y(x,t) = S([xA^{-t}], t), x \in [0, \infty)$.

Theorem 5. *If* $A < 1$ *and (3) is satisfied, then* $Y(x,t) \xrightarrow{\mathcal{D}} Y(x)$ *as* $t \to \infty$, *where* $\mathcal{D}$ *means convergence in Skorohod topology on* $D[0,\infty)$ *and* $Y(x)$ *is the Poisson process with* $EY(x) = Kx \sum_{j>\theta} P_j^*$ *for* $\theta(t) \equiv \theta \in \mathbb{N}_0$ *and it is a zero process if* $\theta(t) \to \infty$.

Now we consider the case $nA^t \to \infty$. Let, as before, $a(t)$ be a positive valued function such that $a(t) \to \infty$ as $t \to \infty$. We define process $W_t^{(3)}(x)$ by the relation

$$W_t^{(3)}(x) = \frac{1}{\sqrt{a(t)}} \left\{ S([xA^{-t}a(t)], t) - [xA^{-t}a(t)]R(\theta(t), t) \right\},$$

where $x \in [0,1]$.

Theorem 6. *If $A < 1$ and (3) is satisfied, then $W_t^{(3)}(x) \xrightarrow{\mathcal{D}} W^{(3)}(x)$ as $t \to \infty$, where $W^{(3)}(x), x \in [0,1]$, is the Brownian motion with zero shift and with diffusion parameter $K \sum_{i>\theta} P_i^*$ for $\theta(t) = \theta \in \mathbb{N}_0$ and it is a zero process if $\theta(t) \to \infty$.*

3. TWO PRELIMINARY RESULTS

Before starting proofs of theorems of part 2 we prove two preliminary theorems. Then the main results can be easily obtained from these theorems (see part 4).

Let $\{\{\xi_{ni}, i = 1, 2, ..., k_n\}, n = 1, 2, ...\}$ be an infinite double array of random variables. Assume that for any n $\xi_{ni}, i = 1, 2, ..., k_n$ are independent and identically distributed Bernoulli random variables with parameter p_n (i.e. have the distribution $b(1, p_n)$). Define a sequence of staochastic processes $\{S^{(n)}\}$ in a Skorohod space D by

$$S^{(n)}(\tau) = \sum_{i=1}^{[\tau k_n]} \xi_{ni}, \tau \in [0, \infty), n = 1, 2, ... \qquad (7)$$

Theorem 7. *If $k_n \to \infty$ and $k_n p_n \to C \in (0, \infty)$ as $n \to \infty$, then $S^{(n)}$ converges in $D[0,\infty)$ to a homogeneous Poisson process with intensity C.*

Proof. First note that $[\tau k_n]p_n \to C\tau, n \to \infty$, and by classical Poisson limit theorem we have that distribution of $S^{(n)}(\tau)$ for any τ converges to the Poisson distribution with the parameter $C\tau$. Since increments of $S^{(n)}$ are independent, this means that joint distributions of increments converge to ones of the Poisson process. According to Corollary 1 to Theorem 5 in [9] (see [9], p. 31) from convergence of increments we obtain that the finite dimensional distributions converge to respective distributions of the Poisson process.

Now we prove convergence in $D[0,\infty)$. It follows from Theorem VI. 16 in [11] that to prove convergence in Skorohod topology on $D[0,\infty)$, besides of convergence of finite dimensional distributions, it suffices to show satisfaction of so called Aldous condition. Namely we have to prove that for each fixed T

$$\Delta S^{(n)} = S^{(n)}(\rho(n)+\delta(n)) - S^{(n)}(\rho(n)) \xrightarrow{P} 0$$

as $n \to \infty$, where $\{\delta(n), n \geq 1\}$ is a sequence of positive numbers converging to zero and $\{\rho(n), n \geq 1\}$ is a seguence of stopping times taking values in $[0,T]$. Recall that the stopping time property means that the event $\{\rho(n) \leq x\}$ should belong to the σ-field generated by random variables $\{S^{(n)}(u), 0 \leq u \leq x\}$.

Let $\varepsilon > 0$. By Chebyshev inequality for positive random variables we have

$$P\{\Delta S^{(n)} > \varepsilon\} \leq \frac{1}{\varepsilon}\left\{ES^{(n)}(\rho(n)+\delta(n)) - ES^{(n)}(\rho(n)\right\}.$$

If we use the definition of $S^{(n)}$ and generalized Wald's identity (see [12], Theorem 3, Ch. VII.2), we obtain that the expression on the right side equals

$$\frac{1}{\varepsilon}\{E[(\rho(n)+\delta(n))k_n] - E[\rho(n)k_n]\}p_n.$$

Taking into account the condition $k_n p_n \to C \in (0,\infty), n \to \infty$, we conclude that the last expression converges to zero as $n \to \infty$ for any sequence of stopping times $\{\rho(n)\}$ and sequence $\delta(n) \to 0, n \to \infty$. Thus $\Delta S^{(n)} \xrightarrow{P} 0, n \to \infty$, i.e. Aldous condition is satisfied. Theorem 7 is proved.

