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Abul Hassan Siddiqi

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

On Current Developments in Evolution Variational Inequalities

Abul Hassan Siddiqi
Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
P.O. Box 1745, Dhahran 31261, Saudi Arabia

Abstract

In this paper we present current researches concerning parallel algorithms of evolution variational inequalities and their sensitivity analysis. The existence of solution, stability and error estimation of rate-independent evolution variational inequality with a nonlinear elliptic part are also presented. These results are closely related to the work of Lions [24], Han, Reddy and Schroeder [14], and Siddiqi and Manchanda [29].

1. Introduction

Variational inequalities are mathematical models of free boundary problems representing important physical phenomenon [1–5, 8–10, 14, 20–22, 24, 26, 27] of science, technology and industry. Attempts were made to develop faster algorithms (see, for example, Hoppe [17–19], Hackbush and Mittelmann [13], Koçvara and Zowe [23]). More recently, in Lions [24] and references therein, parallel algorithms for the solution of evolution variational inequalities have been investigated. Han, Reddy and Schroeder [14] have formulated the quasi-static problem of elastoplasticity as a time-dependent variational inequality of the mixed kind. This variational inequality differs from the standard parabolic variational inequality in that time derivatives of the unknown variable occur in all of its terms. Section 2 is devoted to the formulation of evolution variational inequalities. In Section 3, we present a resumé of Lions results along with our results on parallel algorithms for Han et. al type variational inequality. In Section 4, we study the existence of solutions, stability and error estimation of variational inequality containing derivative in all terms (Han, Reddy and Schroeder type where the bilinear form is replaced by a nonlinear operator). Section 5 deals with the sensitivity analysis of evolution variational inequalities.

2. Formulation of Evolution Variational Inequality

Variational inequalities of evolution were introduced by Lions and Stampacchi (see, for example, Lions [24] for updated references).

Let

$$V \subset H, V \text{dense in } H, \quad V \to H \text{ being continuous.} \tag{2.1}$$

We identify H with its dual so that if V' denotes the dual of V, then

$$V \subset H \subset V'. \tag{2.2}$$

We shall use the notations.

Here, $L_2(0,T;H)$ denote the space of all measurable functions $u : [0,T] \to H$, then it is a Hilbert space with respect to the inner product

$$\langle u, v\rangle_{L_2(0,T;H)} = \int_0^T \langle u(t), v(t)\rangle_H dt. \tag{2.3}$$

If H' is the topological dual of H, then the dual of $L_2(0,T;H) = L_2(0,T;H')$ for any Hilbert space H. $W^{1,2}(0,T;H)$ $(W^{1,1}(0,T;H))$ will denote the space of all those elements of $L_2(0,T;H)$ $\Big((L_1(0,T;H)\Big)$ such that their distributional derivatives D_f also belong to $L_2(0,T;H)$ $\Big(L_1(0,T;H)\Big)$.

We have also a set $K \subset V$ such that

$$K \text{ is a closed convex subset of } V. \tag{2.4}$$

We do not restrict ourselves generally (it suffices to make a translation) by assuming that

$$0 \in K. \tag{2.5}$$

Let f be given such that

$$f \in L_2(0,T;V'). \tag{2.6}$$

We consider now a bilinear form

$$\left.\begin{array}{l}(u,\hat{u}) \to a(u,\hat{u}) \text{ which is continuous on } V \times V \\ a(u,\hat{u}) \text{ is symmetric or not,} \\ a(u,u) \geq \alpha \|u\|_V^2, \quad \alpha > 0 \ \forall\, u \in V\end{array}\right\} \tag{2.7}$$

where we denote by $\|u\|_X$ the norm of u in X.

We look for u such that

$$\left.\begin{array}{l}u \in L_2(0;T;V) \cap L^\infty(0,T;H), u(t) \in K \text{ a.e.,} \\ \left\langle \dfrac{\partial u}{\partial t}, \hat{u} - u \right\rangle + a(u, \hat{u} - u) \geq (f, \hat{u} - u) \ \ \forall \in K \\ u(0) = 0\end{array}\right\}. \tag{2.8}$$

The solution has to be thought of as being a weak solution of (2.8), otherwise the condition $u(0) = 0$ in (2.8) is somewhat ambiguous. This condition becomes precise if we add the condition

$$\frac{\partial u}{\partial t} \in L_2(0,T;V'), \tag{2.9}$$

but this condition can be too restrictive. We can introduce weak solutions in the following form.

We consider smooth functions $\hat{u}$ such that

$$\left.\begin{array}{l}\hat{u} \in L_2(0,T;V), \ \dfrac{\partial \hat{u}}{\partial t} \in L_2(0,T;V') \\ \hat{u} \in K \text{ for a.e.t}, \ \ \hat{u}(0) = 0.\end{array}\right\} \tag{2.10}$$

Then, if u satisfies (2.8) and is supposed to be smooth enough, we have (we write $\langle u, \hat{u} \rangle$ instead of $\langle u, \hat{u} \rangle_H$)

$$\int_0^T \left[\left\langle \frac{\partial \hat{u}}{\partial t}, \hat{u} - u \right\rangle + a(u, \hat{u} - u) - (f, \hat{u} - u) \right] dt$$

$$= \int_0^T \left[\frac{\partial u}{\partial t}, \hat{u} - u) + a(u, \hat{u} - u) - (f, \hat{u} - u) \right] dt$$
$$+ \int_0^T \left(\frac{\partial(\hat{u} - u)}{\partial t}, \hat{u} - u \right) dt.$$

The last term equals $\frac{1}{2}\|\hat{u}(T) - u(T)\|_H^2$ $\left(\text{since } u(0) = 0, \hat{u}(0) = 0\right)$, so that

$$\int_0^T \left[\left\langle \frac{\partial \hat{u}}{\partial t}, \hat{u} - u \right\rangle + a(u, \hat{u} - u) - (f, \hat{u} - u) \right] dt \geq 0, \tag{2.11}$$

for all $\hat{u}$ satisfying (2.10).

