King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 277

May 2002

The Moments of Discrete Distribution Associated with the Principle of Inclusion and Exclusion

A.H. Joarder

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

The Moments of a Discrete Distribution Associated with the Principle of Inclusion and Exclusion

A. H. JOARDER

Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia
Email: anwarj@kfupm.edu.sa

Abstract The number of elements belonging exactly to the intersection of some sets is taught in discrete mathematics and in elementary probability theory. Associated with it there is a probability distribution implicit in most elementary statistics books. The factorial moment structure of the probability distribution has been derived and found to be very elegant. Raw and corrected moments of the distribution are also derived. Some interrelationships among moments of different sample spaces are established.

1. Introduction

Many real world problems involve intersection of finite sets and one needs to find the number of elements belonging only (exactly) to single sets, intersection of two sets, intersection of three sets etc (Kolman, Anton and Averbach, 1992, pp198-200). It is popular to use the Venn Diagram for this kind of problems. However, if the number of sets (k) involved in the problem is 4 or more, the determination of the number of elements belonging exactly to the intersection of some sets becomes a formidable job. We emphasize a different method which is very straightforward for any number of sets.

Suppose that there are k sets $A_1, A_2, \dots, A_k$. Define $N(A)$ as the number of elements belonging to set A. Then for $k=3$ we define $n_{(1)} = N(A_1)+N(A_2)+N(A_3)$, the number of elements in any of the single sets, $n_{(2)} = N(A_1A_2)+N(A_1A_3)+N(A_2A_3)$, the number of elements in any of the 2-combinations intersecting sets formed from the 3 sets, $n_{(3)} = N(A_1A_2A_3)$, the number of elements in the intersection of three sets. We also define $n_1 = N(A_1\overline{A}_2\overline{A}_3)+N(\overline{A}_1A_2\overline{A}_3)+N(\overline{A}_1\overline{A}_2A_3)$, the number of elements belonging only (exactly) to one of the 3 sets, $n_2 = N(A_1A_2\overline{A}_3)+N(A_1\overline{A}_2A_3)+N(\overline{A}_1A_2A_3)$, the number of elements belonging only (exactly) to any of the 2-combinations intersecting sets formed from the 3 sets, $n_3 = N(A_1A_2A_3)$, the number of elements belonging exactly to the intersection of three sets. Similarly we can define other numbers of this kind for k sets. The following theorem is well known (Eisen, 1969, p113).

Theorem 1.1 Suppose that there are k sets $A_1, A_2, \dots, A_k$. Define $N(A)$ as the number of elements belonging to set A. Also define n_i = the number of elements belonging exactly to any of the i-combinations $(i = 1, 2, ..., k)$ intersecting sets, $n_{(j)}$ = the number of elements in any of the j-combinations $(j = 1,2,...,k)$ intersecting sets. Then the following relationship exists between n_i $(i = 1, 2, ..., k)$ and $n_{(j)}$ $(j = 1, 2, ..., k)$.

$$n_i = n_{(i)} - \binom{i+1}{i} n_{(i+1)} + \binom{i+2}{i} n_{(i+2)} - ... + (-1)^{k-i} \binom{k}{i} n_{(k)}, \quad (i = 1, 2, ..., k) \qquad (1.1)$$

Many problems of discrete mathematics (see e.g. Kolman, Anton and Averbach, 1992, pp198-200) can be solved by the above theorem to have better insight.

Eisen (1969, p156) described that the above theorem can be used as a probability distribution (See Theorem 2.1 below) but we have neither found this probability distribution used for solving problems in statistics books nor is there any attempt to study the distribution further. The formidable calculation in the derivation of the moments of the distribution is made much easier and neater by introducing a matrix and exploiting its property. In this paper we derive the factorial moment structure of the probability distribution and find it to be very elegant. Raw and corrected moments of the distribution are also derived. Some interrelationships among moments of different sample spaces are established.

