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The Remainder Method for Sample Quantiles of Even Order

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A new method called the Remainder Method has been proposed for the calculation of sample quantiles of some order say quarties, hexatiles, octatiles, deciles etc. Proof is given for a special case of deciles. These sample characteristics divide the ordered sample observations into desired number of segments leaving the same number of observations in each. The proposed method provides the number of observations in each segment and the number of quantiles having integer ranks. Some open problems are indicated.

Key Words and Phrases: Quartiles, remainders, modulus, hexatiles, octatiles, deciles, percentiles, quantiles.

1. Introduction

A sample quantile is a point below which some specified proportion of the values of a data set lies. The median is the 0.50 quantile because approximately half of all observations lie below this value. The name fractile for quantile is used by some authors (see Lapin, 1975, 52). Quartiles, hexatiles, octatiles, deciles, percentiles are special cases of quantiles.

One method for quartiles, called the Halving Method, is based on finding the median first and then finding the medians of the upper and lower halves of the data. Done so, roughly 25% observations remain below the lower quartile and 25% above the upper quartile. If median is excluded in the calculation of outer quartiles, the method satisfies equi-segmented property. The literature is full of different formulae for sample quartiles with various rounding notions of the corresponding ranks of quartiles . See for example Mendenhall and Sincich (995, 54), and Joarder and Firozzaman (2001) for a detailed survey and illustrations.

In this note we propose, by the use of remainders, the correct form of rounding of the ranks of quartiles, hexatiles , octatiles, deciles for any sample of size n. The method (discussed in Sections 3, 4 and 5) will hereinafter be called the Remainder Method. The resulting ranks of quantiles do satisfy the equi-segmented property that they divide the ordered sample observations having the same number of observations (m) in each segment.

Consider the quantiles of even order say $f = 2, 4, 6, \ldots,$ that divides the ordered sample observations in f divisions with m observations in each segment. Since the sample size can be represented by

$$n = r \bmod f = fm + r, \; (r = 0, 1, 2, \cdots, f-1), \tag{1.1}$$

the number of observations in each of the $f \leq n$ segments is given by

$$m(r) = (n-r)/f \tag{1.2}$$

or m for short. The ranks R_{ir} $(i = 1, 2, \cdots, f-1; \; r = 0, 1, \cdots, f-1)$ for quantiles of order f satisfies equi-segmented property if

$$(i) \;\; \uparrow R_{1r} \uparrow -1 = m \tag{1.3a}$$

$$(ii) \;\; \uparrow R_{ir} \uparrow -[R_{i-1,r}] - 1 = m, \; i = 2, 3, \cdots, f \tag{1.3b}$$

$$(iii) \;\; fm + r - [R_{f-1,r}] = m \tag{1.3c}$$

where $[x]$ and $\uparrow x \uparrow$ are the floor function (largest integer not exceeding x) and the ceiling function (smallest integer at least as large as x) of x. The equation $(1.3a)$ states that the number of observations below the first quantile is m while the equation $(1.3c)$ states that the number of observations above the third quantile is m. The equation $(1.3b)$ states that the number of observations between two consecutive quantiles is m.

It appears that in the disguise of the above property we have rigidly defined sample quantiles for discrete data. The proposed method provides the number of observations in each segment and the number of quantiles having integer ranks. It remains open to come up with a general method for ranks of quantiles of even order. The Remainder Method for quartiles $(f = 4)$, hexatiles $(f = 6)$, octatiles $(f = 8)$ and deciles $(f = 10)$ have been discussed in Sections 3, 4 and 5. The method has been proved for a special case of deciles in Section 4, and argued that proofs for all other cases are similar. A general formula for quantiles of even order, in particular, and quantiles of any order, in general, remains open to be addressed.

2. Popular Method for Sample Quartiles

There is considerable confusion and many definitions are offered for quartiles especially for the extreme quartiles i.e. the lower and the upper quartiles. Different definitions are used by some of the popular statistical computing packages. See for example Joarder and Firozzaman (2001) for a detailed survey and illustrations.

