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On the Behavior of the Interface Separating Fresh and Salt Groundwater in a Heterogeneous Coastal Aquifer

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Introduction

Fresh water and sea water are actually miscible fluids and therefore the zone of contact between them takes the form of a transition zone caused by hydrodynamics dispersion. Across this zone the density of the mixed water varies from that of fresh water to that of sea water.

Under certain conditions the width of this zone is relatively small (ex. when compared with the thickness of the aquifer) so that we assume that each liquid is confined to a well defined portion of the flow domain with an abrupt interface separating the two domains.

However, under certain conditions, this approximation fails and instead of the displacement of the interface as a whole in a regular form, protuberances occur that may advance through the porous medium at velocities much higher than those of the average.

By pumping from the coastal aquifer in excess of replenishment, the interface starts to advance inland. When it reaches inland pumping well, the water become contaminated. So we need to understand the mechanism of sea water intrusion and to learn how to control it in order to improve the yield of coastal aquifers (see [3]).

We shall treat here a two-dimensional model for fresh-salt water in a horizontally extended aquifer. We suppose that the scale of the problem is sufficiently large so that the abrupt interface approximation is applicable. More precisely we consider the model studied by Alt and Van Duijn in [1] yet we assume the aquifer to be heterogeneous and the flow obeying to a nonlinear Darcy's law. In section 1, we indicate briefly how to obtain the existence and some properties of the solutions, the definition of the free boundary $\Gamma = [x = g(x)]$ and the continuity of g

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over its interval of definition $(-h^*, 0)$, $h^* > 0$. All these results generalize previous works in [5] and [4](resp. [6]) where the aquifer was supposed homogeneous (resp. heterogeneous) and the flow governed by a nonlinear (resp. linear) Darcy's law.

The aim of this paper is to study the behavior of the free boundary when $z \to -h^*$ and $z \to 0$. Actually we establish in section 3 that $\lim_{z\to -h^*} g(z) = +\infty$ by generalizing the proof given in [6]. We also give a second proof too much simpler than the first one which works only when the permeability is constant. Then we prove that $\lim_{z\to 0} g(z) = g(0-)$ exists. We recall that in [1] the authors have first proved (in the linear and homogeneous case) that $\liminf_{z\to 0} g(z) > -\infty$ by using blow up arguments. They also proved that $\limsup_{z\to 0} g(z) < 0$ and used this result to prove the existence of the limit $g(0-)$. Our proof does not assume $\limsup_{z\to 0} g(z) < 0$ and is valid for the general case. Moreover we prove that $g(0-)$ is finite in more general cases. Our proofs are systematically based on comparing the solution locally or globally with explicit functions satisfying similar equations. This method of comparison is developed in section 2 to show that the solution increase with respect to the salt water discharge Q_s and decrease with respect to the fresh water discharge Q_f. The uniqueness of the solution is obtained as a direct corollary from this monotonicity result. We also deduce a limit behavior of the solution when Q_f or Q_s goes to zero. Also by a comparison argument, when the permeability is constant, we give a simple proof of the fact that the set filled by fresh water is star shaped with the origin and the free boundary is non increasing in the region $[x > 0]$. These two last results were proved in the linear case in [1] by using blow up arguments. Our proofs are too much simpler than theirs.

1 Description of the model

In this paper we are interested with the study of a stationary flow of fresh and salt water in a heterogeneous coastal aquifer $\Omega = (-h, 0) \times I\!R$ $(h > 0)$ with permeability $A(X)$, $X = (x, y)$. The velocity and the pressure of the fluid are related by the following nonlinear Darcy law:

$$v = -(< A(\nabla p + \gamma e_z), \nabla p + \gamma e_z >)^{\frac{r-2}{2}} A(\nabla p + \gamma e_z) \tag{1.1}$$

with $r > 1$, $e_z = (0, 1)$ and γ given by:

$$\gamma = \gamma_f \chi(\Omega_f) + \gamma_s \chi(\Omega_s) \quad \text{with} \quad 0 < \gamma_f < \gamma_s \tag{1.2}$$

γ_f (resp.γ_s) represents the specific weight of the fresh (resp. salt) water occupying the subset Ω_f (resp. Ω_s) of Ω. $\chi(E)$ denotes the characteristic function of the set E.

Fresh water is injected over the segment $[OA]$ $(A = (0, a)$ with $a > 0)$ uniformly (see Figure 1) with a total amount of Q_f. From infinity at the left of the aquifer, salt water arrives with a total discharge of Q_s. We assume the two fluids unmixed and separated by an interface Γ. The boundary $\partial\Omega \setminus [OA]$ is assumed to be impervious and the flow incompressible. So the velocity satisfies:

$$\begin{aligned} &div v = 0 \quad \text{in} \quad \Omega, \qquad v = -\frac{Q_f}{a} e_z \quad \text{on} \quad [OA], \\ &v.\nu = 0 \quad \text{on} \quad \partial\Omega \setminus [OA], \qquad v_i.\nu = 0 \quad \text{on} \quad \Gamma \quad (i = s, f) \end{aligned} \tag{1.3}$$

v_i denotes the restriction of v to Ω_i and ν is the outward unit normal to $\partial\Omega$ or

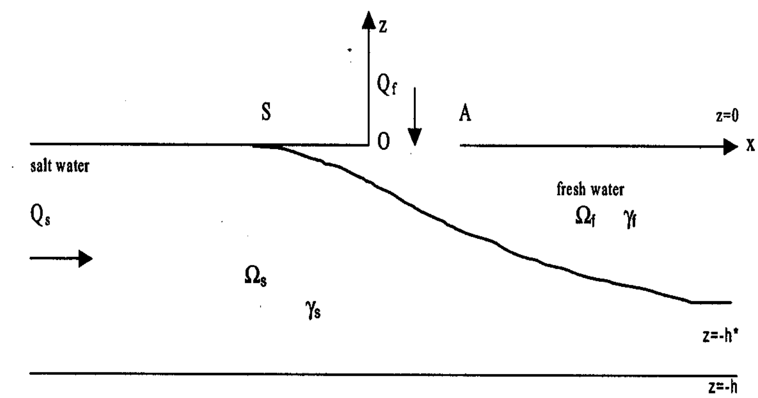

Figure 1:

Γ. We deduce from (1.3) that there exists a stream function ψ satisfying:

$$\begin{aligned} &v = Rot\psi = \Big(-\frac{\partial\psi}{\partial z}, \frac{\partial\psi}{\partial x} \Big) \quad \text{in} \quad \Omega, \qquad \psi = 0 \quad \text{on} \quad \Gamma, \\ &\psi(x, -h) = Q_s \qquad \text{and} \qquad \psi(x, 0) = \phi_0(x) \quad \text{for} \quad x \in I\!R \end{aligned} \tag{1.4}$$

with

$$\phi_0(x) = -Q_f \min\big(\frac{x^+}{a}, 1\big). \tag{1.5}$$

The permeability $A(X)$ is such that:

$$\begin{cases} A(X) \in \left[L^\infty(\Omega)\right]^4, \\ \exists \alpha > 0 \qquad < A(X)\xi, \xi > \geq \alpha|\xi|^2 \qquad \forall \xi \in I\!R^2, \qquad \text{a.e. } X \in \Omega \\ {}^tA = A. \end{cases} \tag{1.6}$$

Then there exists a unique matrix $\mathcal{A}$, symmetric and strictly elliptic satisfying $A(X) = \mathcal{A}^t\mathcal{A} = \mathcal{A}^2$ (see [7]). The velocity v can be written

$$
\begin{aligned}
v &= -\Big(< \mathcal{A}(X)(\nabla p + \gamma e_z), \mathcal{A}(X)(\nabla p + \gamma e_z) > \Big)^{\frac{r-2}{2}} \mathcal{A}(X)\mathcal{A}(X)(\nabla p + \gamma e_z) \\
&= -|\mathcal{A}(X)(\nabla p + \gamma e_z)|^{r-2}\mathcal{A}(X)\mathcal{A}(X)(\nabla p + \gamma e_z).
\end{aligned}
$$

Then:

$$|\mathcal{A}^{-1}(X)v| = |\mathcal{A}(X)(\nabla p + \gamma e_z)|^{r-1}$$

and:

$$\nabla p + \gamma e_z = -|\mathcal{A}^{-1}(X)v|^{\frac{2-r}{r-1}}\mathcal{A}^{-1}(X)(\mathcal{A}^{-1}(X)v). \tag{1.7}$$

Now for $\zeta \in \mathcal{D}(\Omega)$ we get by (1.3) and (1.7),

$$\int_\Omega (\nabla p + \gamma e_z) Rot\zeta = -\int_\Omega |\mathcal{A}^{-1}(X)Rot\psi|^{\frac{2-r}{r-1}}\mathcal{A}^{-1}(X)(\mathcal{A}^{-1}(X)Rot\psi)Rot\zeta.$$

If we set $B(X) = \frac{1}{detA}A$, then there exists a unique matrix b symmetric and strictly elliptic such that $B(X) =^t b.b$ and for which we have:

$${}^t\mathcal{A}^{-1}(X)(\mathcal{A}^{-1}(X)Rot\psi)Rot\zeta =< B(X)\nabla\psi, \nabla\zeta >= b(X)\nabla\psi.b(X)\nabla\zeta$$

and so

$$\int_\Omega |b(X)\nabla\psi|^{\frac{2-r}{r-1}}b(X)\nabla\psi.b(X)\nabla\zeta \ + \gamma e_x\nabla\zeta = 0 \qquad \forall\zeta \in \mathcal{D}(\Omega). \tag{1.8}$$