2. Probability distribution of the number of elements belonging exactly to the intersection of some sets

The relationship between $\pi_i = n_i / n$, $(i = 1, 2, ..., k)$ and $\pi_{(j)} = n_{(j)} / n$, $(j = 1, 2, ..., k)$, where $n = n_0 + n_1 + \cdots + n_k$, is given by the following theorem.

Theorem 2.1 For any integer i with $0 \le i \le k$, the probability π_i that exactly i among k events $A_1, A_2, ..., A_k$ occur simultaneously is given by

$$\pi_i = \pi_{(i)} - \binom{i+1}{i} \pi_{(i+1)} + \binom{i+2}{i} \pi_{(i+2)} - ... + (-1)^{k-i} \binom{k}{i} \pi_{(k)}, \quad (i = 1, 2, ..., k). \qquad (2.1)$$

where $\pi_{(0)} = 1$, $\pi_{(1)} = \sum P(A_i)$, $\pi_{(2)} = \sum P(A_i A_j), \cdots$, $\pi_{(k)} = P(A_1 A_2 \cdots A_k)$ (cf. Eisen,1969, 156).

We now illustrate the idea with some examples.

Example 2.1 It is found that 36% people in a city read newspaper A_1, 27% of them read newspaper A_2, 3% of them read both the newspapers A_1 and A_2. Then $\pi_{(2)} = P(A_1 A_2) = 0.03$, $\pi_{(1)} = P(A_1) + P(A_2) = 0.36 + 0.27 = 0.63$ and consequently the probability that a person read exactly two newspapers is given by $\pi_2 = \pi_{(2)} = 0.03$, the probability that a person read exactly one newspaper is given by

$\pi_1 = \pi_{(1)} - 2\pi_{(2)} = 0.63 - 2(0.3) = 0.57$ and the probability that a person does not read either of the two newspapers is $\pi_0 = 1 - 0.57 - 0.03 = 0.40$.

Example 2.2 Suppose that 36 % of the people in a community read newspaper A_1, 27% of them read newspaper A_2, 22% of them read newspaper A_3, 3% of them read both the newspapers A_1 and A_2, 4% of them read both the newspapers A_1 and A_3, 5% of them read both the newspapers A_2 and A_3, and only 1% of them read all the newspapers.

Let the probability that a randomly selected person read exactly one newspaper, exactly two newspapers, and all the three newspapers be denoted by π_1, π_2 and π_3 respectively. Since $k = 3$, it follows from Theorem 2.1 that

$$\pi_{(3)} = P(A_1A_2A_3) = 0.01$$
$$\pi_{(2)} = P(A_1A_2) + P(A_1A_3) + P(A_2A_3) = 0.03 + 0.04 + 0.05 = 0.12$$
$$\pi_{(1)} = P(A_1) + P(A_2) + P(A_3) = 0.36 + 0.27 + 0.22 = 0.85$$

Then by (2.1) we have

$$\pi_3 = \pi_{(3)} = 0.01$$
$$\pi_2 = \pi_{(2)} - 3\pi_{(3)} = 0.12 - 3(0.01) = 0.09$$
$$\pi_1 = \pi_{(1)} - 2\pi_{(2)} + 3\pi_{(3)} = 0.85 - 2(0.12) + 3(0.01) = 0.64$$
$$\pi_0 = 1 - \pi_1 - \pi_2 - \pi_3 = 0.26$$

Note that $\pi_{(1)}, \pi_{(2)}, \pi_{(3)}$ do not constitute a genuine set of probabilities, but π_3, π_2, π_1 and π_0 do.