Sample quartiles are popularly interpolated linearly by the observations given by the ranks $i\,(n+1)/4$, $(i=1,2,3)$. This method will hereinafter be called the Popular Method. We observe that the ranks provided by this method do not satisfy equi-segmented property if sample sizes are $n=6,10,14$ etc. This led us to conjecture that probably the remainder of the sample size with respect modulus 4 may play a role in the determination of the ranks for quartiles.

Let R_{ir} be the rank of ith quartile with m observations in each of the f (which is 4 for quartiles) segments. Then

$$R_{ir}=i\,\frac{n+1}{f}=\frac{(fm+r)+1}{f}=im+\ i(r+1)/f=im+[u_{ir}]+d/f \tag{2.1}$$

where i and r are integers with $1\le i\le f-1$, $0\le r\le f-1$, $[u_{ir}]$ is the largest integer not exceeding (less than or equal to) $u_{ir}=i(r+1)/f=[u_{ir}]+d/f$.

For simplicity we will often use u for u_{ir}. The quartiles can then be calculated by the simple linear interpolation as

$$Q_{ir}=(1-d/f)\,x_{(im+[u])}+(d/f)\,x_{(im+[u]+1)}\ , \tag{2.2}$$

where $x_{(i)}$ is the i th ordered observation. Note that $u=d/f$ if $[u_{ir}]=0$ i.e. if $u<1$. The quantiles Q_{ir} $(i=1,2,3)$ for a particular r are the usual quartiles and are popularly denoted by simply Q_1,Q_2,Q_3. An example is provided below.

Example 2.1 An independent consumer group tested radial tires from a major brand to determine expected tread life. The data (in thousands of miles) are given below:

50	54	52	47	61
56	51	51	48	56
53	43	56	58	42

(cf. Vinning, 1998, 193).

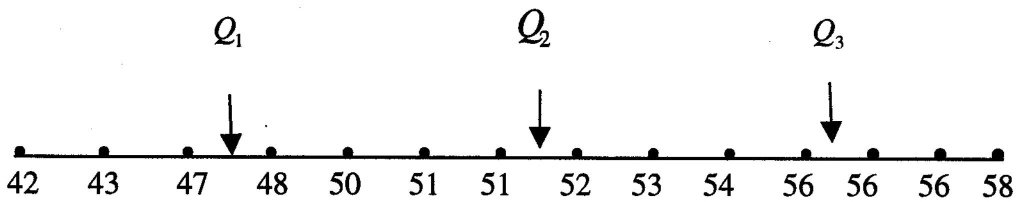
The ordered sample observations are given by

42	43	47	48	50
51	51	52	53	54
56	56	56	58	61

To illustrate the proposed formulae we make four different data sets with the first $n=12$, $n=13$, $n=14$, $n=15$ observations and label them as **Data 1, Data 2, Data 3** and **Data 4** respectively. We observe that the popular method satisfies equi-segmented property for all the above data sets except **Data 3**. This is illustrated below:

Since $n = 14$, $r = 2$ and $m = 3$ for Data 3, the quartiles are given by

$Q_{12} = 3.75$ th obs $= (1-0.75)$ (3 rd obs) $+ 0.75$ (4th obs) $= 47.75$.
$Q_{22} = 7.5$ th obs $= (1-0.50)$ (7 th obs) $+ 0.50$ (8 th obs) $= 0.50\,(51) + 0.50\,(52) = 51.5$.
$Q_{32} = 11.25$ th obs $= (1-0.25)$ 11 th obs $+ (0.25)$12 th obs $= 56$



There are three $(= m)$ observations below the first quartile 48, four observations between the first quartile and the median (51+52)/2=51.5, four observations between the median and the upper quartile 56, three observations above the upper quartile. It appears that if we round up Q_{12} to the nearest integer, and round down Q_{32} to the nearest integer, which happens to be the arithmetic rounding in this case, the resulting ranks of quartiles would satisfy the equi-segmented property.

It may be remarked here that it seems the Halving Method for quartiles (Joarder and Firozzaman, 2001) is difficult to generalize to hexatiles, octiles, deciles, percentiles or to quantiles in general. In the next three sections we try to generalize the above rounding notion to quartiles, hexatiles, octiles, deciles etc.

3. The Remainder Method for Sample Quartiles, Hexatiles and Octatiles

The Remained Method for sample quartiles, hexatiles and octatiles are discussed in this section.