If we set $q = \frac{r}{r-1}$ then $\frac{2-r}{r-1} = q - 2$ and from (1.8) we deduce by taking $\zeta \in \mathcal{D}(\Omega_i)$ $(i = f, s)$:

$$div(\mathcal{B}(X, \psi)) = 0 \quad \text{ in } \mathcal{D}'(\Omega_i) \quad i = f, s \tag{1.9}$$

with $\mathcal{B}(X, \xi) =< B(X)\xi, \xi >^{\frac{q-2}{2}} B(X)\xi$.
If we assume that

$$\psi \le 0 \quad \text{ when } x \to +\infty \quad \text{ and } \quad \psi \ge 0 \quad \text{ when } x \to -\infty \tag{1.10}$$

and since

$$\psi \le 0 \quad \text{ on } \partial\Omega_f \quad \text{ and } \quad \psi \ge 0 \quad \text{ on } \quad \partial\Omega_s \tag{1.11}$$

we deduce by (1.9)-(1.11) and the maximum principle for $\mathcal{B}$-harmonic functions in unbounded domains (see [9], [10]) that:

$$\psi < 0 \quad \text{ in } \Omega_f \quad \text{ and } \quad \psi > 0 \ \text{ in } \ \Omega_s. \tag{1.12}$$

So let us introduce the maximal monotone graph H defined by:

$$H(t) = \gamma_f\chi([t < 0]) + [\gamma_f, \gamma_s]\chi([t = 0]) + \chi([t > 0]).$$

Then we are led to study the following problem:

$$(P)\begin{cases} \text{Find } (\psi,\gamma)\in W^{1,q}_{loc}(\Omega)\times L^\infty(\Omega) \text{ such that:} \\ i) \quad \int_\Omega |b(X)\nabla\psi|^{q-2}b(X)\nabla\psi.b(X)\nabla\zeta+\gamma e_x.\nabla\zeta=0 \\ \qquad \forall\zeta\in W^{1,q}_0(\Omega) \text{ with compact support in } \bar\Omega \\ ii) \quad \gamma\in H(\psi) \\ iii) \quad -Q_f\le\psi\le Q_s \quad \text{a.e. in } \Omega \\ iv) \quad \psi(x,-h)=Q_s, \quad \psi(x,0)=\phi_0(x) \quad \text{for all } x\in I\!R. \end{cases}$$

Adapting technics in [4], [5] and [6] we prove the following theorems:

Theorem 1.1 *i) There exists a solution (ψ,γ) of (P) that satisfies $\psi\in C^{0,\alpha}_{loc}(\bar\Omega)$ for some $\alpha\in(0,1)$. Moreover if $B(X)\in C^{0,\sigma}_{loc}(\Omega)$ then $\psi\in C^{1,\beta}_{loc}(\Omega\setminus[\psi=0])$*
ii) If $b(X)=b(z)$, then there exists a monotone solution (ψ,γ) of (P) in the following sense:

$$\partial_x\psi\le 0 \quad \text{and} \quad \partial_x\gamma\le 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{1.13}$$

From now on, we assume that $b(X)=b(z)$ a.e. in Ω and we will consider only monotone solutions. Moreover we need the following functions defined for $z\in[-h,0]$ by:

$$v_{+\infty}(z)=-Q_f+(Q_s+Q_f)\phi_1(z) \quad \text{and} \quad v_{-\infty}(z)=Q_s\phi_1(z) \tag{1.14}$$

where ϕ_1 is defined by:

$$\phi_1(z)=\frac{\int_z^0 \frac{ds}{(b_{12}^2+b_{22}^2)^{q'/2}(s)}}{\int_{-h}^0 \frac{ds}{(b_{12}^2+b_{22}^2)^{q'/2}(s)}}.$$

Theorem 1.2 *Let (ψ,γ) be a solution of (P).*
i) For $r=\max(q,2)$, we have:

$$\lim_{R\to+\infty}\int_{\Omega_{R,R+1}}|\nabla(\psi-v_{+\infty}|^r=0 \quad \text{and} \quad \lim_{R\to-\infty}\int_{\Omega_{R,R+1}}|\nabla(\psi-v_{-\infty}|^r=0$$

where $\Omega_{m,n}=(m,n)\times(-h,0)$ for $m,n\in I\!R$.
ii) For all $z\in[-h,0]$ one has:

$$\psi(x,z)\to v_{+\infty}(z) \quad (\textit{resp. } v_{-\infty}(z)) \quad \text{as } x\to+\infty \quad (\textit{resp. } x\to-\infty)$$

iii) $\quad v_{+\infty}\le\psi\le v_{-\infty} \quad$ *in* Ω
iv)

$$\gamma(x+R,z)\rightharpoonup\gamma_{+\infty}(z) \quad \text{in} \quad L^{q'}(\Omega_{0,1}) \quad (\textit{resp. } \gamma_{-\infty}(z))$$

as $R\to+\infty$ (resp. $R\to-\infty$) and where $\gamma_{+\infty}\in H(v_{+\infty})$ (resp. $\gamma_{-\infty}\in H(v_{-\infty})$).

Remark 1.1 *From iii), one can see that the strip* $\mathbb{R} \times (-h, -h^*)$ *is contained in* $\psi > 0$, *where* $h^* \in (0, h)$ *is defined by* $\phi_1(-h^*) = \frac{Q_f}{Q_s+Q_f}$.

Arguing as in [4] and [6] we derive the following result for the free boundary $\Gamma = [\psi = 0]$:

Theorem 1.3 *There exists a continuous function* $g : (-h^*, 0) \to \mathbb{R}$ *such that:*

$$[x = g(z)] \subset \Gamma \subset [x = g(z)] \cup [z = -h^*].$$

Corollary 1.1 *i)* $\gamma = \gamma_s \chi([\psi > 0]) + \gamma_f \chi([\psi < 0])$ *a.e. in* Ω.
ii) The sets $[\psi > 0]$ *and* $[\psi < 0]$ *are connected by arcs.*

2 Comparison and Uniqueness

In this section we prove that solutions of (P) increase with respect to Q_s and decrease with respect to Q_f. As a consequence we obtain the uniqueness of the solution of (P).

Let us denote by $(P(Q_s, Q_f))$ the problem (P) corresponding to Q_s and Q_f. Then we have the following comparison result:

Theorem 2.1 *Let* (ψ_1, γ_1) *(resp.* (ψ_2, γ_2)*) be a solution of* $(P(Q_{s_1}, Q_{f_1}))$ *(resp.* $(P(Q_{s_2}, Q_{f_2}))$ *). If* $Q_{s_1} \leq Q_{s_2}$ *and* $Q_{f_2} \leq Q_{f_1}$*, then we have:*

$$\psi_1 \leq \psi_2 \qquad \text{and} \qquad \gamma_1 \leq \gamma_2 \quad \text{a.e. in } \Omega.$$

The proof of this theorem follows an idea in [4] and uses a recent result due to Alessandrini and Sigalotti [2] about isolated zeros of the gradient of $\mathcal{B}$- harmonic functions in planar domains. First we prove the following lemma:

Lemma 2.1 *Under the same assumptions of Theorem 2.1, we have:*

$$T(\zeta) = \int_\Omega \Big(\mathcal{B}(z, \nabla\psi_1) - \mathcal{B}(z, \nabla\psi_0) + (\gamma_1 - \gamma_0)e_x\Big).\nabla\zeta = 0 \qquad \forall \zeta \in \mathcal{D}(\mathbb{R}^2)$$

where $\psi_0 = \min(\psi_1, \psi_2)$ *and* $\gamma_0 = \min(\gamma_1, \gamma_2)$.

Proof. Let $\zeta \in \mathcal{D}(\mathbb{R}^2)$ and let $K = supp\zeta$, $M = \sup_K |\zeta|$. Then there exists $R_0 > a$ such that: $\forall R \geq R_0$, $K \subset (-R, R) \times \mathbb{R}$. Consider ζ_R defined by: $\zeta_R = M.\min\big(1, (-|x| + R + 1)^+\big)$ and set $\zeta_1 = \zeta + \zeta_R$, $\zeta_2 = \zeta_R - \zeta$. Then for $\epsilon > 0$

and $i=1,2$, $\min\left(\zeta_i, \frac{\psi_1-\psi_0}{\epsilon}\right)$ is a test function and one has by the monotonicity of $\mathcal{B}(z,.)$:

$$\int_{[\psi_1-\psi_0\geq\epsilon\zeta_i]}\left(\mathcal{B}(z,\nabla\psi_1)-\mathcal{B}(z,\nabla\psi_0)\right)\nabla\zeta_i+\int_\Omega(\gamma_1-\gamma_0).\zeta_{ix} \leq \int_\Omega(\gamma_1-\gamma_0)\left(\zeta_i-\frac{\psi_1-\psi_0}{\epsilon}\right)^+_x. \tag{2.1}$$

