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On Subprojection Operators

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Abstract

In this paper, we introduce the class of subprojection operators on a Hilbert space H. We give some basic properties of such operators. We give an example to show that the product of two general subprojections is not, in general, a subprojection. We give another example to show that the sum of two commuting subprojections is not, in general, a subprojection. Finally we give some conditions under which a subprojection becomes a projection.

1 Introduction.

Throughout this paper, $L(H)$ is the algebra of all bounded linear operators acting on a Hilbert space H. If $P \in L(H)$ then P is called a projection if $P^2 = P = P^*$ (where P^* is the adjoint of P). We call an operator $S \in L(H)$ a subprojection if $S^2 = S^*$. The class of all subprojection operators acting on H is denoted by $S(H)$. It is clear that if $P \in L(H)$ is a projection then $P \in S(H)$. The converse of the last statement is not in general true as the following example shows:

Example 1.1. Consider the operator $T = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ acting on the two-dimensional space R^2, then one can easily show that $T^2 = T^*$ and $T^* \neq T$.

In the second section of this paper we give some general properties of subprojections. In the third section we give conditions under which the sum and product of two subprojections become subprojections. In the fourth section we give some conditions under which a subprojection becomes a projection.

2. In this section we investigate some general properties of subprojections.

Proposition 2.1. *Let $S \in S(H)$ then*

(i) $S^* \in S(H)$.

(ii) *If S^{-1} exists then $S^{-1} \in S(H)$.*

(iii) *If $T \in L(H)$ such that T and S are unitarily equivalent then $T \in S(H)$.*

(iv) *If M is a closed subspace of H that reduces S, then $S|M \in S(H)$ (where $S|M$ is the restriction of S to M).*

Proof. (i) $(S^*)^2 = (S^2)^* = (S^*)^*$. Thus $S^* \in S(H)$.

(ii) $(S^{-1})^2 = (S^2)^{-1} = (S^*)^{-1} = (S^{-1})^*$. Thus $S^{-1} \in S(H)$.

(iii) Suppose that $T \in L(H)$ is unitarily equivalent to S, then there is a unitary operator U such that $T = U^*SU$. Thus

$$\begin{aligned} T^2 &= U^*SUU^*SU \\ &= U^*S^2U \\ &= U^*S^*U \\ &= (U^*SU)^* \\ &= T^*. \end{aligned}$$

Hence $T \in S(H)$.

(iv) Let M be a closed subspace of H that reduces S then, using the assumption, ([1], Theorem 3, p. 158) and ([1], Theorem 5, p. 159), we have

$$(S|M)^2 = (S^2|M) = (S^*|M) = (S|M)^*.$$

Thus $S|M \in S(H)$.

Proposition 2.2. *If $S \in S(H)$ then S is normal.*

Proof. Since $S \in S(H)$, $S^2 = S^*$. Thus $S^3 = S^*S$ and $S^3 = SS^*$. Hence $SS^* = S^*S$ which implies that S is normal.

The converse of Proposition 2.2 is not in general true as the following example shows:

Example 2.1. Consider the operator $T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ acting on R^2, then direct calculations show that $TT^* = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = T^*T$ while $T^2 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \neq \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = T^*$. Thus T is normal but not subprojection.

In the next example we show that if $T \in L(H)$ is similar to $S \in S(H)$, then it is not necessary that $T \in S(H)$.

Example 2.2. Consider the operators $S = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$, $X = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} -5 & -3 \\ 7 & 4 \end{pmatrix}$ acting on R^2, then it can be shown that $X^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$, $S^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = S^*$ and $T = X^{-1}SX$. Thus $S \in S(H)$ and T is similar to S. Direct calculations show that $TT^* = \begin{pmatrix} -11 & -20 \\ 16 & 29 \end{pmatrix}$ while $T^*T = \begin{pmatrix} 29 & 16 \\ -20 & -11 \end{pmatrix}$ which means that T is not normal. Thus, by Proposition 2.2, T is not a subprojection.

Proposition 2.3. *If $0 \neq S \in S(H)$ then*

(i) $\|S\| = 1$

(ii) $r(S) = 1$, *where* $r(S)$ *is the spectral radius of* S.

Proof. (i) Since $S \in S(H), S^2 = S^*$. Thus $\|S^2\| = \|S^*\| = \|S\|$. By Proposition 2.2 S is normal, thus $\|S^2\| = \|S\|^2$. Hence $\|S\|^2 = \|S\|$ which implies that $\|S\| = 1$.

(ii) Since $S \in S(H)$, S is normal; thus by ([4], Theorem 3.5, p. 351), $r(S) = \|S\|$ which implies by (i) above that $r(S) = 1$.

Proposition 2.4. *If* $0 \neq S \in S(H)$ *and if* $\alpha S \in S(H)$ *for some real* α *then* $\alpha = 0$ *or* $\alpha = 1$.