3.1 The Remainder Method for Quartiles

The refinement of the formulae for quartiles is based on the equi-segmented property discussed in Section 1 and on the notion that it is desirable. With a view to improving upon the rank of quartiles given by the Popular Method so that equi-segmented property is satisfied, a special notion of rounding depending on the remainder r and d of the ranks of quartiles is considered. It is observed that quartiles given by the Popular Method for Data 3 of Section 2 satisfies the equi-segmented property for $r = 0, 1, 3$ but not for $r = 2$. We also observe that the formulae (2.1) satisfy equi-segmented property if R_{12} and R_{32} are rounded i.e. the ranks of quartiles are rounded for $(r = 2, d = 3)$ and $(r = 2, d = 1)$ respectively. The formulae for quartiles given by the Popular Method with these notion of rounding will hereinafter be called the Remainder Method for Quartiles.

Theorem 3.1 Let $m = (n-r)/4,\ n = 4m + r \geq 4$, and R_{ir} be the rank of the i th quartile with m observations in each segment. Then the ranks

$$R_{ir} = \begin{cases} im + [u_{ir}] \text{ if } (r,d) \in A, \text{and } d \leq 2 & (3.1a) \\ im + \uparrow u_{ir} \uparrow \text{ if } (r,d) \in A, \text{and } d > 2 & (3.1b) \\ im + u_{ir} \quad \text{if } (r,d) \notin A & (3.1c) \end{cases}$$

where i and r are integers with $1 \leq i \leq 3$ and $0 \leq r \leq 3$, $u_{ir} = i(r+1)/4 = [u_{ir}] + d/4$, and $A = \{(r,d) : (2,1)(2,3)\}$ satisfy equi-segmented property.

If $(r,d) \notin A$, the quartiles can be calculated by the simple linear interpolation as

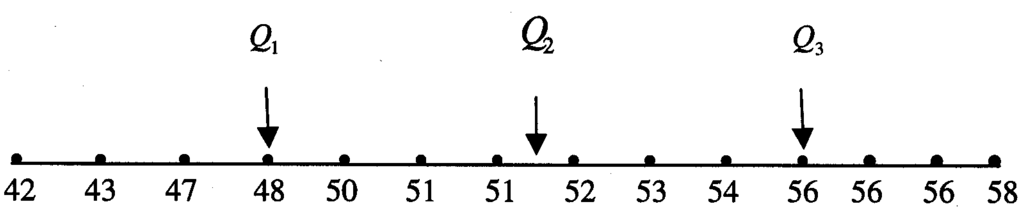
$$Q_{ir} = (1 - d/4)\, x_{(im+[u])} + (d/4)\, x_{(im+[u]+1)}\ , \qquad (3.2)$$

where $x_{(i)}$ is the i th ordered observation.

Example 3.1 Consider Data 3 of Section 2. Here the sample size $n = 14 = 4(3) + 2$ i.e. $m = 3, r = 2$. The ranks for the quartiles are given by R_{12}, R_{22}, R_{32}. Since $u_{12} = 1(2+1)/4 = 3/4$ $(i.e.\, d = 3 > 2)$ and $(r,d) = (2,3) \in A$, it follows from $(3.1b)$ that $R_{12} = 1m + \uparrow u_{12} \uparrow = 3 + 1 = 4$. Again since $u_{22} = 2(2+1)/4 = 1 + 2/4$ $(i.e.\, d = 2)$ and $(r,d) = (2,2) \notin A$, it follows from $(3.1c)$ that $R_{22} = 2m + u_{22} = 6 + 1 + 2/4 = 7 + 2/4$. Also since $u_{32} = 3(2+1)/4 = 2 + 1/4$ $(i.e.\, d = 1 < 2)$ and $(r,d) = (2,1) \in A$, it follows from $(3.1a)$ that $R_{22} = 3m + [u_{12}] = 9 + 2 = 11$. Then the quartiles for Data 3 given by

$Q_{12} = 4$ th obs $= 48$.
$Q_{22} = 7 + 2/4$ th obs $= (1 - 2/4)$ (7 th obs) $+ (2/4)$ (8 th obs) $= 0.50\,(51) + 0.50\,(52) = 51.5$
$Q_{32} = 11$ th obs $= 56$

are shown below:



Clearly the above rounding of ranks guarantees the desirable equi-segmented property. Here there are $m = 3$ observations in each segment.