Since $Q_{s_1}\leq Q_{s_2}$ and $Q_{f_2}\leq Q_{f_1}$ one has:

$$-h_1^*=\phi_1^{-1}\left(\frac{Q_{f1}}{Q_{s1}+Q_{f1}}\right)\leq\phi_1^{-1}\left(\frac{Q_{f2}}{Q_{s2}+Q_{f2}}\right)=-h_2^*$$

and then if we denote by $I=\{z\in(-h_2^*,0)\,/\,g_2(z)<g_1(z)\}$

$$\begin{aligned}\int_\Omega(\gamma_1-\gamma_0)\left(\zeta_i-\frac{\psi_1-\psi_0}{\epsilon}\right)^+_x &= \int_{[\psi_0<0<\psi_1]}(\gamma_s-\gamma_f)\left(\zeta_i-\frac{\psi_1-\psi_0}{\epsilon}\right)^+_x\\ &=(\gamma_s-\gamma_f)\int_I\int_{g_2(z)}^{g_1(z)}\left(\zeta_i-\frac{\psi_1-\psi_0}{\epsilon}\right)^+_x\\ &=(\gamma_s-\gamma_f)\int_I\left(\zeta_i+\frac{\psi_2}{\epsilon}\right)^+(g_1(z),z)-\left(\zeta_i-\frac{\psi_1}{\epsilon}\right)^+(g_2(z),z)\end{aligned}$$

which goes to zero when $\epsilon\to0$. So from (2.1) we get $T(\zeta_i)\leq0$ for $i=1,2$. This leads to:

$$T(\zeta_R)\leq T(\zeta)\leq T(\zeta_R). \tag{2.2}$$

Moreover we have:

$$\begin{aligned}T(\zeta_R)=M&\int_{\Omega_{-R-1,-R}}\left(\mathcal{B}(z,\nabla\psi_1)-\mathcal{B}(z,\nabla\psi_0)\right).e_x+M\int_{\Omega_{-R-1,-R}}(\gamma_1-\gamma_0)\\ &-M\int_{\Omega_{R,R+1}}\left(\mathcal{B}(z,\nabla\psi_1)-\mathcal{B}(z,\nabla\psi_0)\right).e_x+M\int_{\Omega_{R,R+1}}(\gamma_1-\gamma_0).\end{aligned}$$

Using Theorem 1.2 and the fact that we have either $v^1_{+\infty}\equiv v^2_{+\infty}$ or $v^1_{+\infty}<v^2_{+\infty}$ in $(-h,0)$, we deduce (see [6], [4]) that $\lim_{R\to+\infty}T(\zeta_R)=0$ which leads by (2.2) to $T(\zeta)=0$.

Proof of Theorem 2.1. let us denote by D the domain $[\psi_1<0]$ (see Corollary 1.1). First we remark from Lemma 2.1 that (ψ_0,γ_0) is also a solution of $(P(Q_s,Q_f))$, that ψ_0 and ψ_1 are $\mathcal{B}$-Harmonic in D and that $\mathcal{B}(z,\nabla\psi_0).\nu=\mathcal{B}(z,\nabla\psi_1).\nu$ on $(a,+\infty)\times\{0\}$.

Since we have $\psi_0\leq\psi_1$ in D, $\psi_0=\psi_1$ on $(a,+\infty)\times\{0\}$ and $\psi_0,\ \psi_1\in C^1(D\cup(a,+\infty)\times\{0\})$, it is enough according to Lemmas A1, A2, to prove that $\nabla\psi_1$ does not vanish on some part Γ_0 of $(a,+\infty)\times\{0\}$. Assume that for some $x_1>a$ and $0<r<\frac{x_1-a}{2}$ we have $\nabla\psi_1(x,0)=0\quad\forall x\in(x_1-r,x_1+r)$. If we extend ψ_1 by $-Q_f$

and $\mathcal{B}(z,\xi)$ by $\mathcal{B}(0,\xi)$ to $B((x_1,0),r)\setminus\overline{D}$, it is clear that $\psi_1 \in W^{1,q}(B((x_1,0),r))$ and is $\mathcal{B}$-Harmonic in $B((x_1,0),r)$. Since the zeros of the gradient of a nonconstant $\mathcal{B}$-Harmonic function are isolated (see [2]) and $\nabla\psi_1 = 0$ in $B((x_1,0),r)\cap[z>0]$, we deduce that $\psi_1 = -Q_{f1}$ in $B((x_1,0),r)$. By the monotonicity of ψ_1 this leads to $\psi_1 = -Q_{f1}$ in the strip $[x_1,+\infty)\times(-r,0)$ which contradicts the asymptotic behavior of ψ_1 at $+\infty$. So there exists $x_1' \in (x_1-r,x_1+r)$ such that $\nabla\psi_1(x_1',0)\neq 0$. Since $\psi_1 \in C^1(D\cup(a,+\infty)\times\{0\})$ there exists $r'>0$ such that $\nabla\psi_1(x,0)\neq 0$ $\forall x \in (x_1'-r',x_1'+r')$. Thus we get $\psi_0=\psi_1$ in D. In particular $\psi_0=\psi_1$ in $[\psi_0<0]$. Similarly one can prove that $\psi_0=\psi_1$ in $[\psi_0>0]$ and then by continuity $\psi_0=\psi_1$ in Ω. Using Corollary 1.1 we deduce that $\gamma_0=\gamma_1$ in Ω.

As a direct consequence of Theorem 2.1, we have:

Corollary 2.1 *The solution of problem (P) is unique.*

According to Theorem 2.1 the solution of (P) decreases with respect to Q_f and increases with respect to Q_s. Intuitively one would expect that as $Q_f\to 0$ (resp. $Q_s\to 0$) the aquifer would be saturated by salt (resp. fresh) water only. More precisely we have:

Theorem 2.2 *Let $(\psi_{Q_f},\gamma_{Q_f})$ be the solution of $(P(Q_s,Q_f))$. Then we have when $Q_f\to 0$:*

$$(\psi_{Q_f},\gamma_{Q_f}) \longrightarrow (v_{-\infty},\gamma_s) \qquad in \quad W^{1,q}_{loc}(\Omega)\times L^{q'}_{loc}(\Omega).$$

Proof. Using the monotonicity of ψ_{Q_f} and γ_{Q_f} with respect to Q_f and the fact that the two functions are uniformly bounded we deduce by Beppo- Levi's theorem that there exists two functions ψ, γ such that:

$$\begin{aligned} \psi_{Q_f} &\longrightarrow \psi && \text{in } L^r_{loc}(\Omega) \quad \text{and a.e. in } \Omega \\ \gamma_{Q_f} &\longrightarrow \gamma && \text{in } L^r_{loc}(\Omega) \quad \forall r\geq 1. \end{aligned}$$

Now let $m>a$ and $\eta\in W^{1,\infty}(\mathbb{R})$ such that $0\leq\eta\leq 1$, $\eta=1$ in $(-m,m)$, $\eta=0$ for $|x|\geq m+1$ and $|\eta'|\leq 1$. Let $\Phi(x,z)=\phi_0(x)+\phi_1(z)(Q_s-\phi_0(x))$. Then $\eta^q(\psi_{Q_f}-\Phi)$ is a test function for (P) and we get:

$$\begin{aligned} \int_{\Omega_{m+1}} \eta^q|b(z)\nabla\psi_{Q_f}|^q &= \int_{\Omega_{m+1}} \eta^q|b(z)\nabla\psi_{Q_f}|^{q-2}b(z)\nabla\psi_{Q_f}.b(z)\nabla\Phi \\ &- \int_{\Omega_{m+1}} q\eta^{q-1}(\psi_{Q_f}-\Phi)|b(z)\nabla\psi_{Q_f}|^{q-2}b(z)\nabla\psi_{Q_f}.b(z)\nabla\eta \\ &- \int_{\Omega_{m+1}} \gamma_{Q_f}\eta^q\partial_x\psi_{Q_f} + \int_{\Omega_{m+1}} \gamma_{Q_f}\eta^q\partial_x\Phi - \int_{\Omega_{m+1}} \gamma_{Q_f}(\psi_{Q_f}-\Phi)q\eta^{q-1}\eta'. \end{aligned}$$

Since $\psi_{Q_f},\Phi,\nabla\Phi,\eta,\eta',\gamma_{Q_f}$ are uniformly bounded, we deduce by using Hölder's inequality:

$$|\psi_{Q_f}|_{1,q,\Omega_m}\leq c_m$$

where c_m is a constant depending on m. Then by using a diagonal process we get, up to a subsequence:

$$\psi_{Q_f} \rightharpoonup \psi \qquad \text{in } W^{1,q}_{loc}(\Omega).$$

Let us show that this convergence holds strongly. Let ω be an open bounded subset of Ω such that $\omega \subset\subset \Omega_{m_0}$ with $m_0 > a$. Let $\rho \in \mathcal{D}(\Omega)$, $\rho = 1$ in ω and $\rho = 0$ in $\Omega \setminus \Omega_{m_0}$. Then $\rho^q(\psi_{Q_f} - \psi)$ is a test function for (P) and we have:

$$\begin{aligned}\int_{\Omega_{m_0}} \rho^q |b(z)\nabla\psi_{Q_f}|^q &= \int_{\Omega_{m_0}} \eta^q |b(z)\nabla\psi_{Q_f}|^{q-2} b(z)\nabla\psi_{Q_f}.b(z)\nabla\psi \\ &- \int_{\Omega_{m_0}} q\rho^{q-1}(\psi_{Q_f} - \psi)|b(z)\nabla\psi_{Q_f}|^{q-2} b(z)\nabla\psi_{Q_f}.b(z)\nabla\rho \\ &- \int_{\Omega_{m_0}} \rho^q \gamma_{Q_f} \partial_x(\psi_{Q_f} - \psi) - \int_{\Omega_{m_0}} \gamma_{Q_f}(\psi_{Q_f} - \psi) q\rho^{q-1}\partial_x\rho.\end{aligned}$$

