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SEMIGROUP OF ORDER-DECREASING TRANSFORMATIONS: SOME FUNDAMENTAL CONGRUENCES

P.M. Higgins[1] and A. Umar

1. Introduction.

Let S be a semigroup. A relation R on the set S is called *left compatible* (with the operation on S) if

$$(\forall\, s,t,a \in S)(s,t) \in R \Rightarrow (as,at) \in R,$$

and a *right compatible* relation is defined dually. It is compatible if

$$(\forall\, s,t,s',t' \in S)[(s,t) \in R \text{ and } (s',t') \in R] \Rightarrow (ss',tt') \in R.$$

A left [right] compatible equivalence is called a left [right] congruence. An immediate consequence of these definitions is that a relation ρ on a semigroup S is a congruence if and only if it is both a left and right congruence. Congruences are rarely mentioned explicitly in group theory, but they are present in the background. Given a congruence ρ on a group G with identity e, it is easy to verify that $e\rho$ is a normal subgroup of G

$$(x,y) \in \rho \text{ if and only if } xy^{-1} \in N.$$

For each x in G the ρ-class $x\rho$ is simply the coset Nx (or equivalently xN, since N is normal). Thus, the congruence ρ is completely determined by N.

A similar situation arises in a ring R. If ρ is a congruence on R then $O\rho$ is a two-sided ideal of R. Denoting $O\rho$ by I, we see that

$$(x,y) \in \rho \text{ if and only if } x-y \in I.$$

The ρ-classes in this case are the residue classes $x+I$. Here again the congruence is wholly determined by I.

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We do not have this fortunate occurrence in semigroups and so the explicit study of congruences is inescapable. The description of all congruences on a class of semigroups often involves work of a very detailed nature, see for example [**4,5,6,8**].

In this article we wish to initiate a general study of congruences on the semigroups of order-decreasing finite full transformations. (A full transformation $\alpha : X \to X$, where X is totally ordered, is said to be *order-decreasing* if for all x in X we have $x\alpha \leq x$.) As in [**9**] we denote the full transformation semigroup and the semigroup of all order-decreasing finite full transformations of $X_n = \{1, 2, \ldots, n\}$ by T_n and D_n, respectively. For standard concepts in semigroup theory, see for example [**3**] or [**1**].

2. Preliminaries.

We begin by recalling some definitions and results associated with a transformation. For a given (partial) mapping or transformation $\alpha : Y \subseteq X \to X$ we denote its set of fixed points by $F(\alpha) = \{x \in Y : x\alpha = x\}$, its domain Y by Dom α and its image set by Im α. If Dom $\alpha = X$ then α is called a *full* or *total* mapping, otherwise it is called *strictly partial.* An idempotent mapping ϵ is characterized by the property that $F(\alpha) = \operatorname{Im} \alpha$. However, in the semigroup D_n we have this sharper result:

Lemma 1. [**9**, Lemma 2.2]. An order-decreasing mapping α is idempotent if and only if $t = \min t\alpha^{-1}$, for all t in $\operatorname{Im} \alpha$.

Lemma 2. [**7**, Proposition 2.1]. Let S be a subsemigroup of T_n. Then S is $\mathcal{R}$-trivial if and only if for every α, β in S

$$F(\alpha\beta) = F(\alpha) \cap F(\beta) = F(\beta\alpha).$$

Remark. Lemma 2 applies in particular to D_n, since it is known to be $\mathcal{R}$-trivial [**9**, Lemma 2.1].

3. Congruences.

A chain of congruences. For each $y \in X_n$, define an equivalence ρ_y on D_n by the rule that

$$(\alpha, \beta) \in \rho_y \text{ if and only if } x\alpha = x\beta \text{ for all } x \leq y.$$

Then ρ_y is easily seen to be a congruence and $\rho_y \subseteq \rho_x$ if and only if $x \leq y$. In fact, we have this interesting result:

Lemma 3. Let λ be an arbitrary congruence on D_n. Then $\lambda \vee \rho_y = \rho_y \circ \lambda \circ \rho_y$.