The remainder r is also, as expected, the number of quartiles having integer ranks for any sample of size $n \geq 4$.

3.2 The Remainder Method for Hexatiles

Hexatiles are five numbers that divide ordered sample observations into six segments. The following theorem guarantees that the ranks for hexatiles given by the Remainder Method satisfy equi-segmented property.

Theorem 3.2 Let $m = (n-r)/6$, $n = 6m + r \geq 6$, and R_{ir} be the rank of the i th quartile with m observations in each segment. Then the ranks of hexatiles are given by

$$R_{ir} = \begin{cases} im + [u_{ir}] \text{ if } (r,d) \in A, \text{and } d \leq 3 & (3.3a) \\ im + \uparrow u_{ir} \uparrow \text{ if } (r,d) \in A, \text{and } d > 3 & (3.3b) \\ im + u_{ir} \text{ if } (r,d) \notin A & (3.3c) \end{cases}$$

where i and r are integers with $1 \leq i \leq 5$ and $0 \leq r \leq 5$, $u_{ir} = i(r+1)/6 = [u_{ir}] + d/6$ and $A = \{(r,d) : (3,2), (4,1), (4,2), (4,4), (4,5)\}$ satisfy the equi-segmented property.

If $(r,d) \notin A$, the hexatiles can then be calculated by the simple linear interpolation as

$$Q_{ir} = (1 - d/6)\, x_{(im+[u])} + (d/6)\, x_{(im+[u]+1)} \,, \quad (3.4)$$

where $x_{(i)}$ is the i th ordered observation, $u = i(r+1)/6$. The remainder r is also, as expected, the number of hexatiles having integer ranks for any sample size $n \geq 6$.

3.3 The Remainder Rule for Octatiles

Octatiles are seven numbers that divide ordered sample observations into eight segments. The following theorem guarantees that the ranks for octatiles given by the Remainder Method satisfy equi-segmented property.

Theorem 3.3 Let $m = (n-r)/8$, $n = 6m + r \geq 8$, and R_{ir} be the rank of the i th quartile with m observations in each segment. Then the ranks for octatiles are given by

$$R_{ir} = \begin{cases} im + [u_{ir}] \text{ if } (r,d) \in A, \text{and } d \leq 4 & (3.5a) \\ im + \uparrow u_{ir} \uparrow \text{ if } (r,d) \in A, \text{and } d > 4 & (3.5b) \\ im + u_{ir} \text{ if } (r,d) \notin A & (3.5c) \end{cases}$$

where i and r are integers with $1 \leq i \leq 7$ and $0 \leq r \leq 7$, $u_{ir} = i(r+1)/8 = [u_{ir}] + d/8$ and

$$A = \{(r,d) : (2,1),(2,2),(4,1),(4,2),(4,3),(4,4),(5,2),(5,6),(6,1),(6,2),(6,3),(6,5),(6,6),(6,7)\}$$

satisfy the equi-segmented property.

If $(r,d) \notin A$, the octatiles can then be calculated by the simple linear interpolation as

$$Q_{ir} = (1-d/8)\, x_{(im+[u])} + (d/8)\, x_{(im+[u]+1)}\ , \tag{3.6}$$

where $x_{(i)}$ is the i th ordered observation. The remainder r is also, as expected, the number of octatiles having integer ranks.

4. The Remainder Method for Deciles

Deciles are nine numbers that divide ordered sample observations into ten segments. The following theorem guarantees that the ranks for deciles given by the Remainder Method satisfy the equi-segmented property.

Theorem 4.1 Let $m = (n-r)/10,\ n = 10m + r \geq 10$, and R_{ir} be the rank of the i th quartile with m observations in each segment. Then the ranks for deciles are given by

$$R_{ir} = \begin{cases} im + [u_{ir}] \text{ if } (r,d) \in A, \text{and } d \leq 5 & (4.1a) \\ im + \uparrow u_{ir} \uparrow \text{ if } (r,d) \in A, \text{and } d > 5 & (4.1b) \\ im + u_{ir} \text{ if } (r,d) \notin A & (4.1c) \end{cases}$$

where i and r are integers with $1 \leq i \leq 9$ and $0 \leq r \leq 9$, $u_{ir} = i(r+1)/10 = [u_{ir}] + d/10$ and

$$A = \{(r,d) : (2,1),(2,2),(3,2),(5,2),(5,4),(6,1),(6,2),(6,3),(6,4)(6,8),(6,9),$$
$$(7,2),(7,4),(7,8),(8,1),(8,2),(8,3),(8,4),(8,6),(8,7),(8,8),(8,9)\}$$

satisfy equi-segmented property.