Applying Hölder's inequality and letting $Q_f \to 0$, we get:

$$\limsup_{Q_f\to 0} \Big(\int_{\Omega_{m_0}} \rho^q |b(z)\nabla\psi_{Q_f}|^q \Big)^{1/q} \leq \Big(\int_{\Omega_{m_0}} \rho^q |b(z)\nabla\psi|^q \Big)^{1/q}.$$

Hence $\rho^q b(z)\nabla\psi_{Q_f} \to \rho^q b(z)\nabla\psi$ in $L^q(\Omega_{m_0})$ and in particular

$$\nabla\psi_{Q_f} \longrightarrow \nabla\psi \quad \text{in } W^{1,q}_{loc}(\Omega).$$

Now using the monotonicity with respect to Q_f and the continuity of ψ_{Q_f}, it follows by Dini's theorem:

$$\lim_{Q_f\to 0} \Big(\lim_{R\to\pm\infty} \psi_{Q_f}(x+R, z) \Big) = \lim_{R\to\pm\infty} \Big(\lim_{Q_f\to 0} \psi_{Q_f}(x+R, z) \Big)$$

and

$$\lim_{R\to\pm\infty} \psi(x+R, z) = \lim_{Q_f\to 0} v^{Q_f}_{\pm\infty}(z) = v_{-\infty}(z) \qquad \text{for } (x,z) \in \Omega_{0,1}.$$

Since $\partial_x\psi \leq 0$, we deduce that $\psi(x,z) = v_{-\infty}(z) = Q_s\phi_1(z)$. Moreover $\gamma \in H(\psi)$ and $\psi > 0$ in Ω, so $\gamma = \gamma_s$ a.e. in Ω.

We conclude that when $Q_f \to 0$, the aquifer is completely saturated by only salt water.

Theorem 2.3 *Let $(\psi_{Q_s}, \gamma_{Q_s})$ be the solution of $(P(Q_s, Q_f))$. Then we have when $Q_s \to 0$:*

$$(\psi_{Q_s}, \gamma_{Q_s}) \longrightarrow (\psi, \gamma_f) \qquad \text{in } W^{1,q}_{loc}(\Omega) \times L^{q'}_{loc}(\Omega).$$

where ψ is the solution of the following problem:

$$(P(Q_f)) \begin{cases} i) & div(\mathcal{B}(z, \nabla\psi)) = 0 \quad \text{in} \quad \mathcal{D}'(\Omega) \\ ii) & -Q_f < \psi < 0 \quad \text{in} \quad \Omega \\ iii) & \psi(x,-h) = 0, \quad \psi(x,0) = \phi_0(x) \quad \text{for all} \quad x \in I\!R \\ iv) & \lim_{x\to-\infty} \psi(x,z) = 0, \qquad \lim_{x\to+\infty} \psi(x,z) = Q_f(\phi_1(z) - 1). \end{cases}$$

Proof. Taking into account the monotonicity of ψ_{Q_s} and γ_{Q_s} with respect to Q_s and arguing as in the proof of Theorem 2.4, we deduce the existence of two functions $\bar{\psi}$ and $\bar{\gamma}$ such that:

$$\begin{aligned} \psi_{Q_s} &\longrightarrow \bar{\psi} \quad \text{in} \quad W^{1,q}_{loc}(\Omega) \\ \gamma_{Q_s} &\longrightarrow \bar{\gamma} \quad \text{in} \quad L^{q'}_{loc}(\Omega). \end{aligned}$$

It follows that $(\bar{\psi}, \bar{\gamma})$ satisfies:

$$\begin{cases} i) \quad div(\mathcal{B}(z, \nabla\bar{\psi})) = -\partial_x \bar{\gamma} \quad \text{in} \quad \mathcal{D}'(\Omega) \\ ii) \quad \bar{\gamma} \in H(\bar{\psi}) \\ iii) \quad -Q_f \leq \bar{\psi} \leq 0 \quad \text{in} \quad \Omega \\ iv) \quad \bar{\psi}(x, -h) = 0, \quad \bar{\psi}(x, 0) = \phi_0(x) \quad \text{for all} \ x \in I\!R \\ v) \quad \lim_{x \to -\infty} \bar{\psi}(x, z) = 0, \quad \lim_{x \to +\infty} \bar{\psi}(x, z) = Q_f(\phi_1(z) - 1) \\ vi) \quad \partial_x \bar{\psi} \leq 0 \quad \text{and} \quad \partial_x \bar{\gamma} \leq 0 \quad \text{in} \ \mathcal{D}'(\Omega). \end{cases}$$

By the weak maximum principle, we can compare $\bar{\psi}$ with ψ the solution of $P(Q_f)$. This gives $\bar{\psi} \leq \psi$ in Ω. Since ψ satisfies by the strong maximum principle $-Q_f < \psi < 0$, we deduce that $\bar{\psi} < 0$ in Ω and then $\bar{\gamma} = \gamma_f$ a.e in Ω. Consequently $\bar{\psi} = \psi$ in Ω. In this case, the aquifer is completely saturated by fresh water when $Q_s \to 0$.

The end of this section is devoted to study the set $\Omega_f = [\psi < 0]$. Actually, in [1] it was proved in the linear case that this set is star shaped with the origin. The proof is based on blow-up arguments. Here we propose a different proof based on comparison arguments. This method works for the linear case as well as for the nonlinear one with constant permeability. So we assume that $\mathcal{B}$ does not depend on z.

For any $r > 0$, we consider:

$$\psi_r(x, z) = \frac{1}{r}\psi(rx, rz), \qquad \gamma_r(x, z) = \frac{1}{r}\gamma(rx, rz)$$

defined on

$$\Omega_r = I\!R \times (\frac{-h}{r}, 0).$$

Note that (ψ_r, γ_r) is the solution of the following problem:

$$
(P_r)\begin{cases}
\text{Find } (\psi_r,\gamma_r)\in W^{1,q}_{loc}(\Omega_r)\times L^\infty(\Omega_r) \text{ such that:}\\
i)\quad \int_{\Omega_r}|b(X)\nabla\psi_r|^{q-2}b(X)\nabla\psi_r.b(X)\nabla\zeta+\gamma_r e_x.\nabla\zeta=0\\
\qquad \forall\zeta\in W^{1,q}_0(\Omega_r) \text{ with compact support in } \overline{\Omega}_r\\
ii)\quad \gamma_r\in H(\psi_r)\\
iii)\quad \psi_r(x,\frac{-h}{r})=\frac{Q_s}{r},\qquad \psi_r(x,0)=\frac{1}{r}\phi_0(rx)\quad \text{for all } x\in I\!R\\
iv)\quad \partial_x\psi_r\le 0 \quad\text{and}\quad \partial_x\gamma_r\le 0 \quad\text{in}\quad \mathcal{D}'(\Omega_r).
\end{cases}
$$

From the study of problem (P), we know that problem (P_r) has a unique solution with a continuous free boundary g_r and that $\lim_{x\to-\infty}\psi_r(x,z)=\frac{1}{r}v_{-\infty}(rz)$, $\lim_{x\to+\infty}\psi_r(x,z)=\frac{1}{r}v_{+\infty}(rz)$. Moreover since we assume that $\mathcal{B}$ does not depend on z, we have $v_{-\infty}(z)=-\frac{Q_s}{h}z$ and $v_{+\infty}(z)=-Q_f-\frac{Q_f+Q_s}{h}z$.

Theorem 2.4 *For* $0<r_1<r_2$*, we have:*

$$\psi_{r_1}\le\psi_{r_2}\qquad \textit{in } \Omega_{r_2}\subset\Omega_{r_1}.$$

Proof. We remark that $(\psi_{r_1},\gamma_{r_1})$ and $(\psi_{r_2},\gamma_{r_2})$ satisfy the same equation on Ω_{r_2}. Moreover one can check that $\psi_{r_1}\le\psi_{r_2}$ on $\partial\Omega_{r_2}$ and $\lim_{x\to\pm\infty}\psi_{r_1}(x,z)\le\lim_{x\to\pm\infty}\psi_{r_2}(x,z)$. Then we can derive a similar identity as in Lemma 2.1. Since $\psi_{r_1}(x,0)=\psi_0(x,0)=-\frac{Q_f}{r_1}$ where $\psi_0=\min(\psi_{r_1},\psi_{r_2})$, one can argue as in the proof of Theorem 2.1 to see that $\psi_{r_1}=\psi_0$ in $[\psi_0<0]$.