Proof. All that remains to show is that $\rho_y \circ \lambda \circ \rho_y$ is transitive. So suppose that $(\alpha, \beta), (\beta, \gamma) \in \rho_y \circ \lambda \circ \rho_y$. Then there exist $\delta_1, \delta_2, \eta_1, \eta_2$ in D_n such that

$$(\alpha, \delta_1) \in \rho_y, (\delta_1, \delta_2) \in \lambda, \ (\delta_2, \beta) \in \rho_y,$$

$$(\beta, \eta_1) \in \rho_y, (\eta_1, \eta_2) \in \lambda, \ (\eta_2, \gamma) \in \rho_y.$$

Now by the transitivity of ρ_y we have $(\delta_2, \eta_1) \in \rho_y$, and so $x\delta_2 = x\eta_1$ for all $x \leq y$. Thus $\epsilon\delta_2 = \epsilon\eta_1$ where ϵ is given by

$$x\epsilon = x \ (\forall x \leq y) \text{ and } x\epsilon = y \text{ (otherwise)}.$$

By the left compatibility of λ we have $(\epsilon\delta_1, \epsilon\delta_2) \in \lambda$, $(\epsilon\eta_1, \epsilon\eta_2) \in \lambda$ and so $(\epsilon\delta_1, \epsilon\eta_2) \in \lambda$. Moreover, $(\delta_1, \epsilon\delta_1) \in \rho_y$ so that by transitivity $(\alpha, \epsilon\delta_1) \in \rho_y$. It is also clear that by the left compatibility of ρ_y we have $(\epsilon\eta_2, \epsilon\gamma) \in \rho_y$ while $(\epsilon\gamma, \gamma) \in \rho_y$. Thus by transitivity we have $(\epsilon\eta_2, \gamma) \in \rho_y$. Hence, summarizing we have that

$$(\alpha, \epsilon\delta_1) \in \rho_y, \ (\epsilon\delta_1, \epsilon\eta_2) \in \lambda, \ (\epsilon\eta_2, \gamma) \in \rho_y$$

and so $(\alpha, \gamma) \in \rho_y \circ \lambda \circ \rho_y$.

Lemma 4. Let k be a natural number for which $1 \leq k \leq n$. Then $D_n/\rho_k \cong D_k$ and $\iota = \rho_n \subsetneq \rho_{n-1} \subsetneq \cdots \subsetneq \rho_2 \subsetneq \rho_1 = \omega$.

Proof. The proof is similar to that of [**11**, Lemma 1.8].

The minimum semilattice congruence. First we prove the following lemma:

Lemma 5. Let $\alpha \in D_n$ and let α^ω be the idempotent power of α, that is $\alpha^\omega = \alpha^k = \alpha^{k+1}$ (for some $k \geq 1$). Then for all $\alpha, \beta \in D_n$ we have that $\alpha^\omega\beta^\omega = \alpha^\omega$ if $F(\alpha) \subseteq F(\beta)$.

Proof. By Lemma 2,

$$F(\alpha^\omega) = F(\alpha) \subseteq F(\beta) = F(\beta^\omega)$$

which implies (by idempotency), that

$$\operatorname{Im}\alpha^\omega = F(\alpha^\omega) \subseteq F(\beta^\omega) = \operatorname{Im}\beta^\omega$$

and hence $\alpha^\omega\beta^\omega = \alpha^\omega$ as required.

Define a relation $\sigma_{\min}$ on D_n by the rule that: $(\alpha, \beta) \in \sigma_{\min}$ if and only if $F(\alpha) = F(\beta)$. Then we have

Theorem 6. Let $\sigma_{\min}$ be as defined above then $\sigma_{\min}$ is the minimum semilattice congruence on any subsemigroup of D_n.