If $(r,d) \notin A$, the deciles can then be calculated by the simple linear interpolation as

$$Q_{ir} = (1-d/10)\, x_{(im+[u])} + (d/10)\, x_{(im+[u]+1)}\ , \tag{4.2}$$

Proof. By writing out the ranks for deciles by (2.1) with $f = 10$, it is easy to observe that no rounding is needed for $r = 0,1,4,9$. For other cases of $r = 2,5,6,7,8$, some ranks need to be rounded so that the deciles satisfy equisegmentation property . Since proofs are similar in all cases, we prove the theorem only for $r = 6$.

Let $n = 10(m) + 6$ so that $r = 6$. Then by Theorem 4.1 the ranks for deciles are given by

$R_{16} = 1m + 1(6+1)/10 = m + 7/10$, since $d = 7$, $(r,d) = (6,7) \notin A$
$R_{26} = 2m + [2(6+1)/10] = 2m + 1$, since $d = 4 < 5$, $(r,d) = (6,4) \in A$

$R_{36} = 3m + [3(6+1)/10] = 3m + 2$, since $d = 1 < 5$, $(r,d) = (6,1) \in A$
$R_{46} = 4m + \uparrow 4(6+1)/10 \uparrow = 4m + 3$, since $d = 8 > 5, (r,d) = (6,8) \in A$

$R_{56} = 5m + 5(6+1)/10 = 5m + 3 + 5/10$, since $d = 5 \le 5, (r,d) = (6,5) \notin A$
$R_{66} = 6m + [6(6+1)/10] = 6m + 4$, since $d = 2 < 5, (r,d) = (6,2) \in A$

$R_{76} = 7m + \uparrow 7(6+1)/10 \uparrow = 7m + 5$, since $d = 9 > 5, (r,d) = (6,9) \in A$
$R_{86} = 8m + 8(6+1)/10 = 8m + 5 + 6/10$, since $d = 6, (r,d) = (6,6) \notin A$

$R_{96} = 9m + [9(6+1)/10] = 9m + 6$, since $d = 3 < 6, (r,d) = (6,3) \in A$

It is easy to check that

(*i*) $\uparrow R_{16} \uparrow - 1 = \uparrow m + 7/10 \uparrow - 1 = (m+1) - 1 = m$

(*ii*) $\uparrow R_{26} \uparrow - [R_{16}] - 1 = \uparrow 2m + 1 \uparrow - [m + 7/10] - 1 = (2m+1) - (m) - 1 = m,$
$\uparrow R_{36} \uparrow - [R_{26}] - 1 = \uparrow 3m + 2 \uparrow - [2m+1] - 1 = 3m + 2 - (2m+1) - 1 = m,$
$\uparrow R_{46} \uparrow - [R_{36}] - 1 = \uparrow 4m + 3 \uparrow - [3m+2] - 1 = 4m + 3 - (3m+2) - 1 = m,$
$\uparrow R_{56} \uparrow - [R_{46}] - 1 = \uparrow 5m + 3 + 5/10 \uparrow - [4m+3] - 1 = 5m + 4 - (4m+3) - 1 = m,$
$\uparrow R_{66} \uparrow - [R_{56}] - 1 = \uparrow 6m + 4 \uparrow - [5m + 3 + 5/10] - 1 = 6m + 4 - (5m+3) - 1 = m,$
$\uparrow R_{76} \uparrow - [R_{66}] - 1 = \uparrow 7m + 5 \uparrow - [6m+4] - 1 = 7m + 5 - (6m+4) - 1 = m,$
$\uparrow R_{86} \uparrow - [R_{76}] - 1 = \uparrow 8m + 5 + 6/10 \uparrow - [7m+5] - 1 = 8m + 6 - (7m+5) - 1 = m,$
$\uparrow R_{96} \uparrow - [R_{86}] - 1 = \uparrow 9m + 6 \uparrow - [8m + 5 + 6/10] - 1 = 9m + 6 - (8m+5) - 1 = m$

(*iii*) $10m + 6 - [R_{96}] = 10m + 6 - [9m + 6] = 10m + 6 - (9m + 6) = m$.