Example 2.3 It is found that 38% people in a city read newspaper A_1, 49% of them read newspaper A_2, 43% of them read newspaper A_3, 33% of them read newspaper A_4. 22% of them read both the newspapers A_1 and A_2, 11% of them read both the newspapers A_1 and A_3., 11% of them also read both the newspapers A_1 and A_4, 22% of them read both the newspapers A_2 and A_3, 12% of them read both the newspapers A_2 and A_4 and 14% of them read both the newspapers A_3 and A_4. 8% of them read the newspapers $A_{1,} A_2$ and A_3, 6% of them read the newspapers $A_{1,} A_2$ and A_4, 4% of them read the newspapers $A_{1,} A_3$ and A_4, 7% of them read the newspapers $A_{2,} A_3$ and A_4, and only 1% reads all the four newspapers.

To find π_i , the proportion of people who read exactly $i (i = 1,2,3,4)$ newspapers, we proceed as follows.

$$\pi_{(4)} = P(A_1A_2A_3A_4) = 0.01$$

$$\pi_{(3)} = P(A_1A_2A_3) + P(A_1A_2A_4) + P(A_1A_3A_4) + P(A_2A_3A_4)$$
$$= 0.08 + 0.06 + 0.04 + 0.07 = 0.25$$

$$\pi_{(2)} = P(A_1A_2) + P(A_1A_3) + P(A_1A_4) + P(A_2A_3) + P(A_2A_4) + P(A_3A_4)$$
$$= 0.22 + 0.11 + 0.11 + 0.22 + 0.12 + 0.14 = 0.92$$

$$\pi_{(1)} = P(A_1) + P(A_2) + P(A_3) + P(A_4) = 0.38 + 0.49 + 0.43 + 0.33 = 1.63$$

Then by (2.1) we have

$$\pi_4 = \pi_{(4)} = 0.01$$
$$\pi_3 = \pi_{(3)} - 4\pi_{(4)} = 0.25 - 4(0.01) = 0.21$$
$$\pi_2 = \pi_{(2)} - 3\pi_{(3)} + 6\pi_{(4)} = 0.92 - 3(0.25) + 6(0.1) = 0.23$$
$$\pi_1 = \pi_{(1)} - 2\pi_{(2)} + 3\pi_{(3)} - 4\pi_{(4)} = 1.63 - 2(0.92) + 3(0.25) - 4(0.01) = 0.50$$
$$\pi_0 = 1 - (\pi_1 + \pi_2 + \pi_3 + \pi_4) = 1 - (0.50 + 0.23 + 0.21 + 0.01) = 0.05$$

Many problems of elementary probability (see e.g. # 2.4 in Hines and Montgomery, 1990, pp.57-58) can be solved by Theorem 2.1 to have better insight.

3. Moments of the distribution

Let us now calculate the moment a probability distribution discussed the following example (which is made out of Example 2.2).

Example 3.1 Let X = Number of newspapers read by a person. Then the probability density function of X is given by
$P(X = i) = \pi_i = n_i / n, \ (i = 0, 1, 2, 3)$ i.e.

i	0	1	2	3
π_i	0.26	0.64	0.09	0.01

Then the expected number of newspapers read by a person is given by

$$E(X) = \sum_{i=0}^{3} i \ n_i / n = (n_1 + 2n_2 + 3n_3)/n = n_{(1)}/n = 85/100 = 0.85.$$

The second raw moment is given by

$$E(X^2) = \sum_{i=0}^{3} i^2 \ n_i / n = (n_1 + 4n_2 + 9n_3)/n.$$

The variance of X is then given by

$$V(X) = E(X^2) - [E(X)]^2 = (n_{(1)} + 2n_{(2)})/n - (n_{(1)}/n)^2$$
$$= (n_{(1)}/n)(1 - n_{(1)}/n) + 2n_{(2)}/n = 0.85(0.15) + 2(0.12) = 0.3675$$