Proof. First it is clear that $\sigma_{\min}$ is an equivalence and that it is a semilattice congruence follows from the fact that for all α, β in D_n

$$F(\alpha\beta) = F(\alpha) \cap F(\beta) = F(\beta\alpha)$$

and so $F(\alpha^2) = F(\alpha)$. Now let λ be an arbitrary semilattice congruence on D_n and let $(\alpha, \beta) \in \sigma_{\min}$. Then

$$F(\alpha^\omega) = F(\alpha) = F(\beta) = F(\beta^\omega) \text{ and } (\alpha, \alpha^\omega), (\beta^\omega, \beta) \in \lambda.$$

However, by Lemma 5 we have $\alpha^\omega\beta^\omega = \alpha^\omega$ and $\beta^\omega\alpha^\omega = \beta^\omega$ so that

$$(\alpha^\omega, \beta^\omega) = (\alpha^\omega\beta^\omega, \beta^\omega\alpha^\omega) \in \lambda$$

and by transitivity we have $(\alpha, \beta) \in \lambda$, as required.

Next, for a given relation, say $\mathcal{K}$ on D_n we denote by $\mathcal{K}^{\#}$ the congruence generated by $\mathcal{K}$ and notice that α_i for which $X_n \setminus F(\alpha) = S(\alpha_i) = \{j_i\}$ is an idempotent and will be denoted by $\begin{pmatrix} j_i \\ j_i\alpha_i \end{pmatrix}$. Now we are going to show that $\mathcal{D}^{\#} = \sigma_{\min}$. However, first we recall from [**9**, Lemma 2.1] that

Lemma 7. Let $\alpha, \beta \in D_n$. Then $(\alpha, \beta) \in \mathcal{D}$ if and only if $\operatorname{Im}\alpha = \operatorname{Im}\beta$ and $\min z\alpha^{-1} = \min z\beta^{-1}$ for all z in $\operatorname{Im}\alpha$.

Lemma 8. Let $\alpha \in D_n$ be such that $S(\alpha) = \{j_1, j_2, \ldots, j_k\}$ where $1 < j_1 < j_2 < \cdots < j_k \leq n$ and $k \geq 2$. Then

$$\alpha = \begin{pmatrix} j_1 \\ j_1\alpha \end{pmatrix} \begin{pmatrix} j_2 \\ j_2\alpha \end{pmatrix} \cdots \begin{pmatrix} j_k \\ j_k\alpha \end{pmatrix}.$$

Proof. The result follows immediately from the fact that for all $i = 1, 2, \ldots, k-1$; we have by the order-decreasing property and the stated assumptions that $j_i\alpha < j_i < j_{i+1} < \cdots < j_k$.

Corollary 9. Every α in D_n is expressible as a product of at most $n-1$ idempotents.

Theorem 10. $\mathcal{D}^{\#}$ is the minimum semilattice congruence on D_n.

Proof. First note that $\mathcal{J} \subseteq \eta = \sigma_{\min}$ in any semigroup [**1**, p.38], where η is the minimum semilattice congruence. Thus $\mathcal{J}^{\#} = \mathcal{D}^{\#} \subseteq \sigma_{\min}$.

Conversely, let $(\alpha, \beta) \in \sigma_{\min}$, then $F(\alpha) = F(\beta) \Leftrightarrow S(\alpha) = S(\beta)$. Suppose that $S(\alpha) = \{j_1, j_2, \ldots, j_k\} = S(\beta)$, where $1 < j_1 < j_2 < \cdots < j_k \leq n$. Observe that for all $i = 1, 2, \ldots, k$ we have that

$$\alpha_i = \begin{pmatrix} j_i \\ j_i\alpha \end{pmatrix} \mathcal{D} \begin{pmatrix} j_i \\ j_i\beta \end{pmatrix} = \beta_i$$

since $F(\alpha_i) = \operatorname{Im}\alpha_i = X_n \setminus \{j_i\} = \operatorname{Im}\beta_i = F(\beta_i)$ and so $\min z(\alpha_i)^{-1} = z = \min z(\beta_i)^{-1}$

for all z in Im α_i. Thus, by the compatibility of $\mathcal{D}^\#$,

$$\alpha = \begin{pmatrix} j_1 \\ j_1\alpha \end{pmatrix}\begin{pmatrix} j_2 \\ j_2\alpha \end{pmatrix}\cdots\begin{pmatrix} j_k \\ j_k\alpha \end{pmatrix}\mathcal{D}^\#\begin{pmatrix} j_1 \\ j_1\beta \end{pmatrix}\begin{pmatrix} j_2 \\ j_2\beta \end{pmatrix}\cdots\begin{pmatrix} j_k \\ j_k\beta \end{pmatrix} = \beta$$

so that $\sigma_{\min} \subseteq \mathcal{D}^\#$. Hence $\mathcal{D}^\# = \sigma_{\min}$.