Thus it is proved that deciles satisfy equisegmented property.

The idea is also illustrated with a complete example in Firozzaman and Joarder (2001). The remainder r is also, as expected, the number of deciles having integer ranks for any sample of size $n \ge 10$.

5. Quantiles of Even Order

The notion now needs to be generalized to quantiles of even order. By the Remainder Rule the ranks for percentiles are given by

Theorem 5.1 Let $m=(n-r)/f$, $n=fm+r\geq f$, where f is even and R_{ir} be the rank of the ith quartile with m observations in each segment. Then the ranks for deciles are given by

$$R_{ir}=\begin{cases} im+[u_{ir}] \text{ if } (r,d)\in A, \text{and } d\leq f/2 & (5.1a)\\ im+\uparrow u_{ir}\uparrow \text{ if } (r,d)\in A, \text{and } d>f/2 & (5.1b)\\ im+u_{ir} \text{ if } (r,d)\notin A & (5.1c)\end{cases}$$

where i and r are integers with $1\leq i\leq f-1$, $0\leq r\leq f-1$ $u_{ir}=i(r+1)/f=[u_{ir}]+d/f$ and

$$A=\{(r,d)\} \quad (5.2)$$

a set yet to be determined so that the ranks satisfy the equi-segmented property discussed in Section 1.

5.1 Determination of a Compact Form of the Set A

Quantiles of order f (even) are $f-1$ numbers that divide ordered sample observations into f segments. Clearly the set $A=\{(r,d)\}$ is all that we need to generalize the Remainder Method.

(i) The Set A for Hexatiles $f=6$

The set A for hexatiles (See Theorem 3.2) given by

$$A=\{(r,d_{ir}):(r=3,d_{23}=d_{53}=2),(r=4,d_{54}=1),(r=4,d_{44}=2),\\ (r=4,d_{24}=4),(r=4,d_{14}=5)\} \quad (5.3)$$

can be simply written as

$$\begin{aligned} &(a)\ r=2,3;\ 1\leq d\leq r, \text{ or}\\ &(b)\ r=4,\quad d=1,2 \text{ or}\\ &(c)\ r=4,\quad d=4,5 \end{aligned} \quad (5.4)$$

(ii) **The set A for octatiles** $(f = 8)$

The set A for octatiles (See Theorem 3.3) given by

$$A = \{(r, d_{ir}) : (r = 2, d_{32} = 1), (r = 2, d_{62} = 2), (r = 4,\ d_{54} = 1), (r = 4, d_{24} = 2), (r = 4, d_{74} = 3),$$
$$(r = 4, d_{44} = 4), (r = 5, d_{35} = d_{75} = 2), (r = 5, d_{15} = d_{55} = 6), (r = 6, d_{76} = 1),$$
$$(r = 6, d_{66} = 2), (r = 6, d_{56} = 3), (r = 6, d_{36} = 5), (r = 6, d_{26} = 6), (r = 6, d_{16} = 7)\} \quad (5.5)$$

can be simply written as

$(a)\ r = 2, 3, 4;\ 1 \le d \le r$, or
$(b)\ r = 5, 6;\ \ d = 1, 2, 3$ or (5.6)
(c) $r = 5, 6;\ \ 14 - r \le d \le 9$

(iii) **The set A for Deciles** $(f = 10)$

Let d^*_{ir} is the value of d_{ir} corresponding to any r for which the ranks of deciles needs to be rounded to ensure equi-segmented property.

