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On Sum and Product of 2-normal Operators

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Abstract

In this note we show that the product of two commuting 2-normal operators is 2-normal. We also show that the sum of two commuting 2-normal operators is not necessarily 2-normal, and we give a condition under which this sum becomes 2-normal.

1. Introduction.

In [1], the author introduced the class of 2-normal operators acting on a Hilbert space H. If $T = A + iB$ is the cartesian decomposition of an operator in $L(H)$, the algebra of all bounded linear operators acting on H, then T is called 2-normal if, and only if, $AB^2 = B^2A$ and $A^2B = BA^2$. Several properties and characterizations of 2-normal operators are given in [1].

2. Results.

In this note we study the sum and product of two 2-normal operators. We give an example of two non-commuting 2-normal operators whose product is not 2-normal and then we prove that the product of two commuting 2-normal operators is 2-normal. We give another example of two commuting 2-normal operators whose sum is not 2-normal and then we give a condition under which the sum of two 2-normal operator becomes 2-normal.

Example 2.1. Let $S = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ be two operators acting on the two-dimensional space R^2, then direct computations shows that $SS^* = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = S^*S$ which implies that S is normal. Thus, by ([1], Proposition 2.1, p. 192) S is 2-normal. One can easily show that $T^2 = I$ which implies, by ([1], Proposition 2.1, p. 192) that T is 2-normal. Let $K = ST = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}$, then direct computations show that $K^2K^* = \begin{pmatrix} -3 & -1 \\ 4 & 4 \end{pmatrix}$ while $K^*K^2 = \begin{pmatrix} -2 & 2 \\ 1 & 3 \end{pmatrix}$ which implies, by ([1], Proposition 1.1, p. 190), that K is not 2-normal. Clearly $ST \neq TS = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$.

Proposition 2.1. *If $S, T \in L(H)$ such that $ST = TS$ then ST is 2-normal.*

Proof. Since S, T are 2-normal operators, then by ([1], Proposition 1.6, p. 192), S^2 and T^2 are normal operators. Since $ST = TS$, $S^2T^2 = T^2S^2$. Thus S^2T^2 is normal, which implies that $(ST)^2$ is normal. Hence ST is 2-normal.

Corollary 1. *If $S \in L(H)$ is 2-normal then S^n is 2-normal for $n \geq 2$.*

In the following we give an example of two commuting 2-normal operators whose sum is not 2-normal.

Example 2.2. Let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be two operators acting on R^2, then direct computations show that $T^{*^2}T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = TT^{*^2}$ and $S^{*^2}S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = SS^{*^2}$ which implies, by ([1], Proposition 1.1, p. 190), that S and T are 2-normal operators. Now $S + T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and direct computations show that $[(T+S)^*]^2(T+S) = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$ while $(T+S)[(T+S)^*]^2 = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$ which implies, again by ([1], Proposition 1.1, p. 190), that $T+S$ is not 2-normal. It is clear that $TS = ST$ but $TS \neq -ST$.

Proposition 2.2. *Let $S, T \in L(H)$ be two 2-normal operators such that $TS = -ST$, then $T+S$ is 2-normal.*

Proof. Since $TS = -ST$, $(T+S)^2 = T^2 + S^2$. Also one can easily show that $TS = -ST$ implies that $T^2S^2 = S^2T^2$. Thus T^2 and S^2 are two commuting normal operators which implies that $T^2 + S^2$ is normal. Thus $(T+S)^2$ is normal, which implies, by ([1], Proposition 1.6, p. 192), that $T+S$ is 2-normal.

Reference

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