The minimum congruence. Let $\gamma_{\min}$ be a relation on $D_n (n \geq 3)$ whose only non-singleton class is $\{\mathbf{0}, \alpha^{n-2}\}$, where

$$x\mathbf{0} = 1, \ x\alpha = \max\{1, x-1\} \ (x \in X_n)$$

so that

$$n\alpha^{n-2} = 2, x\alpha^{n-2} = 1 \ \ (x \neq n).$$

However, $\{\mathbf{0}, \alpha^{n-2}\}$ is easily seen to be an ideal and so it follows that $\gamma_{\min}$ is a non-trivial (Rees) congruence on $D_n (n \geq 3)$. Let ρ be an arbitrary non-trivial congruence on D_n $(n \geq 3)$ and let α, β in D_n be such that $(\alpha, \beta) \in \rho$ and $\alpha \neq \beta$. We may, without loss of generality choose x in X_n such that $x\alpha > x\beta \geq 1$. Now define α', β' in D_n by

$$\begin{aligned} n\alpha' &= x, \ z\alpha' = 1 \ (z < n); \\ z\beta' &= 1 \ (z < x\alpha), \ z\beta' = 2 \ (z \geq x\alpha). \end{aligned}$$

Then it is clear that $\alpha'\alpha\beta' = \alpha^{n-2}$ and $\alpha'\beta\beta' = \mathbf{0}$ so we deduce that $\gamma_{\min}$ is contained in every non-trivial congruence D_n. Thus we have proved the following result:

Theorem 11. $\gamma_{\min}$ is the minimum congruence on D_n $(n \geq 3)$.

The minimum band congruence. Let α be an arbitrary element of D_n and for each $s \in S(\alpha) = \{x \in X_n : x\alpha \neq x\}$ define

$$C_s(\alpha) = \{x \in X_n : s\alpha \leq x < s\}.$$

Then we have the following simple lemma:

Lemma 12. For all α, γ in S_n^- we have that $C_s(\alpha) \subseteq C_s(\alpha\gamma)$ and $C_{s\gamma}(\alpha) \subseteq C_s(\gamma\alpha)$.

Now define an equivalence relation β^* on S_n^- by the rule that: $(\alpha, \beta) \in \beta^*$ if and only if (for all $s \in S(\alpha) = S(\beta))C_s(\alpha)$ or $C_s(\beta) \subseteq F(\alpha) = F(\beta)$ implies that $s\alpha = s\beta$. Then we have

Proposition 13. β^* is a band congruence.

Proof. Let $(\alpha, \beta) \in \beta^*$ then $F(\alpha) = F(\beta)$ and $C_s(\alpha)$ or $C_s(\beta) \subseteq F(\alpha) = F(\beta)$ implies that $s\alpha = s\beta$. Now suppose that (for all $s \in S(\alpha\gamma) = S(\beta\gamma)$) $C_s(\alpha\gamma)$ or $C_s(\beta\gamma) \subseteq F(\alpha\gamma) = F(\beta\gamma)$ for some γ in D_n. Without loss of generality suppose that $C_s(\alpha\gamma) \subseteq F(\alpha\gamma)$. Then by Lemma 12, if $s \in S(\alpha)$ we have that

$$C_s(\alpha) \subseteq C_s(\alpha\gamma) \subseteq F(\alpha\gamma) \subseteq F(\alpha)$$