We then define a weak solution of (2.8) as a function u such that

$$u \in L_2(0, T; V), \ u(t) \in K \text{ a.e.} \tag{2.12}$$

and which satisfies (2.11) for all $\hat{u}$ satisfying (2.10).

Recently, Han Reddy and Schroeder [14] have formulated the elastic-plastic problem (EP) in the form of an abstract variational inequality. This variational inequality closely resembles the evolution variation inequality discussed above with the important distinction that the rate quantity occurs in the arguments of all the functionals in the inequality. This kind of variational inequality also arises in elasticity with frictional contact, see Duvaut and Lions [8]. We consider here the following classes of evolution of variational inequalities.

Find $w : [0, T] \to H$, $w(0) = 0$, such that for almost all $t \in (0, T), \dot{w}(t) \in K$ and

$$a\Big(w(t), z - \dot{w}(t)\Big) \geq \langle \ell(t), z - \dot{w}(t) \rangle \qquad \forall\, z \in K. \tag{2.13}$$

Here H denotes a Hilbert space, K a nonempty, closed, convex cone in H. The bilinear form $a : H \times H \to R$ is symmetric, bounded, and H-elliptic, $\ell \in W^{1,2}(0, T; H')$. Han et. al have studied the existence and uniqueness of solution, stability and error estimation. In the next section, we study parallel algorithm for this class of variational inequality. In Section 4, we study the existence, stability and error estimation of a class of variation inequality where $a(\cdot,\cdot)$ in (2.13) is replaced by $A : H \to H'$, a nonlinear operator on the Hilbert space H into its dual H'.

3. Decomposition Method and Parallel Algorithms

We introduce N couples of Hilbert spaces V_i and H_i, and N convex sets K_i:

$$V_i \subset H_i \subset V_i' \qquad i = 1, 2, \ldots, N \tag{3.1}$$

$$K_i \subset V_i, \quad K_i \text{ closed convex subset of } V_i, \text{ nonempty.} \tag{3.2}$$

We are given linear operators r_i such that

$$\left.\begin{aligned} & r_i \in \mathcal{L}(H, H_i) \cap \mathcal{L}(V; V_i) \\ & r_i \text{ maps } K \text{ into } K_i \end{aligned}\right\} \qquad i = 1, 2, \ldots, N. \tag{3.3}$$

We are also given a family of Hilbert spaces H_{ij} such that

$$H_{ij} = H_{ji} \quad \forall\, i, \;\; j \in [1, 2, \ldots N], \tag{3.4}$$

and a family of operators r_{ij} such that

$$r_{ij} \in \mathcal{L}(H_j; H_{ij}). \tag{3.5}$$

The following hypotheses are made:

$$r_j r_{ji} \varphi = r_i r_{ij} \varphi \qquad \forall\, \varphi \in V, \tag{3.6}$$

if N elements u_i are given such that

$$u_i \in K_i \qquad \forall\, i, \quad r_{ij} u_j = r_{ji} u_i \qquad \forall\, i, j, \tag{3.7}$$

then there exists $u \in K$ such that

$$u_i = r_i u, \text{ and moreover } \|u\|_V^2 \le c \left(\sum_{i=1}^{N} \|u\|_{V_i}^2 \right). \tag{3.8}$$

The hypothesis

$$K_i = V_i \text{ for a subset of } [1, 2, 3, \ldots, N], \tag{3.9}$$

is perfectly acceptable!

We now proceed with the decomposition of the problem. We introduce the following bilinear forms:

$c_i(u_i, \hat{u}_i)$ is continuous, symmetric on $H_i \times H_i$ and it satisfies

$$c_i(u_i, u_i) \geq \alpha_i \|u_i\|_{H_i}^2, \quad \alpha_i > 0, \qquad \forall\, u_i \in H, \tag{3.10}$$

$a_i(u_i, \hat{u}_i)$ is continuous, symmetric or not, on $V_i \times V_i$, and it satisfies

$$a_i(u_i, u_i) \geq \beta_i \|u\|_{V_i}^2, \quad \beta_i > 0, \quad \forall\, u_i \in H_i, \tag{3.11}$$

We assume that

$$\sum_{i=1}^{N} c_i(r_i u, r_i \hat{u}) = \langle u, \hat{u} \rangle_H, \qquad \forall\, u, \hat{u} \in H, \tag{3.12}$$

$$\sum_{i=1}^{N} a_i(r_i u, r_i \hat{u}) = a(u, \hat{u}), \qquad \forall\, u, \hat{u} \in V. \tag{3.13}$$

Finally we assume that the function f is also decomposed as follows: We are given functions $f_i \in L_2(0, T_i; V_i')$ such that

$$\sum_{i=1}^{N} (f_i, r_i \hat{u}) = \langle f, \hat{u} \rangle = (f, \hat{u}) \qquad \forall\, \hat{u} \in V. \tag{3.14}$$

We are now ready to introduce the decomposed approximation. We look for functions u_i $(i = 1, 2, 3, \ldots, N)$ such that

$$c_i\left(\frac{\partial u_i}{\partial t}, \hat{u}_i - u_i\right) + a_i(u_i, \hat{u}_i - u_i) + \frac{1}{\epsilon} \sum_j \langle r_{ji} u_i - r_{ij} u_j, r_{ji}(\hat{u}_i - u_i \rangle_{H_{ij}}$$

$$\geq (f_i, \hat{u}_i - u) \quad \forall\, \hat{u}_i \in K_i, \tag{3.15}$$

$$u_i \in L_2(0, T; V_i), \qquad u_i(t) \in K_i \text{ a.e. } u_i(0) = 0. \tag{3.16}$$

Remark 3.1. It may be remarked that each of the variational inequality in (3.15) has to be thought of in its weak formulation as introduced in the preceding section.