3.1 The factorial moments of X when there are k sets

Let X be a random variable with the following probability density function:

$$P(X = i) = \pi_i = n_i / n \ , \ i = 0, 1, \ldots, k \tag{3.1}$$

where n_i is the number of elements belonging exactly to the intersection of some sets defined in (1.1). In what follows we introduce a matrix that makes the algebra of finding the moments of the above distribution neater. The formula in (1.1) can be written as

$$n_i = \sum_{i=1}^{k} \sum_{j=1}^{k} (-1)^{j-i} \binom{j}{i} n_{(j)}, \quad (i = 1, 2, \ldots, k)$$

or simply by $\underline{n} = C\,\underline{n}_{(.)} = C\underline{m}$ (3.2)

where $\underline{n}' = (n_1, n_2, \cdots, n_k)$, $C = ((c_{ij}))$, $c_{ij} = (-1)^{j-i}\binom{j}{i}$, $i = 1,2,\ldots,k;\ j = 1,2,\ldots,k$ with $c_{ij} = 0$ if $i > j$ and 1 if $i = j$. Further let $n_{(i)} = m_i$, $(i = 1,2,\ldots,k)$ so that $\underline{m}' = (m_1, m_2, \ldots, m_k)$. The matrix C can then be written as

$$C = \begin{bmatrix} 1 & -2 & +3 & -4 & \ldots & (-1)^{k-2}(k-1) & (-1)^{k-1}k \\ 0 & 1 & -3 & 6 & \ldots & (-1)^{k-3}\binom{k-1}{2} & (-1)^{k-2}\binom{k}{2} \\ 0 & 0 & 1 & -4 & \ldots & (-1)^{k-4}\binom{k-1}{3} & (-1)^{k-3}\binom{k}{3} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & 1 & (-1)^{-1}\binom{k}{k-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Let $C(l,l)$, $(l = 1,2,\ldots,k)$ be the principal submatrix of order l i.e. a matrix formed from C by deleting the first l rows and the first l columns. For example, $C(2,2)$ is a matrix formed from C by deleting the first two rows and the first two columns. Similarly $\underline{m}(l)$ is a vector formed from $\underline{m}$ after deleting the first l columns.

Lemma 3.1 With the above notations, and $r = 1,2,\cdots,k$, the following holds:

$$\begin{pmatrix} \frac{r!}{0!} & \frac{(r+1)!}{1!} & \cdots & \frac{k!}{(k-r)!} \end{pmatrix} C(r-1,r-1) = (r!\,0\,0 \cdots 0)$$

Proof. The proof is straightforward.

In what follows we derive factorial moments of X by the use of Lemma 3.1.

Theorem 3.1. The r th factorial moment of the distribution of X is given by

$$\mu'_{(r)} = E\left[X^{(r)}\right] = \begin{cases} r! \dfrac{n_{(r)}}{n} = r!\pi_{(r)}, & r \le k \\ 0 & r \ge k+1 \end{cases} . \tag{3.3}$$

Proof.

$$E[X^{(1)}] = \sum_{i=0}^{k} i \frac{n_i}{n} = \frac{1}{n}(1\ 2\ 3\ \ldots\ k)\ \underline{n}$$
$$= \frac{1}{n}(1\ 2\ 3\ \ldots\ k)\ C\ \underline{m} = \frac{1}{n}(1\ 0\ \ldots\ 0)\ \underline{m}$$
$$= \frac{m_2}{n} = \frac{n_{(1)}}{n} = \pi_{(1)},$$

$$E[X^{(2)}] = \sum_{i=0}^{k} i(i-1) \frac{n_i}{n}$$
$$= \frac{1}{n}[(2)(1)n_2 +)(3)(2)n_3 + (4)(3)n_3 + \ldots + k(k-1)n_k]$$
$$= \frac{1}{n}\left[\frac{2!}{0!}\ \frac{3!}{1!}\ \frac{4!}{2!} \ldots \frac{k!}{(k-2)!}\right] \underline{n}(1)$$
$$= \frac{1}{n}\left[\frac{2!}{0!}\ \frac{3!}{1!}\ \frac{4!}{2!} \ldots \frac{k!}{(k-2)!}\right] C(11)\ \underline{m}(1)$$
$$= \frac{1}{n}(2!\ 0\ 0 \ldots 0)\ \underline{m}(1)$$
$$= \frac{2!}{n} m_2 = \frac{2!}{n}\ n_{(2)} = 2!\pi_{(2)}$$