To prove that $\psi_{r_1}=\psi_0$ in $[\psi_0>0]$ it is enough to verify that $\nabla\psi_{r_1}$ does not vanish on some part of the left hand side of the line $[z=-\frac{h}{r_2}]$. So assume that for some x_0, we have $\nabla\psi_{r_1}(x_0,-\frac{h}{r_2})=0\ \forall x\le x_0$. Then $\psi_{r_1}(x_0,-\frac{h}{r_2})=C\in I\!R\quad\forall x\le x_0$. By the asymptotic behavior $C=\frac{Q_s}{r_2}>0$. So by continuity and monotonicity, ψ_{r_1} is positive in a strip $D_\epsilon=(-\infty,x_0)\times(-\frac{h}{r_2}-\epsilon,-\frac{h}{r_2}+\epsilon)$ for some small $\epsilon>0$. Thus ψ_{r_1} is $\mathcal{B}$-Harmonic in D_ϵ and its gradient has non-isolated zeros, so $\psi_{r_1}=C$ in D_ϵ which contradicts the asymptotic behavior.

Corollary 2.2 *Ω_f is star shaped with the origin.*

Proof. Let $X_0\in\Omega_f$ and $t\in(0,1]$. We have by Theorem 2.4, $\psi_t(X_0)\le\psi_1(X_0)=\psi(X_0)<0$. So $\psi(tX_0)<0$, which means that $tX_0\in\Omega_f$.

Corollary 2.3 *i) There exists* $z_0 \in (-h^*, 0)$ *such that* $g(z) \geq 0 \quad \forall z \in (-h^*, z_0)$.
ii) g is non increasing where it is nonnegative.

Proof. *i)* First note that the set $[g > 0]$ is nonempty. Indeed if $g(z) \leq 0 \; \forall z \in (-h^*, 0)$, then for any $(x, z) \in (0, +\infty) \times (-h^*, 0)$ we have $\psi(x, z) < 0$ which contradicts the fact that $\psi(x, z) > 0$ in $(0, +\infty) \times (-h, -h^*)$. Let $z_0 = \inf\{ z \in (-h^*, 0) \,/\, g(z) > 0\}$ and take $z \in (-h^*, z_0)$. Since $r = \frac{z_0}{z} < 1$, we have:

$$0 = \psi(g(z_0), z_0) = \frac{1}{r}\psi\Big(r\Big(\frac{g(z_0)}{r}\Big), rz\Big) = \psi_r\Big(\frac{g(z_0)}{r}, z\Big)$$
$$\leq \psi_1\Big(\frac{g(z_0)}{r}, z\Big) = \psi\Big(\frac{g(z_0)}{r}, z\Big) \leq \psi(g(z_0), z)$$

since $g(z_0) \geq 0$ and $\frac{1}{r} > 1$. Thus $\psi(g(z_0), z) \geq 0$ and then $g(z) \geq g(z_0) \geq 0$.

ii) Let $z_1, z_2 \in (-h^*, z_0)$ such that $z_1 < z_2$. Since $g(z_2) \geq 0$, we can argue as in *i)* and obtain $g(z_1) \geq g(z_2)$.

3 Behavior of the free boundary near $z = -h^*$ and $z = 0$

In [6] we proved for the linear case ($q = 2$) that the free boundary has the line $[z = -h^*]$ as an asymptote. Here we generalize the proof to the nonlinear case. Before this, we give a second proof which is too much simpler but works only when the permeability is constant.

Theorem 3.1 *i) The set* $S = \{x \in I\!R \,/\, \psi(x, -h^*) = 0\}$ *is empty and* $\Gamma = [x = g(z)]$.
ii) $\lim_{z \to -h^*} g(z) = +\infty$.

$1^{st}Proof$: Case of constant permeability.

ii) Since g is non increasing in $(-h^*, z_0)$ (see Corollary 2.3), there exists a limit L for g as $z \to -h^*$. Assume that L is finite. By the monotonicity of g we get $g(z) \leq L \; \forall z \in (-h^*, z_0)$ and then:

$$\forall x > L, \quad \forall z \in (-h^*, z_0) \quad \psi(x, z) < 0.$$

Since $\psi > 0$ for $z < -h^*$ we deduce by continuity that we have necessarily $\psi(x, -h^*) = 0 \; \forall x > L$. This leads to a contradiction by Lemma 5.1 of [4]. Thus $L = +\infty$.

i) Assume that $S \neq \emptyset$. Then there exists $x_0 \in I\!R$ such that $\psi(x_0, -h^*) = 0$. For $A > x_0$ there exists $\delta > 0$ by *ii)* such that: $\forall z \in (-h^*, -h^* + \delta)$, $g(z) > A$. So for $(x, z) \in (-\infty, A) \times (-h^*, -h^* + \delta)$ we have $\psi(x, z) > 0$. By monotonicity of ψ, we deduce that $\psi(x,-h^*) = 0\ \forall x \geq x_0$ since one has $v_{+\infty}(-h^*) = 0 \leq \psi(x,-h^*) \leq \psi(x_0,-h^*) = 0$. Hence we have $\psi > 0$ in $(x_0, A) \times \big((-h^*, -h^* + \delta) \setminus \{-h^*\}\big)$ and $\psi(x,-h^*) = 0\ \forall x \in (x_0, A)$ which contradicts Lemma 5.1 of [4].

$2^{nd}Proof$: The general case

Since we follow the proof given in [6] for the linear case, we will only give an outline of it.

i) Assume that $S \neq \emptyset$. Then $S = [\alpha, +\infty)$ with $\alpha = \inf S > -\infty$. We need some lemmas.

Lemma 3.1 *There exists $u \in L^\infty(-h,0)$ such that $u > 0$ for a.e $z \in (-h,0)$ and:*

$$E(u(z)) = \big(B_{22}u^2(z) - 2B_{12}u(z) + B_{11}\big)^{\frac{q-2}{2}}\big(B_{22}u(z) - B_{12}\big) - C_0 = 0$$

for some constant $C_0 > -\big(B_{11}^{\frac{q-2}{2}} B_{12}\big)(z)$ for a.e $z \in (-h,0)$.

Proof. The function $E : I\!R \longrightarrow I\!R$ is continuous and satisfies $\lim_{t\to+\infty} E(t) = +\infty$, $E(0) = -B_{11}^{\frac{q-2}{2}} B_{12} - C_0 < 0$. So we deduce that for a.e $z \in (-h,0)$, there exists $u(z) > 0$ such that $E(u(z)) = 0$. This clearly defines a positive and uniformly bounded function u on $(-h,0)$.
Set $\alpha' = \alpha + \frac{1}{2}$ and define $f(z)$ by:

$$f(z) = \int_{-h}^{z} u(s)ds + \alpha'.$$

For $k > 0$ define v and θ by:

$$\begin{cases} v(x,z) = \big(k(\gamma_s - \gamma_f)\big)^{\frac{1}{q-1}}(f(z) - x)^+ \\ \theta(x,z) = \gamma_s\chi([x < f(z)]) + \gamma_f\chi([x > f(z)]) \end{cases}$$

for $(x,z) \in D(z_1) = (\alpha', +\infty) \times (-h^*, z_1)$ with $z_1 \in (-h^*, 0)$. Then we have:

Lemma 3.2 *There exists $k > 0$ such that:*

$$\int_{D(z_1)} \big(\mathcal{B}(z,\nabla v) + \theta e_x\big)\nabla\xi \geq 0 \qquad \forall \xi \in \mathcal{D}(D(z_1)).$$

Proof. See [6] Lemma 6.3.

Lemma 3.3 *Let (ψ,γ) be the solution of (P). Then there exists $z_0 \in (-h^*,0)$ such that:*

$$\int_{D(z_0)} \big(\mathcal{B}(z,\nabla\psi^+) - \mathcal{B}(z,\nabla v_0) + (\gamma - \theta_0)e_x\big)\nabla\zeta = 0 \qquad \forall \zeta \in \mathcal{D}(I\!R)$$

where $v_0 = \min(\psi^+, v)$ and $\theta_0 = \min(\gamma, \theta)$.

Proof. See [6] Lemma 6.4.

Now let $z_* \in (-h^*, z_0)$.

If $\psi^+(\alpha', z_*) = 0$, then by monotonicity we have $\psi^+(x, z_*) = 0 \ \forall x \geq \alpha'$.

If $\psi^+(\alpha', z_*) > 0$, then since we also have $\psi^+(\alpha', z_*) < v(\alpha', z_*)$ we deduce by continuity that there exists a small ball $B_r(\alpha', z_*)$ in which one has $\psi^+ > 0$ and $\psi^+ < v$. Then $\psi^+ = v_0$ in $B_r(\alpha', z_*)$. Let us denote by C_* the connected component of $D(z_0) \cap [\psi^+ > 0]$ which contains $B_r(\alpha', z_*) \cap D(z_0)$.