Remark 3.2. In (3.15), ϵ is positive and small. The corresponding term in (3.15) is a penalty term.

Remark 3.3. In the examples $\|r_{ji}\|$ is a sparse matrix.

Theorem [24]. *The set of (decomposed) variational inequalities (3.15) and (3.16) admits a unique solution $u_i = u_i^\epsilon$ $(i = 1, 2, 3, \dots N)$. Further, as $\epsilon \to 0$, one has*

$$u_i^\epsilon \to u_i \textit{ in } L_2(0, T; V_i) \textit{ weakly} \tag{3.17}$$

and

$$u_i = r_i u,$$

where u is the solution of (2.8) (weak form (2.11)).

Assuming $H = V$ and keeping in mind the above terminology decomposed approximation of (2.13) can be written as

$$a_i\Big(u_i, \hat{u}_i - \dot{u}_i(t)\Big) \geq \Big(\ell(t), \hat{u}_i - \dot{u}_i(t)\Big) + \cdots \tag{3.18}$$
$$+\frac{1}{\epsilon}\sum_j \langle (r_{ji}\dot{u}_i - r_{ij}\dot{u}_j, r_{ji}(\hat{u}_i - \dot{u}_i)\rangle_{H_{ij}}$$

$$u_i \in L_2(0, T; H_i), \dot{u}_i(t) \in K_i \text{ a.e. } \ u_i(0) = 0. \tag{3.19}$$

We prove the following theorem.

Theorem 3.1. The set of (decomposed) variational inequalities (3.18), (3.19) admits a unique solution $u_i = u_i^\epsilon (i = 1, 2, \dots, N)$. Further, as $\epsilon \to 0$, we have

$$u_i^\epsilon \to u_i \text{ in } L_2(0, T; H) \text{ weakly}$$

and

$$u_i = r_i u \qquad \text{where } u \text{ is the solution of (2.13).} \tag{3.20}$$

Proof of Theorem 3.1. We can assume without loss of generality that $0 \in K_i$. Therefore, taking $\hat{u}_i = 0$ in (3.18) is allowed. Using penalty arguments we get

$$a_i\Big(u_i, \dot{u}_i(t)\Big) + \frac{1}{\epsilon} X_i \leq \Big(f_i, \dot{u}_i(t)\Big) \qquad i = 1, 2, 3, \dots N \tag{3.21}$$

$$\text{where } \ X_i = \sum_j \langle r_{ji}\dot{u}_i - r_{ij}\dot{u}_j, r_{ji}\dot{u}_i\rangle_{H_{ij}} \, . \tag{3.22}$$

We can write

$$\sum X_i = \frac{1}{2} \sum_{i,j} \|r_{ji}\dot{u}_i - r_{ij}\dot{u}_j\|^2_{H_{ij}}. \tag{3.23}$$

Integration in t, in the interval $(0, t)$ of (3.21), and by summing in i, using (3.23), we obtain

$$\left.\begin{aligned} &\sum_i \int_0^t a_i\Big(\dot{u}_i(s)\Big) ds + \frac{1}{2\epsilon} \sum_{i,j} \int_0^t \|r_{ji}\dot{u}_i(s) - r_{ij}\dot{u}_j(s)\|^2_{H_{ij}} ds \\ &\qquad \leq \sum_i \int_0^t (f_i, \dot{u}_i) ds. \end{aligned}\right\} . \tag{3.24}$$

It follows from (3.24), (3.11) that as $\epsilon \to 0$, u_i^ϵ remains in a bounded set of $L_2(0, T; H_i) \cap L^\infty(0, T; H)$, $\dot{u}_i^\epsilon(t) \in K_i$.

$$\frac{1}{\sqrt{\epsilon}} \left(r_{ji}\dot{u}_i^\epsilon - r_{ij}\dot{u}_j^\epsilon\right) \text{ remains in a bounded set of } L_2(0, T; H_{ij}). \tag{3.25}$$

Therefore, we can extract a subsequence, still denoted by u_i^ϵ, such that

$$\left.\begin{aligned} &u_i^\epsilon \to u_i \text{ in } L_2(0, T; V_i) \text{ weakly} \\ &\dot{u}_i(t) \in K_i, \end{aligned}\right\} \tag{3.26}$$

and, by virtue of (3.25), we have

$$r_{ji}u_i = r_{ij}u_j \qquad \forall\, i, j. \tag{3.27}$$

Notice that we have not used the fact that $\dot{u}_j^\epsilon$ remains in a bounded set of $L^\infty(0, T; H_i)$. It follows from (3.26), (3.27), and the hypothesis (3.7) that

$$u_i = r_i u_*, \dot{u}_*(t) \in K \text{ a.e. } u_* \in L_2(0, T, H). \tag{3.28}$$

It remains to show that $u_* = u$, the solution (2.13). To avoid complication, let

$$\int_0^T \Big[a\Big(\hat{u}, \hat{u} - \dot{u}(t)\Big) - \Big(f, \hat{u} - \dot{u}(t)\Big)\Big] dt \geq 0, \tag{3.29}$$

for all $\hat{u}$ satisfying (2.10).