so that $s\alpha = s\beta$ which in turn implies that $s\alpha\gamma = s\beta\gamma$, otherwise if $s \notin S(\alpha) = S(\beta)$ then $s\alpha = s = s\beta$ which also implies that $s\alpha\gamma = s\beta\gamma$. Similarly, suppose that (for all $s \in S(\gamma\alpha) = S(\gamma\beta)$) $C_s(\gamma\alpha)$ or $C_s(\gamma\beta) \subseteq F(\gamma\alpha) = F(\gamma\beta)$ for some γ in D_n. Without loss of generality suppose that $C_s(\gamma\alpha) \subseteq F(\gamma\alpha)$. Then again by Lemma 12, if $s\gamma \in S(\alpha) = S(\beta)$ we have that

$$C_{s\gamma}(\alpha) \subseteq C_s(\gamma\alpha) \subseteq F(\gamma\alpha) \subseteq F(\alpha)$$

so that $(s\gamma)\alpha = (s\gamma)\beta$ while if $s\gamma \notin S(\alpha) = S(\beta)$ then $(s\gamma)\alpha = s\gamma = (s\gamma)\beta$. Thus β^* is a congruence.

Finally, to show that β^* is a band congruence, we suppose that $C_s(\alpha^2) \subseteq F(\alpha^2) = F(\alpha)$. Then by Lemma 12, $C_s(\alpha) \subseteq C_s(\alpha^2) \subseteq F(\alpha)$ so that $(s\alpha)\alpha = s\alpha \Leftrightarrow s\alpha^2 = s\alpha$ as required.

The following lemma is necessary to show that β^* is the minimum band congruence on D_n:

Lemma 14. Let $\alpha \in D_n$ and let $G(\alpha) = \{s \in S(\alpha) : C_s(\alpha) \subseteq F(\alpha)\}$. Define γ in D_n by

$$x\gamma = x\alpha \text{ (if } x \in F(\alpha) \cup G(\alpha)), \; x\gamma = x^- \text{ (otherwise)},$$

where x^- is the immediate predecessor of x in $S(\alpha)$. Then $(\alpha, \gamma) \in \beta^*$, $\gamma \circ \gamma^{-1} \subseteq \alpha \circ \alpha^{-1}$ and for all β such that $(\alpha, \beta) \in \beta^*, x\beta \leq x\gamma$ for all x in X_n.

Proof. First notice that by construction $F(\alpha) = F(\gamma), G(\alpha) \subseteq G(\gamma)$ since $x\alpha = x\gamma$ for all x in $F(\alpha) \cup G(\alpha)$. Moreover, if $x \notin F(\alpha) \cup G(\alpha)$ then $C_x(\gamma) \nsubseteq F(\gamma)$ and so $G^c(\alpha) \subseteq G^c(\gamma)$ ($\Leftrightarrow G(\gamma) \subseteq G(\alpha)$). Thus $G(\gamma) = G(\alpha)$. Now if $C_s(\alpha) \subseteq F(\alpha)$ then $s \in G(\alpha)$ so that $s\alpha = s\gamma$. Similarly, if $C_s(\gamma) \subseteq F(\gamma)$ then $s \in G(\gamma)$ so that $s\gamma = s\alpha$.

Next let $(\alpha, \beta) \in \beta^*$ whence for all x in $F(\alpha) \cup G(\alpha)$ we have $x\beta = x\alpha = x\gamma$. Thus let $x \in S(\alpha) \setminus G(\alpha)$ then $x \neq x\beta \leq x^- = x\gamma$, as required.

Finally, notice that for all β in D_n such that $(\alpha, \beta) \in \beta^*$, β and γ differ only on $S(\alpha) \setminus G(\alpha)$. However, since γ maps $F(\alpha) \cup G(\alpha)$ into $F(\alpha)$ and maps $S(\alpha) \setminus G(\alpha)$ into $S(\alpha)$ in a one-one fashion, by construction, it follows that $\gamma \circ \gamma^{-1} \subseteq \alpha \circ \alpha^{-1}$. Note also that if $S(\alpha) \setminus G(\alpha)$ is empty then the β^*-class containing α is singleton.