In $C_* \cap D^+(z_0)$ ($D^+(z_0) = D(z_0) \cap [x < f(z)]$), we have $\psi^+ = \psi > 0$ and then $div\big(\mathcal{B}(z, \nabla\psi^+)\big) = 0$. By Lemma 3.3 we have:

$$div\big(\mathcal{B}(z, \nabla\psi^+) - \mathcal{B}(z, \nabla v_0)\big) + (\gamma - \theta_0)_x = 0 \quad \text{in } \mathcal{D}'(C_* \cap D^+(z_0)).$$

But since in $C_* \cap D^+(z_0)$ we have $\psi^+ > 0$, $v > 0$, it follows that $\gamma = \theta = \theta_0 = \gamma_s$ and then:

$$div\big(\mathcal{B}(z, \nabla v_0)\big) = 0 \quad \text{in } \mathcal{D}'(C_* \cap D^+(z_0)).$$

So we have:

$$\begin{cases} div\big(\mathcal{B}(z, \nabla\psi^+)\big) = 0, & div\big(\mathcal{B}(z, \nabla v_0)\big) = 0 \quad \text{in } \mathcal{D}'(C_* \cap D^+(z_0)) \\ \psi^+ \geq v_0 & \text{in } C_* \cap D^+(z_0) \\ \psi^+ = v_0 & \text{in } B_r(\alpha', z_*) \cap D^+(z_0) \end{cases}$$

which leads by Lemma A.1 (see Appendix) to $\psi^+ = v_0$ in $C_* \cap D^+(z_0)$. We then conclude as in the end of the proof of Theorem 6.1 in [6].

ii) See Corollary 6.6 of [6].

Now we study the free boundary near $z = 0$. We first prove the existence of a limit $g(0-) \leq 0$ as z goes to zero. When $q = 2$ and b_{21} is non increasing we prove that $g(0-)$ is finite. The same result is established when $\mathcal{B}(z, \xi) = |\xi|^{q-2}\xi$. Moreover when $\mathcal{B}(z, \xi) = \xi$ we prove that $g(0-) < 0$.

Theorem 3.2 *g admits a limit when z goes to zero.*

Proof. It suffices to show that $\liminf_{z\to 0} g(z) = \limsup_{z\to 0} g(z)$.
First if $\liminf_{z\to 0} g(z) = -\infty$, then $\liminf_{z\to 0} g(z) = \limsup_{z\to 0} g(z) = -\infty$ and $\lim_{z\to 0} g(z) = -\infty$.
Assume that $\limsup_{z\to 0} g(z) = a^+ > -\infty$. Note that $a^+ \leq 0$. Indeed if $a^+ > 0$, then $\psi(a^+, 0) < 0$ and by continuity of ψ there exists $\epsilon > 0$ such that $\psi < 0$ in $(a^+ - \epsilon, a^+ + \epsilon) \times (-\epsilon, 0)$. So $g(z) \leq a^+ - \epsilon \quad \forall z \in (-\epsilon, 0)$ and then $a^+ \leq a^+ - \epsilon$ which is impossible.

Set $a^- = \liminf_{z\to 0} g(z)$ and assume that $a^- < a^+$. Let $x_1 \in (a^-, a^+)$, $x_2 \in (x_1, a^+)$ and let $(z_n)_n$ be a sequence that satisfies $\lim_{n\to+\infty} z_n = 0$ and $\lim_{n\to+\infty} g(z_n) = a^-$. So there exists $n_1 \in \mathbb{N}$ such that $g(z_n) \le x_1 \quad \forall n \ge n_1$ and then:

$$\psi^+(x, z_n) = 0 \qquad \forall n \ge n_1. \tag{3.1}$$

Arguing as in Lemma 3.1, we prove the existence of a negative function $u \in L^\infty(-h, 0)$ such that for a.e $z \in (-h, 0)$:

$$E(u(z)) = \left(B_{22}u^2(z) - 2B_{12}u(z) + B_{11}\right)^{\frac{q-2}{2}}\left(B_{22}u(z) - B_{12}\right) - C_1 = 0 \tag{3.2}$$

for some constant $C_1 < -\left(B_{11}^{\frac{q-2}{2}} B_{12}\right)(z)$ for a.e $z \in (-h, 0)$. We then set

$$f(z) = \int_0^z u(s)ds + x_1' \qquad \text{for } x_1' \in (x_1, x_2).$$

Since f is continuous and nonincreasing there exists $n_2 \in \mathbb{N}$ such that:

$$f(0) = x_1' < f(z) < x_2 \qquad \forall z \in (z_n, 0) \quad \forall n \ge n_2. \tag{3.3}$$

Now for $k > 0$ we define v and θ by:

$$\begin{cases} v(x,z) = \left(k(\gamma_s - \gamma_f)\right)^{\frac{1}{q-1}}(f(z) - x)^+ \\ \theta(x,z) = \gamma_s\chi([x < f(z)]) + \gamma_f\chi([x > f(z)]) \end{cases} \tag{3.4}$$

for $(x, z) \in D(z_n) = (x_1, +\infty) \times (z_n, 0)$ with $n \ge n_2$. Then as in Lemma 3.2, one can deduce from (3.2) and(3.4) the existence of $k > 0$ such that:

$$\int_{D(z_n)} \left(\mathcal{B}(z, \nabla v) + \theta e_x\right)\nabla\xi \ge 0 \qquad \forall \xi \in \mathcal{D}(D(z_n)).$$

Using the fact that $\psi^+(x_1, 0) = 0$, the continuity of ψ, the monotonicity of f, (3.1) and (3.3), there exists $n \ge \sup(n_1, n_2)$ such that: $\psi^+(x_1, z) \le v(x_1, z) \quad \forall z \in (z_n, 0)$. Moreover one can check that:

$$\psi^+ \le v \quad \text{on } \partial D(z_n) \qquad \text{and} \qquad \lim_{x\to+\infty} \psi^+(x,z) = 0 = \lim_{x\to+\infty} v(x,z).$$

Arguing as in the proof of Theorem 3.1, we obtain $\psi^+ \le v$ in $D(z_n)$ from which we deduce that $\psi^+(x, z) = 0 \quad \forall (x, z) \in (x_2, a^+) \times (z_n, 0)$ and then $g(z) \le x_2 \quad \forall z \in (z_n, 0)$. So $a^+ = \limsup_{z\to 0} g(z) \le x_2 < a^+$ and we get a contradiction. Thus we have $a^+ = a^-$ and $\lim_{z\to 0} g(z)$ exists.

Theorem 3.3 *If $q = 2$ and $b_{21}' \le 0$ in $\mathcal{D}'(-h, 0)$, then* $\lim_{z\to 0} g(z) > -\infty$.

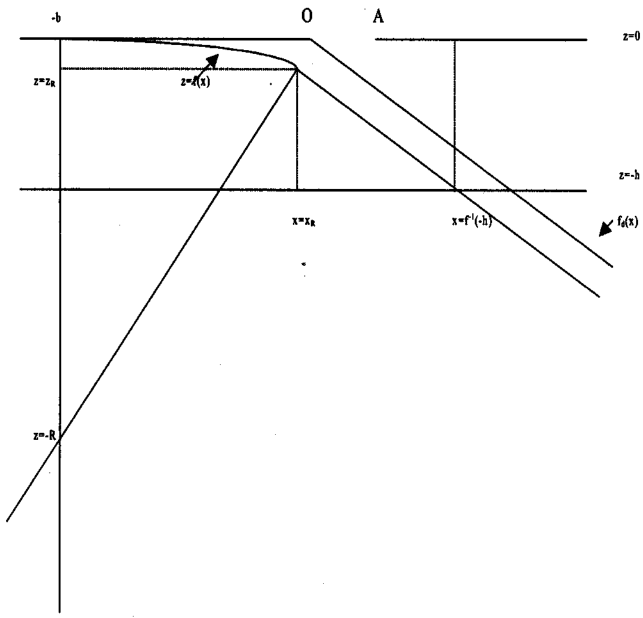

Figure 2:

Proof. The idea is to compare ψ with a suitable function. Let $R, b > 0$ and consider:

$$f(x) = \begin{cases} -R + \sqrt{R^2 - (x+b)^{+2}} & \text{if } x \leq x_R = \frac{R(a+b)}{\sqrt{R^2+(a+b)^2}} - b \\ z_R - \frac{a+b}{R}(x - x_R) & \text{if } x \geq x_R \end{cases}$$

with $z_R = \frac{R(a+b)}{\sqrt{R^2+(a+b)^2}} - R$ (see Figure 2). Set:

$$\omega(z) = h \frac{\int_0^z \frac{ds}{b_{22}(s)}}{\int_{-h}^0 \frac{ds}{b_{22}(s)}} = \kappa \int_0^z \frac{ds}{b_{22}(s)} \quad \text{for } z \in (-h, 0).$$

Consider for $t > 0$, $G(t) = \lambda[(t+1)^2 - 1]$ with $\lambda = \frac{Q_s}{h^2+2h}$ and define v_1 by;

$v_1(x,z) = G(f(x) - \omega(z))$ for $(x,z) \in D_1 = \{x,z) \in \Omega / -h < z < \omega^{-1} o f(x)\}$.

For $d > 0$ set $f_d(x) = f(x) + d$ and consider for $t > 0$, $K(t) = \mu Log(1+t)$ with $\mu = \frac{Q_f}{Log(1+d)}$. Then we define v_2 by:

$$v_2(x,z) = -K(\omega(z) - f(x)) \quad \text{for } (x,z) \in D_2$$
$$\text{with } D_2 = \{x,z) \in \Omega / \omega^{-1} o f(x) < z < \omega^{-1} o f_d(x)\}.$$

Now set:

$$v = \chi(D_1)v_1 + \chi(D_2)v_2 \qquad \text{and} \qquad \theta = \chi(D_1)\gamma_s + \chi(D_1)\gamma_f.$$

We will compare (ψ,γ) with (v,θ) on the domaine $D = (\overline{D_1 \cup D_2}) \cap \Omega$. Remark that $v \in H^1_{loc}(D)$ since $v_1(x, \omega^{-1} o f(x)) = G(0) = 0$ and $v_2(x, \omega^{-1} o f(x)) = -K(0) = 0$.