By introducing $\hat{u}_i$ such that

$$\left.\begin{aligned} &\hat{u}_i \in L_2(0,T;H), \frac{\partial \hat{u}_i}{\partial t} \in L_2(0,T;H_i'), \\ &\hat{u}_i(t) \in K_i \text{ for a.e.t.}, \ \hat{u}_i(0) = 0. \end{aligned}\right\} \tag{3.30}$$

and we replace (3.24) by its weak form

$$\left.\begin{aligned} &\int_0^T a_i(\hat{u}_i, \hat{u}_i - \dot{u}_i^\epsilon) + \frac{1}{\epsilon}\sum_j \Big[\langle r_{ji}\hat{u}_i - r_{ij}\hat{u}_j), r_{ji}(\hat{u}_i - \dot{u}_i^\epsilon\rangle_{H_{ij}}\Big] dt \\ &\geq \int_0^T (f_i, \hat{u}_i - \dot{u}_i^\epsilon). \end{aligned}\right\} \tag{3.31}$$

Let us now assume that

$$\left.\begin{aligned} &\hat{u}_i = r_i\varphi, \\ &\varphi \in L_2(0,T;V), \frac{\partial \varphi}{\partial t} \in L_2(0,T;H'), \varphi(0) = 0, \quad \varphi(t) \in K. \end{aligned}\right\} \tag{3.32}$$

Since $r_{ji}r_i\varphi = r_{ij}r_j\varphi$, the $\frac{1}{\epsilon}$ terms in (3.31) drop out, so that

$$\int_0^T a_i(r_i\varphi, r_i\varphi - \dot{u}_i^\epsilon) dt \geq \int_0^T (f_i, r_i\varphi - \dot{u}_i^\epsilon) dt. \tag{3.33}$$

Because of (3.28) and taking limit of ϵ we get

$$\int_0^T a_i(r_i\varphi, r_i\varphi - r_i\dot{u}_*) dt \geq \int_0^t (f_i, r_i\varphi - r_i\dot{u}_*) dt. \tag{3.34}$$

Summing (3.34) in i and using (3.13) and (3.14), we obtain

$$\int_0^T \Big[a(\varphi, \varphi - \dot{u}_*) - (f, \varphi - \dot{u}_*)\Big] dt \geq 0;$$

so that $u_* = u$. This proves the theorem.

Parallel Algorithm

We introduce the time step Δt and a semi-discretization. We denote by u_i^n an approximation of $u_i(n\Delta t)$. We then define u_i^n by

$$\left.\begin{aligned} & a_i\left(u_i^n, \hat{u}_i - \frac{u_i^n - u_i^{n-1}}{\Delta t}\right) + \frac{1}{\epsilon}\sum_j \langle r_{ji}u_i^n - r_{ij}u_j^{n-1}, r_{ji}(\hat{u}_i - u_i^n)\rangle_{H_{ij}} \\ & \quad \geq \left(f_i^n, \hat{u}_i - \frac{u_i^n - u_i^{n-1}}{\Delta t}\right) \\ & u_i^n \in K_i \qquad (n = 1, 2, 3, \ldots) \text{ where } f_i^n = \frac{1}{\Delta t}\int_{(n-1)\Delta t}^{n\Delta t} f_i(t)dt, u_i^0 = 0. \end{aligned}\right\} \tag{3.35}$$

The algorithm (3.35) is parallel. Each u_i^n is computed through the solution of a stationary variational inequalities. Once the u_j^{n-1} are computed, in the computation u_i^n, only those j such that $r_{ij} \neq 0$ are used. The stability of algorithm will be studied.

4. Rate-independent Evolution Variational Inequality with a Nonlinear Elliptic Part

In this section, we discuss the existence of solution, stability and error estimation of the variational inequality (2.13) where the bilinear form $a(\cdot, \cdot)$ is replaced by a nonlinear operator $A : H \to H^*$, so that the problem takes the form

Find $w : [0, T] \to H, w(0) = 0$ such that for almost all $t \in (0, T)$ $\dot{w}(t) \in K$ and $\langle Aw(t), z - \dot{w}(t)\rangle \geq \langle \ell(t), z - \dot{w}(t)\rangle, \quad \forall\, z \in K \qquad \dot{w}(t) \in K, w(0) = 0.$

We prove here the following theorems.

Theorem 4.1. *Let H be a Hilbert space, $K \subseteq H$, $K \neq \phi$, closed and convex cone; $A : H \to H^*$ be monotone coercive and Lipschitz continuous. Furthermore, let $\ell \in W^{1,2}(0, T; H^*)$ with $\ell(0) = 0$, then there exists at least one $w : [0, T] \to H$, $w \in$*

$W^{1,2}(0,T;H)$ *satisfying*

$$\langle Aw(t), z - \dot{w}(t)\rangle \geq \langle \ell(t), z - \dot{w}(t)\rangle; \tag{4.1}$$

w is unique if, in addition, A is strictly monotone.