Theorem 15. β^* is the minimum band congruence on S_n^-.

Proof. Let λ be an arbitrary band congruence on D_n and for each α in D_n let α^ω be the idempotent power of α, that is $\alpha^\omega = \alpha^k = \alpha^{k+1}$ (for some $k \geq 1$). Suppose also that $(\alpha, \beta) \in \beta^*$ and let γ be as defined in the statement of Lemma 14 above. Next for all δ in D_n define e_δ (also in D_n) by

$$(x^-)e_\delta = x\delta^\omega \; (x \in S(\delta) \setminus G(\delta)), \quad ye_\delta = y \; (y \neq x^-).$$

Then by Lemma 14, we deduce that $\gamma e_\alpha = \alpha^\omega$ and $\gamma e_\beta = \beta^\omega$. Thus since $(\gamma, \gamma^\omega) \in \lambda$, it follows by right compatibility that

$$(\alpha^\omega, \gamma^\omega) = (\gamma e_\alpha, \gamma^\omega e_\alpha) \in \lambda \text{ and } (\beta^\omega, \gamma^\omega) = (\gamma e_\beta, \gamma^\omega e_\beta) \in \gamma.$$

Thus, by transitivity we have $(\alpha^\omega, \beta^\omega) \in \lambda$, and hence $(\alpha, \beta) \in \lambda$, as required.

Another chain of congruences. Define λ_k on D_n by the rule that

$$(\alpha, \beta) \in \lambda_k \text{ if and only if } \max(i\alpha, i\beta) \leq k \text{ whenever } i\alpha \neq i\beta \ (i = 2, 3, \ldots, n).$$

Lemma 16. Let λ_k be as defined above. Then λ_k is a congruence and

$$\iota = \lambda_1 \subseteq \lambda_2 \cdots \lambda_{n-1} \subseteq \lambda_n = \omega.$$

Proof. Clearly λ_k is reflexive and symmetric. To show transitivity, let $\alpha\lambda_k\beta\lambda_k\gamma$ and suppose that $i\alpha \neq i\gamma$. Now, suppose also by way of contradiction that $i\alpha > k$. Then $k < i\alpha$ and $\alpha\lambda_k\beta$ implies that $i\alpha = i\beta$ and so $i\beta > k$. However, $i\gamma \neq i\alpha = i\beta$ and $\beta\lambda_k\gamma$ implies that $\max(i\beta, i\gamma) \leq k$ and so $i\beta \leq k$ which is a contradiction. Thus if $i\alpha \neq i\gamma$, we must have $\max(i\alpha, i\gamma) \leq k$ and so $\alpha\lambda_k\gamma$. Hence λ_k is transitive.

Next we show that for all $k > 0$, λ_k is left compatible. To see this, suppose that $\alpha\lambda_k\beta$ then for all γ in D_n we have either $i\gamma\alpha = i\gamma\beta$ or $i\gamma\alpha \neq i\gamma\beta$. The latter may be written as

$$\begin{aligned} & j\alpha \neq j\beta \ (j = i\gamma) \\ \Rightarrow\ & \max(j\alpha, j\beta) \leq k \\ \Leftrightarrow\ & \max(i\gamma\alpha, i\gamma\beta) \leq k \end{aligned}$$

and so $\gamma\alpha\lambda_k\gamma\beta$.

Similarly, for all γ in D_n if $i\alpha\gamma \neq i\beta\gamma$ then $i\alpha \neq i\beta$ and so $\max(i\alpha, i\beta) \leq k$ which in turn implies that $\max(i\alpha\gamma, i\beta\gamma) \leq k$, by the order-decreasing property. Therefore λ_k is a congruence. Finally, it is obvious that

$$\iota = \lambda_1 \subseteq \lambda_2 \cdots \lambda_{n-1} \subseteq \lambda_n = \omega.$$

Lemma 17. Let $\alpha, \beta \in D_n$ and let $m+1 = \min(F(\alpha) \setminus \{1\})$. Then $\alpha \lambda_m \beta$ implies $\alpha^\omega = \beta^\omega$.