Proof. Suppose that $S \in S(H)$ and $\alpha S \in S(H)$ then $(\alpha S)^2 = (\alpha S)^*$ which implies that $\alpha^2 S^2 = \alpha S^* = \alpha S^2$. Thus $\alpha^2 S^2 = \alpha S^2$ which implies that $(\alpha^2 - \alpha) S^2 = 0$. Since S is normal and $S \neq 0$, $S^2 \neq 0$. Thus $\alpha^2 - \alpha = 0$ which implies that $\alpha = 0$ or $\alpha = 1$.

Definition 2.1. $T \in L(H)$ is called partially isometric if T^*T is a projection.

Proposition 2.5. *If* $S \in S(H)$ *then* S *is a partial isometry.*

Proof. Since $S \in S(H)$ then S is normal which implies that $SS^* = S^*S$. Thus we have

$$(S^*S)^2 = S^*SS^*S = S^{*^2}S^2 = (S^2)^*S^2 = (S^*)^*S^2 = SS^* = S^*S = (S^*S)^*.$$

By ([1], Theorem 2(3), p. 148) S^*S is self-adjoint. Thus S^*S is a projection which means that S is a partial isometry.

Proposition 2.6. *If* $S \in S(H)$ *then* $S = SS^*S$.

Proof. Since $S \in S(H)$, S is normal, which implies that $SS^* = S^*S$. Thus

$$SS^*S = S^2S^* = S^{*^2} = (S^2)^* = (S^*)^* = S.$$

Proposition 2.7. *If $S \in S(H)$ is invertible then S is unitary.*

Proof. Since $S \in S(H)$ then, by Proposition 2.6, $S = SS^*S$. Thus $SS^{-1} = SS^*SS^{-1}$ which implies that $SS^*(= S^*S) = I$. Thus S is unitary.

Proposition 2.8. *The class $S(H)$ is strongly closed i.e. $\overline{S(H)}^{sot} = S(H)$.*

Proof. Let (S_n) be a sequence of operators in $S(H)$ such that (S_n) converges strongly to $S \in L(H)$ i.e. $\|S_n x - Sx\| \to 0$ as $n \to \infty$. Thus

$$\|S_n^* x - S^* x\| = \|(S_n - S)^* x\| \leq \|(S_n - S)^*\| \, \|x\| = \|S_n - S\| \, \|x\| \to 0 \quad \text{as } n \to \infty.$$

Thus (S_n^*) converges strongly to S^*. Since the product of operators is sequentially continuous in the strong operator topology ([2], p. 62), and since (S_n) converges strongly to S, one concludes that (S_n^2) converges strongly to S^2. Thus (S_n^*) converges strongly to S^2 which implies that $S^2 = S^*$. Thus $S \in S(H)$, which implies that $S(H)$ is strongly closed.

3. In Section three of this paper we study the sum and the product of two subprojections. First we give an example of two noncommuting subprojections whose product is not a subprojection.

Example 3.1. Let $T = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$, $S = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ be two operators acting on R^2, then direct computations show that $T^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = T^*$ and $S^2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = S^*$. Thus T and S are subprojections. However one can easily show that $(TS)^2 = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}$ while $(TS)^* = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}$. Thus TS is not a subprojection. We notice that $TS \neq ST$.

Proposition 3.1. *If $S, T \in S(H)$ such that $ST = TS$, then $ST \in S(H)$.*

Proof. $(ST)^2 = STST = S^2T^2 = S^*T^* = (TS)^* = (ST)^*$. Thus $ST \in S(H)$.

Corollary. *If $S \in S(H)$ then $S^n \in S(H)$ for $n > 1$.*

Proposition 3.2. *The sum of two commuting subprojections in $L(H)$ is not necessarily a subprojection.*

Proof. Let $T \in S(H)$ then

$$(T+T)^2 = (2T)^2 = 4T^2 = 4T^* \neq (T+T)^*.$$

Proposition 3.3. *If $S, T \in S(H)$ then $S+T \in S(H)$ if and only if $ST = -TS$.*

Proof. Suppose first that $ST = -TS$ then

$$(S+T)^2 = S^2 + ST + TS + T^2 = S^2 + T^2 = S^* + T^* = (S+T)^*.$$

Thus $S+T$ is a subprojection.

Suppose now that $S+T$ is a subprojection, then

$$S^2 + ST + TS + T^2 = (S+T)^2 = (S+T)^* = S^* + T^* = S^2 + T^2$$

which implies that $ST + TS = 0$. Hence $ST = -TS$.

Proposition 3.4. *If $S, T \in S(H)$ such that $S+T \in S(H)$ then $ST = 0$.*

Proof. Let S, T and $S+T$ be subprojections then, by Proposition 3.3,

$$ST + TS = 0. \tag{1}$$

Multiplying (1) on the left by S we get

$$S^2T + STS = 0. \tag{2}$$

Multiplying (1) on the right by S we get

$$STS + TS^2 = 0. \tag{3}$$

Subtracting (3) from (2) we get

$$S^2T = TS^2.$$

Since T is normal therefore, by Fuglede's theorem, the last equation gives

$$(S^2)^*T = T(S^2)^*.$$

Thus we have $(S^*)^*T = T(S^*)^*$ which implies that $ST = TS$ which, together with (1) above, gives $ST = 0$.