1^{st} *step* There exists $\alpha_1 > 0$ such that:

$$\forall \alpha \in (0, \alpha_1), \quad \forall R \geq \max\Big(\frac{2M^2}{\kappa^2}, \frac{a}{\alpha}\Big), \quad div(b(z)\nabla v_1) \geq 0 \quad \text{in } \mathcal{D}'(D_1). \tag{3.5}$$

Indeed let C_1 be a constant such that $b_{21}(z) + C_1 > 0$ for a.e $z \in (-h, 0)$. Then we have:

$$\begin{aligned}
div(b(z)\nabla v_1) &= \frac{\partial}{\partial x}\Big(b_{11}\frac{\partial v_1}{\partial x} + b_{12}\frac{\partial v_1}{\partial z}\Big) + \frac{\partial}{\partial z}\Big(- C_1\frac{\partial v_1}{\partial x} + b_{22}\frac{\partial v_1}{\partial z}\Big) \\
&+ \frac{\partial}{\partial z}\Big((b_{21} + C_1)\frac{\partial v_1}{\partial x}\Big) \\
&= b_{11}(G''(f(x) - \omega(z))(f')^2 + G'(f(x) - \omega(z))f'') \\
&+ (b_{12} - C_1)\Big(\frac{-\kappa}{b_{22}} f'G''(f(x) - \omega(z)) \\
&+ \frac{\kappa^2}{b_{22}}G''(f(x) - \omega(z)) - \frac{\kappa}{b_{22}}(b_{21} + C_1)G''(f(x) - \omega(z))f' \\
&+ b'_{21}G'(f(x) - \omega(z))f' \\
&\geq 2\lambda b_{11}\Big((f')^2 + \frac{\kappa^2}{b_{22}b_{11}} - \frac{\kappa}{b_{22}b_{11}}(b_{12} - C_1)f' + (f(x) - \omega(z) + 1)f''\Big) \\
&= 2\lambda b_{11} I_1
\end{aligned} \tag{3.6}$$

since $b_{21} + C_1 > 0$, $G'' = 2\lambda$, $f'(x) \leq 0$, $b'_{21} \leq 0$ in $\mathcal{D}'(-h,0)$ and $G'(t) = 2\lambda(t+1) > 0$ for $t > 0$. We shall distinguish three cases:

. For $x \leq -b$, we have $f(x) = 0$ and then $I_1 = \frac{\kappa^2}{b_{22}b_{11}} > 0$.

. For $x \in (x_R, f^{-1}(-h))$, we have $f''(x) = 0$, $f'(x) = -\frac{a+b}{R} = -\alpha$, $m \le b_{11}, b_{22} \le M$, $|b_{12}| \le M$ and:

$$\begin{aligned} I_1 &= \alpha^2 + \frac{\kappa^2}{b_{22}b_{11}} + \frac{\alpha\kappa}{b_{22}b_{11}}(b_{12} - C_1) \\ &\ge \alpha^2 + \frac{\kappa^2}{M^2} - \frac{\alpha\kappa}{m^2}(M + C_1) \\ &\longrightarrow \frac{\kappa^2}{M^2} > 0 \qquad \text{as } \alpha \to 0. \end{aligned}$$

So there exists $\alpha_0 > 0$ such that: $I_1 \ge 0 \quad \forall \alpha \in (0, \alpha_0)$.

. For $x \in (-b, x_R)$, we have:

$$f'(x) = \frac{-(x+b)}{\sqrt{R^2 - (x+b)^2}}, \quad f''(x) = -\frac{R^2}{(R^2 - (x+b)^2)^{3/2}}$$

$$\text{and } 1 + f'^2(x) = \frac{R^2}{R^2 - (x+b)^2}.$$

Then:

$$I_1 = \frac{R^2}{(R^2 - (x+b)^2)^{3/2}}\Big(\sqrt{R^2 - (x+b)^2} - f(x) + \frac{(R^2 - (x+b)^2)^{3/2}}{R^2}\big(\frac{\kappa^2}{b_{22}b_{11}} - 1\big) + \frac{\kappa(b_{12} - C_1)}{b_{22}b_{11}}\frac{(R^2 - (x+b)^2)}{R^2}(x+b) + \omega(z) - 1\Big).$$

Note that:

$.\sqrt{R^2 - (x+b)^2} - f(x) = R$

$.\frac{(R^2 - (x+b)^2)^{3/2}}{R^2}\big(\frac{\kappa^2}{b_{22}b_{11}} - 1\big) \ge R\big(\frac{\kappa^2}{M^2} - 1\big)$ provided that $M > \kappa$

$.\frac{\kappa(b_{12} - C_1)}{b_{22}b_{11}} > -C_2$ for some positive constant C_2, so

$.\frac{\kappa(b_{12} - C_1)}{b_{22}b_{11}}\frac{(R^2 - (x+b)^2)}{R^2}(x+b) \ge -C_2\frac{\alpha R}{\sqrt{1+\alpha^2}} \ge -C_2(x_R + b)$

$.\omega(z) \ge -h.$

Then:

$$\begin{aligned} I_1 &\ge \frac{R^2}{(R^2 - (x+b)^2)^{3/2}}\Big(R\frac{\kappa^2}{M^2} - \frac{C_2\alpha}{\sqrt{1+\alpha^2}}R - h - 1\Big) \\ &\ge \frac{R^3}{(R^2 - (x+b)^2)^{3/2}}\Big(\big(\frac{\kappa^2}{2M^2} - C_2\alpha\big) + \big(\frac{\kappa^2}{2M^2} - \frac{h+1}{R}\big)\Big) \\ &\ge 0 \qquad \text{provided that } \alpha \le \frac{\kappa^2}{2M^2C_2} \text{ and } R > \frac{2M^2}{\kappa^2}(h+1). \end{aligned}$$

Finally for $\alpha \in (0, \alpha_1 = \min(\alpha_0, \frac{\kappa^2}{2M^2C_2}))$ and $R > \frac{2M^2}{\kappa^2}(h+1)$, one has $I_1 \geq 0$ and then by (3.6) we obtain $div(b(z)\nabla v_1) \geq 0 \quad \text{in } \mathcal{D}'(D_1)$.

2nd step There exists $\alpha_2 > 0$ such that:

$$\forall \alpha \in (0, \alpha_2), \quad \forall R \geq \max\Big(\frac{2M^2}{\kappa^2}, \frac{a}{\alpha}\Big), \quad div(b(z)\nabla v_2) \geq 0 \quad \text{in } \mathcal{D}'(D_2). \tag{3.7}$$

Indeed we have:

$$\begin{aligned}
div(b(z)\nabla v_2) &= \frac{\partial}{\partial x}\Big(b_{11}\frac{\partial v_2}{\partial x} + b_{12}\frac{\partial v_2}{\partial z}\Big) + \frac{\partial}{\partial z}\Big(- C_1\frac{\partial v_2}{\partial x} + b_{22}\frac{\partial v_2}{\partial z}\Big) \\
&+ \frac{\partial}{\partial z}\Big((b_{21} + C_1)\frac{\partial v_2}{\partial x}\Big) \\
&= b_{11}\big(- K''(\omega(z) - f(x))(f')^2 + K'(\omega(z) - f(x))f''\big) \\
&+ (b_{12} - C_1)\frac{\kappa}{b_{22}} f' K''(\omega(z) - f(x)) \\
&- \frac{\kappa^2}{b_{22}} K''(\omega(z) - f(x)) + \frac{\kappa}{b_{22}}(b_{21} + C_1)K''(\omega(z) - f(x))f' \qquad (3.8) \\
&+ b_{21}' K'(\omega(z) - f(x))f' \\
&\geq \frac{\mu b_{11}}{(1 + \omega(z) - f(x))^2}\Big((f')^2 + \frac{\kappa^2}{b_{22}b_{11}} - \frac{\kappa(b_{12} - C_1)}{b_{22}b_{11}} f' \\
&+ (1 + \omega(z) - f(x))f''\Big) \\
&= \frac{\mu b_{11}}{(1 + \omega(z) - f(x))^2} I_2
\end{aligned}$$

since $b_{21} + C_1 > 0$, $K''(t) = -\frac{\mu}{(1+t)^2} < 0$, $K'(t) = \frac{\mu}{1+t} > 0$, $f'(x) \leq 0$, $b_{21}' \leq 0$ in $\mathcal{D}'(-h, 0)$. We shall distinguish two cases:

. For $x \in (x_R, f_d^{-1}(-h))$, we have $f''(x) = 0$, $f'(x) = -\frac{a+b}{R} = -\alpha$ and then:

$$\begin{aligned}
I_2 &= \alpha^2 + \frac{\kappa^2}{b_{22}b_{11}} + \frac{\alpha\kappa}{b_{22}b_{11}}(b_{12} - C_1) \\
&\geq \alpha^2 + \frac{\kappa^2}{M^2} - \frac{\alpha\kappa}{m^2}(M + C_1) \\
&\longrightarrow \frac{\kappa^2}{M^2} > 0 \qquad \text{as } \alpha \to 0.
\end{aligned}$$

So there exists $\alpha_0 > 0$ such that: $I_2 \geq 0 \quad \forall \alpha \in (0, \alpha_0)$.