Theorem 4.2 (Stability). *Under the assumption of Theorem 1, the solution of the variational inequality in* (4.1) *depends continuously on ℓ, namely for $\ell_1, \ell_2 \in W^{1,2}(0,T;H)$ with $\ell_1(0) = \ell_2(0)$, the corresponding solutions w_1, w_2 satisfy*

$$\|w_1 - w_2\|_{L^\infty(0,T;H)} \leq C\left(\|\ell_1 - \ell_2\|_{L^\infty(0,T;H^*)} + \|\dot{\ell}_1 - \dot{\ell}_2\|_{L^1(0,T;H^*)}\right). \tag{4.2}$$

Theorem 4.3 (Error Estimation). *Let P_h denote the approximate problem*

P_h : Find $w^h : [0,T] \to H^h$, where $w^h(0) = 0$ and H^h is a finite-dimensional subspace of H, such that for almost all $t \in (0,T)$, $\dot{w}^h(t) \in K^h$ (K^h is a non-empty, closed and convex cone for a given h) and

$$\left\langle Aw^h(t), z^h - \dot{w}^h(t)\right\rangle \geq \left\langle \ell(t), z^h - \dot{w}^h(t)\right\rangle \quad \textit{for all} \quad z^h \in K^h. \tag{4.3}$$

(i) Let $w_1^h(t)$ and $w_2^h(t)$ be two solutions of the problem P_h given by (4.3), then

$$\left\|w_1^h - w_2^h\right\|_{L^\infty(0,T;H)} \leq C\left(\|\ell_1 - \ell_2\|_{L^\infty(0,T;H')} + \|\dot{\ell}_1 - \dot{\ell}_2\|_{L^1(0,T;H')}\right).$$

(ii) $\|w(t) - w^h(t)\|_{L^\infty(0,T;H)} \leq C \inf_{z^h \in L_2(0,T;K^h)} \|w - z^h\|^{1/2}_{L_2(0,T;H)}$.

We require the following lemmas:

Lemma 4.1 [33]

$$\|f(t) - f(s)\|_X \leq \int_s^t \|\dot{f}(\tau)\|_X d\tau \tag{4.4}$$

for $s < t$ and $f \in W^{1,2}(0,T;H)$.

Lemma 4.2. *For any given $\{\ell_n\} \in H', \ell_0 = 0$ there exists $\{w_n\}_{n=0}^N \subset H$, with $w_0 = 0$ such that for $n = 1, 2, 3, \ldots, N$, $\Delta w_n \in K$ and*

$$A(w_n, z - \Delta w_n) - \langle \ell_n, z - \Delta w_n \rangle \geq 0 \qquad \forall\, z \in H. \tag{4.5}$$

Furthermore, there exists a constant c independent of $\epsilon = t_n - t_{n-1}$ $(0 = t_1 < t_2 < t_3 < \cdots < t_N = T)$ is any partition of the time interval $[0, T]$ such that

$$\|\Delta w_n\| \leq c\|\Delta \ell_n\|_{H'}, \qquad n = 1, 2, 3, \ldots, N. \tag{4.6}$$

Lemma 4.3. *Assume $\ell \in W^{1,2}(0,T;H')$, $\ell(0) = 0$, then the solution $\{w_n\}_{n=0}^N$ defined in Lemma 4.2 satisfies*

$$\max_{1 \leq n \leq N} \|w_n\|_H \leq c\|\dot{\ell}\|_{L_2(0,T;H')} \tag{4.7}$$

and

$$\sum_{n=1}^{N} \|\Delta w_n\|_H^2 \leq c\epsilon\|\dot{\ell}\|^2_{L_2(0,T;H')} \tag{4.8}$$

Proof of Lemma 4.2. It can be easily seen that (4.5) has a unique solution. We prove here (4.6). We have

$$\langle A w_n, z - w_n \rangle \geq \langle \ell_n, z - w_n \rangle \quad \forall\, z \in K_n, \qquad w_n \in K_n \tag{4.9}$$

Set $K_n = w_{n-1} + K$, K_n is closed and convex. Let w_n be a solution of (4.9). Then

$$\Delta w_n = w_n - w_{n-1} \in K. \tag{4.10}$$

(4.9) is equivalent to

$$\langle A w_n, (z - w_{n-1} - (w_n - w_{n-1})) \rangle \geq \langle \ell_n, (z - w_{n-1}) - w_n - w_{n-1}) \rangle \qquad \forall\, z \in K_n. \tag{4.11}$$

(4.11) is equivalent to

$$\langle Aw_n, z - \Delta w_n\rangle \geq \langle \ell_n, z - \Delta w_n\rangle \qquad \forall\, z \in K. \tag{4.12}$$

Choosing $z = 0$ and $z = \Delta w_n + \Delta_{n-1}$ in (4.12) and by subtracting the resultant equations we get

$$\langle Aw_n - Aw_{n-1}, \Delta w_n\rangle \geq \langle \Delta\ell_n, \Delta w_n\rangle. \tag{4.13}$$

By the coercivity of A, we get

$$\frac{1}{c_n}\|\Delta w_n\|^2 \leq \|\Delta\ell_n\|\;\|\Delta w_n\|$$

or

$$\|\Delta_n\|^2 \leq c_n\|\Delta\ell_n\|_H, \qquad n = 1, 2, 3, \ldots, N.$$

Proof of Lemma 4.3.

$$\begin{aligned}
\|w_n\| &= \|w_n - w_{n-1} + w_{n-1} - w_{n-2} + \cdots + (w - w_0)\| \\
&\leq \sum_{k=1}^{n} \|\Delta_k\| \\
&\leq c\sum_{k=1}^{n} \|\Delta_k\| \\
\|\Delta\ell_n\| &= \|\ell(t_n) - \ell(t_{n-1})\| = \left\|\int_{t_{n-1}}^{t_n} \dot{\ell}(t)dt\right\| \\
&\leq c\sum_{k=1}^{n}\left\|\int_{t_{k-1}}^{t_k} \dot{\ell}(t)dt\right\| \leq c\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} \|\dot{\ell}(t)\|dt \\
&= c\int_0^{t_n} \|\dot{\ell}(t)\|dt \\
&\leq c\int_0^{T} \|\dot{\ell}(t)\|dt = c\|\dot{\ell}\|_{L^1(0,T;H')} \\
\text{or} \quad &\max_{1\leq n\leq N} \|w_n\| \leq c\|\dot{\ell}\|_{L^1(0,T;H')}.
\end{aligned}$$