Proposition 3.4. *The direct sum and the direct product of two subprojections are subprojections.*

Proof. Let $S, T \in S(H)$ and let $x = x_1 \oplus x_2 \in H \oplus H$, then $(S \oplus T)^2(x) = (S \oplus T)^2(x_1 \oplus x_2) = S^2(x_1) \oplus T^2(x_2) = S^*(x_1) \oplus T^*(x_2) = (S^* \oplus T^*)(x_1 \oplus x_2) = (S \oplus T)^*(x)$. Thus $(S \oplus T)^2 = (S \oplus T)^*$ which implies that $S \oplus T \in S(H)$. Also $(S \otimes T)^2(x) = (S^2 \otimes T^2)(x_1 \otimes x_2) = S^2(x_1) \otimes T^2(x_2) = S^*(x_1) \otimes T^*(x_2) = (S^* \otimes T^*)(x_1 \otimes x_2) = (S + T)^*(x)$. Thus $(S \otimes T)^2 = (S \otimes T)^*$ which implies that $S \otimes T \in S(H)$.

4. In the fourth and last section of this paper we give some conditions under which a subprojection becomes a projection.

Proposition 4.1. *Let $S \in S(H)$. If $I - S \in S(H)$ then S is a projection.*

Proof. Let S and $I - S$ be subprojections, then

$$I - 2S + S^* = I - 2S + S^2$$

$$
\begin{aligned}
&= (I-S)^2 \\
&= (I-S)^* \\
&= I-S^*.
\end{aligned}
$$

Thus $2S = 2S^*$ which implies that $S = S^*$. Thus S is a projection.

Definition 4.1. Let $T \in L(H)$ have the cartesian decomposition $T = A + iB$, then T is called skew-normal if and only if $AB = -BA$.

It is clear from the definition that T is skew-normal if and only if T^2 is hermitian ([4], p. 431).

Proposition 4.2. *If $S \in S(H)$ is skew-normal then S is a projection.*

Proof. Since S is skew-normal, S^2 is hermitian. Since $S \in S(H)$, $S^2 = S^*$. Thus S^* is a hermitian which means that $(S^*)^* = S^*$. Thus $S = S^*$ which implies that S is a projection.

If $T \in L(H)$ is a projection then it is clear that $T = T^3$. The converse of this statement is not in general true. An example of an operator $T \in L(H)$ with $T = T^3$ which is not a projection is a general idempotent.

Proposition 4.3. *If $S \in S(H)$ such that $S = S^3$ then S is a projection.*

Proof. Let $S = S^3$ then $S = S^2S = S^*S$. Since S^*S is always hermitian, S is hermitian which implies that S is a projection.

Proposition 4.4. *Let $S, T \in L(H)$ such that $S \in S(H)$ and T is an idempotent. If S and T are similar then S is a projection.*

Proof. Since S and T are similar, there is an invertible operator $N \in L(H)$ such that $S = N^{-1}TN$. Thus $S^2 = N^{-1}TNN^{-1}TN = N^{-1}T^2N = N^{-1}TN = S$. Hence S is a projection.

Proposition 4.5. *Let $S \in L(H)$ be a subprojection which is unitarily equivalent to its adjoint, then S is a projection.*

Proof. By assumption, there is a unitary operator $U \in L(H)$ such that $S = US^*U^*$. Thus

$$S^2 = US^*U^*US^*U^* = US^{*^2}U^* = US^{2^*}U^* = U(S^*)^*U^* = USU^* = S^*.$$

Thus S is a projection.

Proposition 4.6. *Let $S = A+iB$ be the cartesian decomposition of a subprojection $S \in L(H)$. If A is a projection then S is a projection.*

Proof. Since $S \in S(H)$, $S^2 = S^*$. Thus $(A^2 - B^2) + i(AB + BA) = A - iB$ which implies that $A^2 - B^2 = A$. Since A is a projection, $A^2 = A$ which implies that $B = 0$. Thus S is hermitian which implies that S is a projection.

Proposition 4.7. *Let $A \in L(H)$ such that either $0 \notin W(A)$ – the numerical range of A – or $\sigma(A) \cap \sigma(-A) = \phi$. Suppose that $S \in S(H)$ such that $AS = S^*A$, then S is a projection.*

Proof. The proof follows immediately from Proposition 2.2 and ([2], Corollary 7, p. 334).

The class of all partially isometric operators can be extended in the following way:

Definition 4.3. $T \in L(H)$ is called almost partially isometric if T^*T is a subprojection.

However, since T^*T is always selfadjoint, therefore the two classes of partially isometric operators and almost partially isometric operators are equivalent.

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