r	d^*_{ir}	Type
2	$d_{72} = 1, d_{42} = 2$	(5.1a)
3	$d_{32} = d_{83} = 2$	(5.1a)
5	$(d_{25} = d_{75} = 2, d_{45} = d_{95} = 4)$	(5.1a)
6	$(d_{36} = 1, d_{66} = 2, d_{96} = 3, d_{26} = 4)$, $(d_{46} = 8, d_{76} = 9)$	(5.1b) (5.1c)
7	$(d_{47} = d_{97} = 2, d_{37} = d_{87} = 4)$, $(d_{17} = d_{67} = 8)$	(5.1b) (5.1c)
8	$(d_{98} = 1, d_{88} = 2, d_{78} = 3, d_{68} = 4)$, $(d_{46} = 6, d_{38} = 7, d_{28} = 8, d_{18} = 9)$	(5.1b) (5.1c)

Thus the Remainder Method satisfies the equi-segmented property if ranks are rounded for values of (r, d) given in the following table:

Table 5.1

r	d_{ir}	d^*_{ir}	Adjoining set	Larger set	Type	Count
0	$1 \le d \le 9$	No need				
1	0, 2, 4, 6, 8	No need				
2	$1 \le d \le 9$	1, 2		$1 \le d \le r$	5.1 (a)	2
3	0, 2, 4, 6, 8	2	3	$1 \le d \le r$	5.1 (a)	1
4	0, 5	No need	1, 2, 3, 4	$1 \le d \le r$	5.1 (a)	
5	0, 2, 4, 6, 8	2, 4	1, 3	$1 \le d \le r$	5.1 (a)	2
6	$1 \le d \le 9$	1, 2, 3, 4 8, 9		$1 \le d \le 4$ $14 - r \le d \le 9$	5.1 (b) 5.1 (c)	4 2
7	0, 2, 4, 6, 8	2,4 8	1,3 7,9	$1 \le d \le 4$ $14 - r \le d \le 9$	5.1 (b) 5.1 (c)	2 1
8	$1 \le d \le 9$	1, 2, 3, 4 6, 7, 8, 9		$1 \le d \le 4$ $14 - r \le d \le 9$	5.1 (b) 5.1 (c)	4 4
9	0	No need				

In view of these relationships between r and d as seen in Table 5.1, or the modulus matrix (See Section 5.3), the superset of $A = \{(r, d)\}$ defined by

$(a)\ 2 \le r \le 5,\ 1 \le d \le r$, or
$(b)\ 6 \le r \le 8,\ 1 \le d \le 4$ or (5.3)
(c) $6 \le r \le 8,\ 14 - r \le d \le 9$

(See Column 5 of Table 5.1 for specific explanations) also does the same rounding as done by the set given in Table 5.1 (See Column 3 of Table 5.1). There are 22 distinct sets of (r, d) corresponding to 28 ranks for which rounding are needed to ensure equidivision property.

(iv) The set A for $f = 20$

We have also checked the method for $f = 20$ divisions, and determined the set A of $\{(r, d)\}$ points for which rounding is needed to satisfy equi-segmented property (See Appendix 1). The distinct number of (r, d) points for which rounding is nedded for quartiles, hexatiles, octatiles, deciles, and quantiles with $f = 20$ divisions are given by 2, 6, 15, 22 and 137 respectively.

The results reported here are based on Joarder (2002) who has written out explicitly the ranks for all quantiles of order $f = 4, 6, 8, 10, 20$ by the formula given by (2.1) and inspected manually a set of $A = \{(r,d)\}$ where

$$\begin{aligned}
&(a)\ 2 \le r \le 10,\ 1 \le d \le r, \text{ or} \\
&(b)\ 11 \le r \le 18,\ 1 \le d \le 9, \text{or} \qquad (5.4) \\
&(c)\ 11 \le r \le 18,\ 29 - r \le d \le 19
\end{aligned}$$

satisfies the equi-segmented property.

5.2 The Modulus Matrix for the Ranks of Quantiles

Let us now define some matrix which may shed some light towards finding a compact form of the set A of (r,d) points. The product matrix for the ranks of deciles $(f = 10)$ is defined by $((d_{ir}))$, where d_{ir} is defined by $i(r+1) = d_{ir}$; $i = 1,2,\cdots,f-1$; $r = 0,1,2,\cdots,5,f-2$. The product matrix for the ranks of deciles is given by

$$\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\
3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 \\
4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 36 \\
5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 \\
6 & 12 & 18 & 24 & 30 & 36 & 42 & 48 & 54 \\
7 & 14 & 21 & 28 & 35 & 42 & 49 & 56 & 63 \\
8 & 16 & 24 & 32 & 40 & 48 & 56 & 64 & 72 \\
9 & 18 & 27 & 36 & 45 & 54 & 63 & 72 & 81
\end{bmatrix}$$