and

$$E[X^{(3)}]=\sum_{i=0}^{k} i(i-1)(i-2)\ \frac{n_i}{n}$$
$$=\frac{1}{n}[(2)(1)n_2+)(3)(2)n_3+(4)(3)n_3+...+k(k-1)n_k]$$
$$=\frac{1}{n}\left[\frac{3!}{0!}\ \frac{4!}{1!}\ \frac{5!}{2!}...\ \frac{k!}{(k-3)!}\right]\ \underline{n}(2)$$
$$=\frac{1}{n}\left[\frac{3!}{0!}\ \frac{4!}{1!}\ \frac{5!}{2!}...\ \frac{k!}{(k-3)!}\right]\ C(22)\ \underline{m}(2)$$
$$=\frac{1}{n}(3!\ 0\ 0...\ 0)\,\underline{m}(2)$$
$$=\frac{3!}{n}\ m_3=\frac{3!}{n}\ n_{(3)}=3!\pi_{(3)}$$

In general for $r \leq k$ we have

$$E[X^{(r)}]=\sum_{i=0}^{k} i(i-1)(i-2)...(i-r+1)\ \frac{n_i}{n}=\sum_{i=0}^{k}\frac{i!}{(i-r)!}\ \frac{n_i}{n}$$
$$=\frac{1}{n}\left[\frac{r!}{0!}\ n_r+\frac{(r+1)!}{1!}\ n_{r+1}+\cdots+\frac{k!}{(k-r)!}\ n_k\right]$$
$$=\frac{1}{n}\left[\frac{r!}{0!}\ \frac{(r+1)!}{1!}\ \frac{(r+2)!}{2!}...\ \frac{k!}{(k-r)!}\right]\ \underline{n}(r-1)$$
$$=\frac{1}{n}\left[\frac{r!}{0!}\ \frac{(r+1)!}{1!}\ \frac{(r+2)!}{2!}...\ \frac{k!}{(k-r)!}\right]\ C(r-1,r-1)\ \underline{m}(r-1)$$
$$=\frac{1}{n}\ (r!\ 0\ 0\ \cdots\ 0)\ \underline{m}(r-1)$$
$$=\frac{r!}{n}\ n_{(r)}=r!\pi_{(r)}$$

3.2 The Moments of X for $k=1,2,3,4$

The raw moments of X are given by

$$\mu'_r=\sum_{i=1}^{r} S(r,i)\ \mu'_{(i)}=S(r,1)\mu'_{(1)}+S(r,2)\mu'_{(2)}+...+S(r,r)\mu'_{(r)}$$

where $S(r,i)$ is the Stirling number of the second kind (see Johnson, Kotz and Kemp, 1993, 44). For simplicity $\mu'_{(1)}$ is traditionally denoted by μ. In particular we have

$$\begin{aligned}
\mu_1' &= \mu_{(1)}' = \mu \\
\mu_2' &= \mu + \mu_{(2)}' \\
\mu_3' &= \mu' + 3\mu_{(2)}' + \mu_{(3)}' \\
\mu_4' &= \mu_4' - 7\mu_{(2)}' + 6\mu_{(3)}' + \mu_{(4)}'
\end{aligned} \tag{3.4}$$

The corrected moments of X can be calculated by the raw moments by the following well known relations:

$$\begin{aligned}
\mu_2 &= \mu_2' - \mu^2 \\
\mu_3 &= \mu_3' - 3\mu_2'\mu + 2\mu^3 \\
\mu_4 &= \mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4
\end{aligned} \tag{3.5}$$