Let $\epsilon = t_n - t_{n-1}$. Define

$$w_\epsilon(t) = w_{n-1} + \frac{\Delta w_n}{\epsilon}(t_n - t_{n-1}).$$

By Lemma 4.1 and above results, we get

$$\sum_{n=1}^{N} \|\Delta w_n\|_H^2 \leq c\epsilon \|\dot{\ell}\|_{L_2(0,T;H^*)}.$$

Proof of Theorem 4.1. Proceeding on the lines of the proof of Theorem 4.3 [14] and keeping in mind conditions on A of Theorem 4.1, we find that

$$\begin{aligned} J_\epsilon = \int^T &\Big[\langle Aw_\epsilon(t), z - \dot{w}_\epsilon(t)\rangle - \langle \ell_\epsilon(t), z - \dot{w}_\epsilon(t)\rangle\Big] dt \\ &+ \frac{1}{2} c\epsilon \int_0^T \|\dot{\ell}(t)\|_{H'}^2 \, dt \geq 0 \end{aligned} \tag{4.14}$$

where

$$\left.\begin{aligned} &w_\epsilon(t) = w_{n-1} + \frac{\Delta w_n}{\epsilon}(t - t_{n-1}) \quad \text{for } t_{n-1} \leq t \leq t_n, \\ &w_\epsilon \in L^\infty(0,T;H), \dot{w}_\epsilon \in L_2(0,T;H), \\ &z(t) = z_n \text{ for } t_{n-1} \leq t \leq t_n, \quad n = 1,2,3,\ldots,N-1, \quad z_n \in H \\ &z(t) = z_N \text{ for } t_{N-1} \leq t \leq t_N \text{ and} \\ &z_{N+1} = 0, \\ &\delta w_n = \Delta w_n \\ &c \text{ is the constant of inequality (4.8).} \end{aligned}\right\} \tag{4.15}$$

By (4.7) and (4.8) and the definition of w_ϵ we see by direct evaluation that

$$\|w_\epsilon\|_{L^\infty(0,T;H)} \leq c_1 \text{ and } \|\dot{w}_\epsilon\|_{L_2(0,T;H)} \leq c_2.$$

If we fix a stepsize $\epsilon_0 > 0$ and consider the sequence of stepsize $\epsilon_k = 2^{-k}\epsilon_0, k = 0,1,\ldots,$ we get a sequence $\{w_{\epsilon_{k_i}}\}$ of sequence $\{w_{\epsilon_k}\}$ and a $w \in \dot{H}^{1,2}(0,T;H)$ such that $w_{\epsilon_{k_i}} \overset{*}{\rightharpoonup} w$ in $L^\infty(0,T;H)$ and $\dot{w}_{\epsilon_{k_i}} \rightharpoonup \dot{w}$ in $L_2(0,T;H)$ as $i \to \infty$ such that for $\dot{w}(t) \in K$

$$0 \leq \limsup_{i\to\infty} J_{\epsilon_{k_i}} \leq \int_0^T \langle Aw(t), z - \dot{w}(t)\rangle - \langle \ell(t), z - \dot{w}(t)\rangle] dt \tag{4.16}$$

for any step function z corresponding to a step-size $\epsilon_{k_i}, i = 1, 2, 3, \ldots$. Approximating any $z \in L_2(0,T;K)$ by its piecewise averaging step functions $z_{\epsilon_{k_i}}$, it follows that

$$\int_0^T \Big[\langle A(w(t)), z - \dot{w}(t)\rangle - \langle \ell(t), z - \dot{w}(t)\rangle\Big] dt \geq 0$$

$$\forall\, z \in L_2(0,T;K) \Rightarrow z_{\epsilon_{k_i}}(t) \in K \text{ a.e.t.} \tag{4.17}$$

This is equivalent to showing that w satisfies the variational inequality (4.1) a.e. on $[0,T]$. By the Sobolev embedding theorem $W^{1,2}(0,T;H) \subset C([0,T];H)$, and we observe that $w \in L_\infty(0,T;H)$ and $\dot{w} \in L_2(0,T;H)$ is equivalent to $w \in W^{1,2}(0,T;H)$.

Proof of Theorem 4.2. Let $\ell_1, \ell_2 \in W^{1,2}(0,T;H')$ be given $\ell_1(0) = \ell_2(0)$ and let w_1 and w_2 be corresponding solutions whose existence is assured by Theorem 4.1. Thus, for almost all $t \in (0,T), \dot{w}_1(t)$ and $\dot{w}_2(t) \in K$, and

$$\langle Aw_1(t), z - \dot{w}_1(t)\rangle \geq \langle \ell_1(t), z - \dot{w}_1(t)\rangle \quad \forall\, z \in K \tag{4.18}$$

$$\langle Aw_2(t), z - \dot{w}_2(t)\rangle \geq \langle \ell_2(t), z - \dot{w}_2(t)\rangle \quad \forall\, z \in K. \tag{4.19}$$

Take $z = \dot{w}_2(t) \in K$ in (4.18) and $z = \dot{w}_1(t)$ in (4.19) and by adding the resultant inequalities we get

$$-\frac{1}{2}\frac{d}{dt}\langle Aw_1(t) - Aw_2(t), w_1(t) - w_2(t)\rangle \geq \langle \ell_1(t) - \ell_2(t), \dot{w}_1(t) - \dot{w}_2(t)\rangle \tag{4.20}$$

by properties of A.