The modulus matrix for the ranks of deciles $(f = 10)$ is defined by $D = ((d_{ir}))$, where d_{ir} is defined by $i(r+1) = d_{ir} \bmod f$; $i = 1,2,\cdots,f-1$; $r = 0,1,2,\cdots,5,f-2$. Note that $d_{ir} + d_{1,8-r} = 0$ or f $(i = 1,2,\ldots,f-1;\ r = 0,1,,\ldots,f-2)$. The modulus matrix for the ranks of deciles is given by

$$D = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & (8) & (9) \\ 2 & 4 & 6 & 8 & 0 & (2) & (4) & 6 & (8) \\ 3 & 6 & 9 & (2) & 5 & 8 & (1) & (4) & (7) \\ 4 & 8 & (2) & 6 & 0 & (4) & (8) & (2) & (6) \\ 5 & 0 & 5 & 0 & 5 & 0 & 5 & 0 & 5 \\ 6 & 2 & 8 & 4 & 0 & 6 & (2) & (8) & (4) \\ 7 & 4 & (1) & 8 & 5 & (2) & (9) & 6 & (3) \\ 8 & 6 & 4 & (2) & 0 & 8 & 6 & (4) & (2) \\ 9 & 8 & 7 & 6 & 5 & (4) & (3) & (2) & (1) \end{bmatrix}$$

The numbers in the braces are the values of d_{ir} for which rounding is needed for the corresponding ranks of deciles to satisfy equi-segmented property,

One limitation of the Remainder Method is that one needs $n \geq f$ observations to satisfy equi-segmented property. We recommend tȯ use the two popular quantiles namely, quartiles or deciles and not any other quartiles to overcome the above difficulty. The idea of percentiles $(f = 100)$ though sounds exciting, there is no rigid formulae for them in the discrete case available in the literature, and the determination of set A for the Remainder Method is still an open question..

We conclude with three open problems:

(1) Determination of the set A of (r,d) points for which rounding is needed to satisfy equi-segmented property.

(2) Determination of the number of (r,d) points for which rounding is essential by the Remainder Method to ensure equi-segmented property.

(3) Determination of formula for quantiles with odd number of divisions that satisfy equi-segmented property.

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Appendix 1 (Set A for Quantiles of order $f = 20$)

r	d	d^*	Type	Count
0	$1 \le d \le 19$			0
1	$0 \le d \le 18$, all even			0
2	$1 \le d \le 19$	1, 2	(5.1a)	2
3	0, 4, 8, 12, 16			0
4	0, 5, 10, 15			0
5	$0 \le d \le 18$, all even	2, 4	(5.1a)	4
6	$1 \le d \le 19$	$1 \le d \le 6$	(5.1a)	6
7	0, 4, 8, 12, 16	4	(5.1a)	4
8	$1 \le d \le 19$	$1 \le d \le 8$	(5.1a)	8
9	0, 10			0
10	$1 \le d \le 19$	$1 \le d \le 9$	(5.1a)	9
11	0, 4, 8, 12, 16	4, 8	(5.1b)	8
12	$1 \le d \le 19$	$1 \le d \le 9$ $d = 17, 18, 19$	(5.1*b*) (5.1*c*)	9 3
13	$0 \le d \le 18$, all even	2, 4, 6, 8 16, 18	(5.1*b*) (5.1*c*)	8 4
14	0, 5, 10, 15	5 15	(5.1*b*) (5.1*c*)	5 5
15	0, 4, 8, 12, 16	4, 8 16	(5.1*b*) (5.1*c*)	8 4
16	$1 \le d \le 19$	$1 \le d \le 9$ $13 \le d \le 19$	(5.1*b*) (5.1*c*)	9 7
17	$0 \le d \le 18$, all even	2, 4, 6, 8, 12, 14, 16, 18	(5.1*b*) (5.1*c*)	8 8
18	$1 \le d \le 19$	$1 \le d \le 9$ $11 \le d \le 19$	(5.1*b*) (5.1*c*)	9 9
19	0			