It follows from (3.3) and (3.4) that the raw moments of X for $k = 2$ are given by

$$\begin{aligned}
\mu_1'(k=2) &= \mu_{(1)}' = \mu = \pi_{(1)} \\
\mu_2'(k=2) &= \mu + \mu_{(2)}' = \pi_{(1)} + 2!\pi_{(2)} = \pi_{(1)} + 2\pi_{(2)} \\
\mu_3'(k=2) &= \pi_{(1)} + 3(2!\,\pi_{(2)}) + 0 = \pi_{(1)} + 6\pi_{(2)} \\
\mu_4'(k=2) &= \mu + 7\,\mu_{(2)}' + 0 + 0 = \pi_{(1)} + 7(2!\pi_{(2)}) = \pi_{(1)} + 14\pi_{(2)}
\end{aligned} \tag{3.6}$$

It follows from (3.5) and (3.6) that the corrected moments of X for $k = 2$ are given by

$$\begin{aligned}
\mu_2(k=2) &= \pi_{(1)} - \pi_{(1)}^2 + 2\pi_{(2)} \\
\mu_3(k=2) &= \pi_{(1)} - 3\pi_{(1)}^2 + 2\pi_{(1)}^3 + 6\pi_{(2)} - 6\pi_{(1)}\pi_{(2)} \\
\mu_4(k=2) &= \pi_{(1)} - 4\pi_{(1)}^2 + 6\pi_{(1)}^3 - 3\pi_{(1)}^4 + 14\pi_{(2)} - 24\pi_{(1)}\pi_{(2)} + 12\pi_{(2)}\pi_{(1)}^2
\end{aligned} \tag{3.7}$$

It follows from (3.3) and (3.4) that the raw moments of X for $k = 3$ are given by

$$\begin{aligned}
\mu_1'(k=3) &= \mu = \pi_{(1)} \\
\mu_2'(k=3) &= \mu + \mu_{(2)}' = \pi_{(1)} + 2\pi_{(2)} \\
\mu_3'(k=3) &= \pi_{(1)} + 3(2!\,\pi_{(2)}) + 3!\,\pi_{(3)} = \pi_{(1)} + 6\pi_{(2)} + 6\pi_{(3)} \\
\mu_4'(k=3) &= \pi_{(1)} + 7(2!\,\pi_{(2)}) + 6(3!\pi_{(3)}) + 0 = \pi_{(1)} + 14\pi_{(2)} + 36\pi_{(3)}
\end{aligned} \tag{3.8}$$

It follows from (3.5) and (3.8) that the corrected moments of X for $k = 3$ are given by

$$\begin{aligned}
\mu_2(k=3) &= \pi_{(1)} - \pi_{(1)}^2 + 2\pi_{(2)} \\
\mu_3(k=3) &= \pi_{(1)} - 3\pi_{(1)}^2 + 2\pi_{(1)}^3 + 6\pi_{(2)} + 6\pi_{(3)} - 6\pi_{(1)}\pi_{(2)} \\
\mu_4(k=3) &= \pi_{(1)} - 4\pi_{(1)}^2 + 6\pi_{(1)}^3 - 3\pi_{(1)}^4 + 14\pi_{(2)} + 36\pi_{(3)} \\
&\quad - 24\pi_{(1)}\pi_{(2)} - 24\pi_{(1)}\pi_{(3)} + 12\pi_{(2)}\pi_{(1)}^2
\end{aligned} \tag{3.9}$$

It follows from (3.3) and (3.4) that the raw moments of X for $k=4$ are given by

$$\begin{aligned}
\mu_1'(k=4) &= \mu = \pi_{(1)} \\
\mu_2'(k=4) &= \mu + \mu_{(2)}' = \pi_{(1)} + 2\pi_{(2)} \\
\mu_3'(k=4) &= \pi_{(1)} + 3\left(2!\ \pi_{(2)}\right) + 3!\ \pi_{(3)} = \pi_{(1)} + 6\pi_{(2)} + 6\pi_{(3)} \\
\mu_4'(k=4) &= \pi_{(1)} + 7\left(2!\ \pi_{(2)}\right) + 6\left(3!\pi_{(3)}\right) + 4!\pi_{(4)} = \pi_{(1)} + 14\pi_{(2)} + 36\pi_{(3)} + 24\pi_{(4)} + 0
\end{aligned} \tag{3.10}$$