Theorem 8. *Assume that there exists a sequence $\{l_n, n \geq 1\}$ such that $l_n p_n(1-p_n) \to C \in (0,\infty)$ and $k_n/l_n \to \infty$. Then the sequence of processes*

$\{X^{(n)}\}_{n=1,2,\ldots}$ *defined by*

$$X^{(n)}(\tau) = \frac{S^{(n)}(\tau) - [\tau k_n]p_n}{\sqrt{Ck_n/l_n}}, \tau \in [0,1],$$

converges in $D[0,1]$ *to the Wiener process.*

Proof. Since $S^{(n)}(t)$ for any fixed τ is binomial $b([\tau k_n], p_n)$ random variable, by the central limit theorem it follows that the distribution of

$$\frac{S^{(n)}(\tau) - [\tau k_n]p_n}{\sqrt{[\tau k_n]p_n(1-p_n)}}$$

converges to the standard normal distribution. Now, taking into account that $[\tau k_n]p_n(1-p_n)/(Ck_n/l_n) \to \tau$ as $n \to \infty$ and independence of increments of $X^{(n)}$, we obtain that the joint distribution of it's increments converges as $n \to \infty$ to joint distributions of increments of the Wiener process. Hence due to the mentioned above Corollay 1 in [9] (see [9], p. 31), the finite dimensional distributions converge to respective finite dimensional distributions of the Wiener proces.

Now we prove convergence in Skorohod topology. According to Theorem 15.6 in [9], besides of convergence of finite dimensional distributions, it suffices to show that there exists a positive constant C_0 such that for $0 \le r < s < \tau \le 1$

$$E(X^{(n)}(s) - X^{(n)}(r))^2(X(n)(\tau) - X^{(n)}(s))^2 \le C_0(\tau - r)^2. \qquad (8)$$

In fact because the increments of $X^{(n)}(\tau)$ are independent, the expectation on the left side of (8) is non greater than

$$([sk_n] - [rk_n])([\tau k_n] - [sk_n])p_n^2(1-p_n)^2 l_n^2/C^2k_n^2 \le C_0(t-r)^2.$$

The theorem is proved.

4. PROOFS OF MAIN THEOREMS

Proof of Theorem 1. First we prove that, if conditions of Theorem 1 are satisfied, then

$$R(\theta(t),t) \sim \frac{2}{\sigma^2 t}\exp\left\{-\frac{2\theta}{\sigma^2}\right\}, t \to \infty. \tag{9}$$

For it we use the following well known results for critical branching processes (see [10], Ch II). If $A = 1, 0 < \sigma^2 < \infty$, then for any fixed $x > 0$

$$P\{Q(t)X(t) > x | X(t) > 0\} \sim e^{-x}, \quad Q(t) \sim 2/\sigma^2 t, \qquad \text{as } t \to \infty. \tag{10}$$

Taking into account condition (1) and using (10) we obtain that

$$R(\theta(t),t) = P\{X(t) > \theta(t) | X(t) > 0\} Q(t)$$

is equivalent as $t \to \infty$ to the expression on the right side of (9). Thus assretion of Theorem 1 follows from Theorem 7 by taking $k_t = t, t \in \mathbb{N}_0, C = (2/\sigma^2) exp\{-2\theta/\sigma^2\}$ and ξ_{ti} =indicator of $\{X_i(t) > \theta(t)\}$.

Proof of Theorem 2. It follows from (9) that $p_t(1-p_t)l_t \to C$ as $t \to \infty$, where $p_t = R(\theta(t),t), l_t = t$ and C as in the proof of Theorem 1. Thus we obtain the assertion of Theorem 2 from Theorem 8 by taking $k_t = [ta(t)], t \in \mathbb{N}_0$. The theorem is proved.

Proof of Theorem 3. First we prove that, if conditions of Theorem 3 are satisfied, then

$$R(\theta(t),t) \to 1 - \pi(\theta), \tag{11}$$

as $t \to \infty$. To do it we consider the estimate

$$|P\{X(t) \le \theta(t)\} - \pi(\theta)| \le \sup_x |P\{X(t)A^{-t} \le x\} - \pi(x)| + |\pi(\theta(t)A^{-t}) - \pi(\theta)|. \tag{12}$$

First term on the right side of (12) tends to zero as $t \to \infty$ due to the limit theorem for supercritical processes. It follows from condition (2) and continuity of $\pi(x)$ that the limit as $t \to \infty$ of the second term is also zero.

Recall that $S(n,t), n = 1,2,...,t \in \mathbb{N}_0$, is binomial $b(n,p_t)$ random variable with $p_t = R(\theta(t),t)$. Since $p_t \to 1-\pi(\theta)$ as $t \to \infty$ due to (11), the distribution of $S(n,t)$ for any fixed n as $t \to \infty$ converges to $b(n, 1-\pi(\theta))$. Consequently the finite dimensional distributions of $S(n,t)$ converge to respective finite dimensional distributions of the proces $\xi(n)$. Theorem 3 is proved.

Proof of Theorem 4. It follows from (11) that $R(\theta(t),t)(1-R(\theta(t),t)) \to \pi(\theta)(1-\pi(\theta))$ as $t \to \infty$. Consequently the assertion of Theorem 4 follows from Theorem 8, if we take $p_t = R(\theta(t),t), l_t = 1, k_t = [a(t)]$ and $C = \pi(\theta)(1-\pi(\theta))$.

Proof of Theorem 5. First we note that, if conditions of Theorem 5 are satisfied, then

$$R(\theta(t),t) \sim KA^t \sum_{j>\theta} P_j^*, \quad t \to \infty. \tag{13}$$

In fact, since $R(\theta(t),t) = P\{X(t) > \theta(t) | X(t) > 0\}Q(t)$, relation (13) follows from the limit theorem for subcritical processes and (6).

Now we obtain the proof of the theorem from Theorem 7 by taking $p_t = R(\theta(t),t), k_t = [A^{-t}]$ and $C = K\sum_{j>\theta} P_j^*$. The theorem is proved.

Proof of Theorem 6. It follows from (13) that $p_t(1-p_t) \to C$ as $t \to \infty$, where p_t and C as in the proof of Theorem 5. Thus the assertion of Theorem 6 follows from Theorem 8, if we take $k_t = [a(t)A^{-t}]$ and $l_t = A^{-t}$. The theorem is proved.

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