Proof. Let $i \in D(\alpha, \beta) = \{x \in X_n : x\alpha \neq x\beta\}$ and $\alpha\lambda_m\beta$. Then

$$m \geq i\alpha \geq i\alpha^2 \geq \cdots$$

which must terminate at $i\alpha^m = 1$, at most, since there are no fixed points strictly between 1 and $m+1$. Similarly, we deduce that $i\beta^m = 1$. It now follows that $\alpha^\omega = \beta^\omega$, as required.

The maximum idempotent-separating congruence. Define μ on D_n by the rule that

$$(\alpha, \beta) \in \mu \text{ if } F(\alpha) = F(\beta) = \{1\} \text{ or } F(\alpha) = F(\beta) \text{ and } \alpha\lambda_m\beta$$

where $m+1 = \min(F(\alpha) \setminus \{1\})$. Then it is not difficult to show that μ is a congruence. Moreover, we have this interesting result:

Theorem 18. μ is the maximum idempotent-separating congruence on D_n.

Proof. That μ is idempotent-separating follows from Lemma 17. To show that μ is the maximum amongst all idempotent-separating congruences we consider an arbitrary idempotent-separating congruence ν and let $(\alpha, \beta) \in \nu$. Then $\alpha^\omega = \beta^\omega$ and $(\alpha\gamma)^\omega = (\beta\gamma)^\omega$ for all γ in D_n. The former equality implies also that $F(\alpha) = F(\alpha^\omega) = F(\beta^\omega) = F(\beta)$. Now suppose that $i\beta \neq i\alpha > m$ for some $i \in D(\alpha, \beta)$, where $m+1 = \min(F(\alpha) \setminus \{1\})$. Define γ in D_n by

$$(i\alpha)\gamma = (m+1)\gamma = m+1, \;\; x\gamma = 1 \text{ (otherwise)}.$$

If $i\beta \neq m+1$, then $i\beta\gamma = 1$ whereas $i\alpha\gamma = m+1 \in F(\alpha\gamma)$ and so we deduce that $(\alpha\gamma)^\omega \neq (\beta\gamma)^\omega$, a contradiction. If $i\beta = m+1$, we replace α with β in the definition of γ above. This leads to a similar contradiction as above. Therefore $i\alpha \neq i\beta$ $(i \in D(\alpha, \beta))$ implies that $\max(i\alpha, i\beta) \leq m$ and so $(\alpha, \beta) \in \lambda_m$. Hence $\nu \subseteq \mu$, as required.

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REFERENCES

1. P.M. Higgins, *Techniques of semigroup theory* (Oxford University Press, 1992).
2. P. M. Higgins, A proof of Simon's theorem on piecewise testable languages, *Theoretical Computer Science* **178** (1997), 257-264.
3. J.M. Howie, The maximum idempotent-separating congruence on an inverse semigroup, *Proc. Edinburgh Math. Soc.* (2) **14**(1964), 71-79.
4. J.M. Howie, *Fundamentals of semigroup theory* (Oxford: Clarendon Press, 1995).
5. S.L. Limpscomb, The alternating semigroup: congruences and generators, *Semigroup Forum* **44** (1992), 96-106.
6. A.I. Maclev, Symmetric groupoids, *Mat. Sbornik* (N.S.) **3** (1952) 136-151 (MR 14, 349).
7. T. Saito, $\mathcal{J}$-trivial subsemigroups of finite full transformation semigroups, *Semigroup Forum* **57** (1998), 60-68.
8. H.E. Scheiblich, Concerning congruences on symmetric inverse semigroups, *Czech math. Journal* **23** (1973), 1-10.
9. A. Umar, On the semigroups of order-decreasing finite full transformations, *Proc. Roy. Soc. Edinburgh Sect.* A **120** (1992), 129-142.

10. A. Umar, *Semigroups of order-decreasing transformations* (Ph.D. Thesis, University of St. Andrews, 1992).

11. A. Umar, A Class of (0)-Idempotent-free Transformation Semigroups, *Semigroup Forum* **59** (1999), 74-78.