It follows from (3.5) and (3.10) that the corrected moments of X for $k=4$ are given by

$$\begin{aligned}
\mu_2(k=4) &= \pi_{(1)} - \pi_{(1)}^2 + 2\pi_{(2)} \\
\mu_3(k=4) &= \pi_{(1)} - 3\pi_{(1)}^2 + 2\pi_{(1)}^3 + 6\pi_{(2)} + 6\pi_{(3)} - 6\pi_{(1)}\pi_{(2)} \\
\mu_4(k=4) &= \pi_{(1)} - 4\pi_{(1)}^2 + 6\pi_{(1)}^3 - 3\pi_{(1)}^4 + 14\pi_{(2)} + 36\pi_{(3)} \\
&\quad + 24\pi_{(4)} - 24\pi_{(1)}\pi_{(2)} - 24\pi_{(1)}\pi_{(3)} + 12\pi_{(2)}\pi_{(1)}^2
\end{aligned} \tag{3.11}$$

3.3 Interrelationships among moments

It follows from (3.6) and (3.8) that the raw moments of X for $k=2,3$ have the following interrelationships:

$$\begin{aligned}
\mu_1'(k=3) &= \mu_1'(k=2) \\
\mu_2'(k=3) &= \mu_2'(k=2) \\
\mu_3'(k=3) &= \mu_3'(k=2) + 6\pi_{(3)} \\
\mu_4'(k=3) &= \mu_4'(k=2) + 36\pi_{(3)}
\end{aligned}$$

It follows from (3.8) and (3.10) that the raw moments of X for $k=3,4$ have the following interrelationships:

$$\begin{aligned}
\mu_1'(k=4) &= \mu_1'(k=3) \\
\mu_2'(k=4) &= \mu_2'(k=3) \\
\mu_3'(k=4) &= \mu_3'(k=3) \\
\mu_4'(k=4) &= \mu_4'(k=3) + 24\pi_{(4)}
\end{aligned}$$

It follows from (3.7) and (3.9) that the corrected moments of X for $k=2,3$ have the following interrelationships:

$$\begin{aligned}
\mu_2(k=3) &= \mu_2(k=2) \\
\mu_3(k=3) &= \mu_3(k=2) + 6\pi_{(3)} \\
\mu_4(k=3) &= \mu_4(k=2) + 36\pi_{(3)} - 24\pi_{(1)}\pi_{(3)}
\end{aligned}$$

It follows from (3.9) and (3.11) that the corrected moments of X for $k = 3,4$ have the following interrelationships:

$$\mu_2(k=4) = \mu_2(k=3)$$
$$\mu_3(k=4) = \mu_3(k=3)$$
$$\mu_4(k=4) = \mu_4(k=3) + 24\pi_{(4)}$$

Acknowledgement

The author gratefully acknowledges the excellent research facilities available at King Fahd University of Petroleum and Minerals, Saudi Arabia.

References

Eisen, M. (1969). *Elementary Combinatorial Analysis.* Gordon and Breach Science Publishers. New York.

Hines, W.W. and Montgomery, D.C. (1990). Probability and Statistics for Engineering and Management Science. John Wiley. New York.

Johnson, N.L. ; Kotz, S. and Kemp, A.W. (1993). Univariate Discrete Distributions. John Wiley and Sons, Inc. New York.

Kolman, B.; Anton, H. and Averbach, B. (1992). Mathematics with Applications for the Management, Life and Social Sciences. Saunders College Publishing, New York.

File: c \ paper \ p46a.doc