Or

$$\frac{1}{2}\frac{d}{dt}\langle Aw_1(t) - Aw_2(t), w_1(t) - w_2(t)\rangle \leq \langle \ell_1(t) - \ell_2(t), \dot{w}_1(t) - \dot{w}_2(t)\rangle \tag{4.21}$$

Let $e = w_1 - w_2$ and $e(0) = 0$. Then we get from (4.21) and keeping in mind conditions on A

$$\begin{aligned}\|e(t)\|_H^2 &\leq c\|\ell_1(t) - \ell_2(t)\|_{H'}\|e(t)\|_H \\ &+c\int_0^t \|\dot{\ell}_1(t) - \dot{\ell}_2(t)\|_H\|e(t)\|_H dt.\end{aligned}$$

Let $M = \sup_{0 \le t \le T} \|e(t)\|_H$, then

$$\|e(t)\|_H^2 \le c\|\ell_1(t) - \ell_2(t)\|_{H'} M + c\int_0^t \|\dot{\ell}_1(t) - \dot{\ell}_2\|_{H'}^M \, dt.$$

Hence,

$$M \le c\|\ell_1 - \ell_2\|_{L^\infty(0,T;H')} + \|\dot{\ell}_1 - \dot{\ell}_2\|_{L'(0,T;H')}.$$

This proves the theorem.

Proof of Theorem 4.3. Approximate problem. Find $w^h : 0, T] \to H^h$, $w^h(0) = 0$ such that for almost all $t \in (0, T, \dot{w}^h(t) \in K^h$ and

$$A(w^h(t), z^h - \dot{w}^h(t) \ge \langle \ell(t), z^h - \dot{w}^h(t)\rangle \ge 0 \qquad \forall\, z^h \in K^h. \tag{4.22}$$

The existence of a unique solution w^h of the approximate problem (4.22) follows from Theorem 4.1 with H and K replaced by H^h and K^h. We also note that $w^h \in W^{1,2}(0, T; H)$. This implies that $w^h \in C([0, T; H)$; in particular, the value $w^h(0)$ is well defined. From Theorem 4.2, we get the result

$$||w_1^h - w_2^h||_{L^\infty(0,T;H)} \le (\|\ell_1 - \ell_2)_{L^\infty(0,T;H')} + \|\dot{\ell}_1 - \dot{\ell}_2\|_{L_1(0,T;H')}$$

where w_1^h and w_2^h are approximate solutions corresponding to ℓ_1 and ℓ_2.

Let us prove the estimation of error $w - w^h$ similar to Cea's error estimation and its extension by Falk and Gwinner, see for example Gwinner [11].

Set $z = \dot{w}^h(t) \in K$ in (2.13) to get

$$\langle Aw(t), \dot{w}^h(t) - \dot{w}(t)\rangle \ge \langle \ell(t), \dot{w}^h(t) - \dot{w}(t)\rangle \tag{4.23}$$

By adding (4.22) and (4.23), we obtain

$$\langle A(wt), \dot{w}^h - \dot{w}(t)\rangle + \langle Aw^h(t), z^h - \dot{w}^h(t)\rangle \ge \langle \ell(t), z^h - \dot{w}(t)\rangle. \tag{4.24}$$

Using (4.24) and Theorem 4.1, we have for any $z^h \in K^h$

$$\left.\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\langle Aw(t) - Aw^h(t), w(t) - w^h(t)\rangle = \langle Aw(t) - w^h(t), \dot{w}(t) - \dot{w}^h(t)\rangle \\
&= \langle Aw(t) - Aw^h(t), \dot{w}(t) - z^h\rangle + \langle Aw(t) - Aw^h(t), z^h - \dot{w}^h(t)\rangle \\
&\leq \langle Aw - Aw^h(t), \dot{w}(t) - z^h\rangle + \langle Aw(t), z^h - \dot{w}^h(t)\rangle \\
&+\langle Aw(t), \dot{w}^h(t) - \dot{w}(t)\rangle - \langle \ell(t), z^h - \dot{w}(t)\rangle \\
&\leq \langle Aw(t) - Aw^h(t), \dot{w}(t) - z^h\rangle - \langle w^*(t), z^h - \dot{w}(t)\rangle, \quad w^* \in H
\end{aligned}\right\} \tag{4.25}$$

or

$$\frac{1}{2}\frac{d}{dt}\langle Aw(t) - Aw^h(t), w(t) - w^h(t)\rangle \leq \langle Aw(t) - Aw^h(t), \dot{w}(t) - z^h\rangle \tag{4.26}$$
$$+c\|z^h - \dot{w}(t)\|_H \qquad \forall\, z^h \in K^h$$

Since

$$\langle Aw(t) - Aw^h(t), \dot{w}(t) - z^h\rangle$$
$$\leq c(\|w(t) - Aw^h(t)\| + \|\dot{w}(t) - z^h(t)\|_H^2)$$

From (4.26), we find that for any $z^h = z^h(t) \in K^h$

$$\frac{d}{dt}\langle Aw(t) - Aw^h(t), w(t) - w^h(t)\rangle$$
$$\leq c(Aw(t) - Aw^h(t), w(t) - w^h(t)\| + \|\dot{w}(t) - z^h(t)\|_H^2 + \|\dot{w}(t) - z^h(t)\|_H). \tag{4.27}$$

We multiply the inequality (4.27) by e^{-ct} and integrate from 0 to t to obtain

$$|\langle Aw(t) - Aw^h(t), w(t) - w^h(t)\rangle|^2 \leq ce^{ct}\int_0^t e^{-cs}(\|\dot{w}(s) - z^h(s)\|_H^2$$
$$+\|\dot{w}(s) - z^h(s)\|_H)ds.$$

Conditions on A imply that

$$\|w(t) - w^h(t)\|_{L^\infty(0,T;H)} \leq c \inf_{z^h \in L_2(0,T,K^h)} \|\dot{w} - z^h\|_{L_2(0,T;H)}^{1/2}.$$

This inequality could be the basis for various convergence order estimates.

5. Sensitivity of an Evolution Variational Inequality

Let H be a Hilbert space, $K \subset H$ be a closed convex set. We consider the evolution variational inequality

$$\langle \dot{u}(t), v - u(t) \rangle \geq \langle f(t), v - u(t) \rangle, \qquad \forall\, v \in K, \tag{5.1}$$

$$u(t) \in K, \qquad u(0) = u^0 \in K. \tag{5.2}$$

This evolution variational inequality appears as a basic element in the construction of stress-strain laws of linear and nonlinear kinematic hardening type. It has been also studied (in a more general case) by Moreau under the name of 'sweeping process'; for current references see, for example, [29]. Denote its solution operator by

$$u = \mathcal{F}(f). \tag{5.3}$$

It is well known that $\mathcal{F} : L^1(0,T;H) \to W^{1,1}(0,T;H)$ is well defined, and the basic uniqueness and stability argument gives

$$|(\mathcal{F}f_1)(t) - (\mathcal{F}f_2)(t)| \leq \int_0^t |f_1(s) - f_2(s)|ds, \tag{5.4}$$

so that $\mathcal{F} : L^1(0,T;H) \to C(0,T;H)$ is Lipschitz continuous. We are interested in the (open) question whether $\mathcal{F}$ possesses a directional derivative $w \in W^{1,1}(0,T;H)$,

$$\mathcal{F}(f + \epsilon g) = \mathcal{F}(f) + \epsilon w + o(\epsilon) \tag{5.5}$$

for given functions $f, g \in L^1(0,T;H)$, in an appropriate sense. Let us denote $u_\epsilon = \mathcal{F}(f + \epsilon g)$. The object to be studied is the finite difference

$$\Delta_\epsilon u = \frac{1}{\epsilon}(u_\epsilon - u). \tag{5.6}$$

The following procedure is natural. It has been developed in the context of elliptic variational inequalities by Mignot and others and has been used successfully by several others, in particular in the paper of Brokate and Siddiqi [3] on the rigid punch problem. It consists of two steps.

Step 1. Show that $\{\Delta_\epsilon u\}_{\epsilon>0}$ is weakly precompact.

Step 2. Show that every limit point w of $\{\Delta_\epsilon u\}_{\epsilon>0}$ solves a certain associated variational inequality (A).

If the solution of (A) is unique, the directional differentiabilty of $\mathcal{F}$ is proved.

Step 1. Precompactness of the difference quotients. In the literature on rate independent hyteresis, one usually considers the operators $\mathcal{S}$ and $\mathcal{P}$, called the stop and the play respectively, defined by

$$\mathcal{S}(x) = \mathcal{F}(\dot{x}), \mathcal{P}(x) = x - \mathcal{S}(x), \qquad x \in W^{1,1}(0,T;H). \tag{5.7}$$

These operators obviously map $W^{1,1}(0,T;H)$ into themselves. For the case where K equals the ball around 0 with radius r, it has been proved by Brokate and Krejci that $\mathcal{S}$ and $\mathcal{P}$ are Lipschitz continuous on bounded subsets of $W^{1,1}(0,T;H)$. We have used this result to prove that $\{\Delta_\epsilon u\}_{\epsilon>0}$ is bounded in $L^1(0,T;H)$. $\{\Delta_\epsilon u\}_{\epsilon>0}$ is weak star precompact in $C(0,T,H)^*$. Another open problem is to prove the local Lipschitz continuity of $\mathcal{S}$ in $W^{1,1}(0,T;H)$ for other convex sets K; it would be particularly useful for polyhedral K since Step 2 below becomes much easier. The polyhedral case has been recently studied by Desch [7].

Step 2. The associated variational inequality. A natural candidate for the associated variational problem (A) to be satisfied by the weak limit w of the difference quotients $\{\Delta_\epsilon u\}_{\epsilon>0}$ is given by

$$\langle \dot{w}(t), v - w(t)\rangle \geq \langle g(t), v - w(t)\rangle, \qquad \forall\, v \in \Sigma_K(t), \tag{5.8}$$

$$w(t) \in \Sigma_K(t), \qquad w(0) = 0, \tag{5.9}$$

where $\Sigma_k(t) = T_K(u(t)) \cap \{\dot{u}(t) - f(t)\}^{\perp}$ depends upon u and $T_K(u)$ denotes the tangent cone to K at u. If the set K is polyhedral in the sense used by Haraux [15] and Mignot [25], indeed the corresponding elliptic version is the correct one, and a passage to the limit can be proved. In our case, we are interested (because of the result mentioned in Step 1) in the case where K is a ball; it seems that one has to modify the variational inequality (A) in order to take into account the curvature of K; in the form above it does not appear to be satisfied for $v = aw, \ \ a > 1$ (For all other directions v, it holds). This again is an open problem; related results concerning the differentiability of the projection can be found in the paper of Haraux [15]. For polyhedral K, the arguments of the elliptic case can be carried over.

Since $\|\Delta_\epsilon u\}_{\epsilon > 0}$ is weak star precompact in $C(0,T;H)^*$, there exists a measure μ with $\Delta_\epsilon u \to \mu$ weak star in $C(0,T,H)^*$, $\ u_\epsilon \to \mu$ pointwise a.c. in BV, $du = \mu$. The directional derivative of $\mathcal{F}$ in general is a measure and thus one needs a weak form of (5.8). Results contained in Vladimirov [31, 32] may prove helpful for further investigation in this area.

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