. For $x \in (-b, x_R)$, we have:

$$I_2 \;=\; \frac{R^2}{(R^2 - (x+b)^2)} + \frac{\kappa^2}{b_{22}b_{11}} - 1 + \frac{\kappa(b_{12} - C_1)}{b_{22}b_{11}} \frac{(x+b)}{\sqrt{R^2 - (x+b)^2}}$$

$$
\begin{aligned}
- & \frac{R^2}{(R^2-(x+b)^2)^{3/2}}(1+\omega(z)-f(x)) \\
= & \frac{R^2}{(R^2-(x+b)^2)^{3/2}}\Big(\sqrt{R^2-(x+b)^2}+\frac{(R^2-(x+b)^2)^{3/2}}{R^2}\big(\frac{\kappa^2}{b_{22}b_{11}}-1\big) \\
+ & \frac{\kappa(b_{12}-C_1)}{b_{22}b_{11}}(x+b)\frac{(R^2-(x+b)^2)}{R^2}-(1+\omega(z)-f(x))\Big).
\end{aligned}
$$

Taking into account the fact that $x \in (-b, x_R]$ and that the coefficients $b_{ij} \in L^\infty(\Omega)$, it follows that:

$$
\begin{aligned}
I_2 \geq & \frac{R^2}{(R^2-(x+b)^2)^{3/2}}\Big(-R+2\sqrt{R^2-(x_R+b)^2} \\
+ & R(\frac{\kappa^2}{M^2}-1)-C_2(x_R+b)-1\Big) \\
= & \frac{R^2}{(R^2-(x+b)^2)^{3/2}}\Big(-R+\frac{2R}{\sqrt{1+\alpha^2}}+R(\frac{\kappa^2}{M^2}-1)-C_2\frac{\alpha}{\sqrt{1+\alpha^2}}R-1\Big) \\
= & \frac{R^3}{(R^2-(x+b)^2)^{3/2}}\Big(\frac{2}{\sqrt{1+\alpha^2}}-2+(\frac{\kappa^2}{2M^2}-\frac{C_2\alpha}{\sqrt{1+\alpha^2}})+\frac{\kappa^2}{2M^2}-\frac{1}{R}\Big) \geq 0
\end{aligned}
$$

provided that $R > \frac{2M^2}{\kappa^2}$ and $\alpha \in (0, \alpha_0')$ for a small $\alpha_0' > 0$.
Finally for $\alpha \in (0, \alpha_2 = \min(\alpha_0, \alpha_0'))$, we obtain $div(b(z)\nabla v_2) \geq 0$ in $\mathcal{D}'(D_2)$.

<u>3^{rd} *step*</u> : There exists $\alpha_* > 0$ and $R_* > 0$ such that:

$$
\forall \alpha \in (0, \alpha_*), \quad \forall R \geq R_*, \quad div(b(z)\nabla v + \theta e_x) \geq 0 \quad \text{in } \mathcal{D}'(D). \tag{3.9}
$$

Indeed let $\xi \in \mathcal{D}(D)$, $\xi \geq 0$. We have:

$$
\begin{aligned}
I(\xi) &= \int_D (b(z)\nabla v + \theta e_x)\nabla\xi \\
&= \int_{D_1} (b(z)\nabla v_1 + \gamma_s e_x)\nabla\xi + \int_{D_2} (b(z)\nabla v_2 + \gamma_f e_x)\nabla\xi \\
&= < -div(b(z)\nabla v_1), \xi > + < -div(b(z)\nabla v_2), \xi > \\
&+ \int_{[z=\omega^{-1} o f(x)]\cap\Omega} (b(z)(\nabla v_1 - \nabla v_2).\nu + (\gamma_s - \gamma_f)\nu_x)\xi \\
&\leq \int_{[z=\omega^{-1} o f(x)]\cap\Omega} (b(z)(\nabla v_1 - \nabla v_2).\nu + (\gamma_s - \gamma_f)\nu_x)\xi
\end{aligned}
$$

if we choose $\alpha \in (0, \min(\alpha_1, \alpha_2))$ and $R > \frac{2M^2}{\kappa^2}(h+1)$. Moreover one has:

$$\nabla v_1(x,\omega^{-1}of(x)) = G'(0)^t\Big(f'(x), -\frac{\kappa}{b_{22}(\omega^{-1}of(x))}\Big)$$
$$= \frac{2Q_s}{h^2+2h}^t\Big(f'(x), -\frac{\kappa}{b_{22}(\omega^{-1}of(x))}\Big)$$
$$\nabla v_2(x,\omega^{-1}of(x)) = K'(0)^t\Big(f'(x), -\frac{\kappa}{b_{22}(\omega^{-1}of(x))}\Big)$$
$$= \frac{Q_f}{Log(1+d)}^t\Big(f'(x), -\frac{\kappa}{b_{22}(\omega^{-1}of(x))}\Big)$$
$$\nu(x,\omega^{-1}of(x)) = \frac{1}{\sqrt{1+(\omega^{-1}of)'^2(x)}}^t\big(-(\omega^{-1}of)'(x), 1\big)$$
$$= \frac{1}{\sqrt{\big(\frac{\kappa}{b_{22}(\omega^{-1}of(x))}\big)^2 + f'^2(x)}}^t\Big(-f'(x), \frac{\kappa}{b_{22}(\omega^{-1}of(x))}\Big).$$

Then:

$$J_1 = b(z)(\nabla v_1 - \nabla v_2).\nu + (\gamma_s - \gamma_f)\nu_x$$
$$= \frac{-1}{\sqrt{\big(\frac{\kappa}{b_{22}(\omega^{-1}of(x))}\big)^2 + f'^2(x)}}\Big(\frac{2Q_s}{h^2+2h} - \frac{Q_f}{Log(1+d)}\Big)$$
$$.b(z)^t\Big(f'(x), \frac{-\kappa}{b_{22}(\omega^{-1}of(x))}\Big)^t\Big(f'(x), \frac{-\kappa}{b_{22}(\omega^{-1}of(x))}\Big)$$
$$- \frac{f'(x)}{\sqrt{\big(\frac{\kappa}{b_{22}(\omega^{-1}of(x))}\big)^2 + f'^2(x)}}(\gamma_s - \gamma_f)$$
$$= \frac{-1}{\sqrt{\big(\frac{\kappa}{b_{22}(\omega^{-1}of(x))}\big)^2 + f'^2(x)}}\Big(\frac{Q_s}{h^2+2h} - \frac{Q_f}{Log(1+d)}\Big)$$
$$.b(z)^t\Big(f'(x), \frac{-\kappa}{b_{22}(\omega^{-1}of(x))}\Big)^t\Big(f'(x), \frac{-\kappa}{b_{22}(\omega^{-1}of(x))}\Big)$$
$$- \frac{1}{\sqrt{\big(\frac{\kappa}{b_{22}(\omega^{-1}of(x))}\big)^2 + f'^2(x)}}\Big(\frac{Q_s}{h^2+2h}b(z)^t\Big(f'(x), \frac{-\kappa}{b_{22}(\omega^{-1}of(x))}\Big)$$
$$.^t\Big(f'(x), \frac{-\kappa}{b_{22}(\omega^{-1}of(x))}\Big) + (\gamma_s - \gamma_f)f'(x)\Big).$$

By the coercivity of b, we have:

$$J_2 = -\frac{1}{\sqrt{\left(\frac{\kappa}{b_{22}(\omega^{-1}of(x))}\right)^2 + f'^2(x)}}\left(\frac{Q_s}{h^2+2h} b(z)^t\left(f'(x), \frac{-\kappa}{b_{22}(\omega^{-1}of(x))}\right)\right.$$

$$\left. .^t\left(f'(x), \frac{-\kappa}{b_{22}(\omega^{-1}of(x))}\right) + (\gamma_s - \gamma_f) f'(x)\right)$$

$$\leq \sqrt{\left(\frac{\kappa}{b_{22}(\omega^{-1}of(x))}\right)^2 + f'^2(x)}\left(\frac{-mQ_s}{h^2+2h} + (\gamma_s - \gamma_f)\frac{-f'(x)}{f'^2(x) + \left(\frac{\kappa}{b_{22}(\omega^{-1}of(x))}\right)^2}\right).$$

Note that $F(x) = \frac{-f'(x)}{f'^2(x) + \left(\frac{\kappa}{b_{22}(\omega^{-1}of(x))}\right)^2} \leq -\frac{M^2}{\kappa^2} f'(x)$ and then:

. If $x \in (x_R, f_d^{-1}(-h))$ we have $f'(x) = -\alpha$ and $F(x) \leq \frac{\alpha M^2}{\kappa^2}$.

. If $x \in (-b, x_R)$

$$F(x) \leq -\frac{M^2}{\kappa^2} f'(x) = \frac{M^2}{\kappa^2}\frac{x+b}{\sqrt{(R^2 - (x+b)^2)}}$$
$$\leq \frac{M^2}{\kappa^2}\frac{x_R + b}{R} = \frac{M^2}{\kappa^2}\frac{\alpha}{\sqrt{1+\alpha^2}} \leq \alpha \frac{M^2}{\kappa^2}.$$

So if $\alpha < \frac{mQ_s}{h^2+2h}\frac{1}{\gamma_s - \gamma_f}\frac{\kappa^2}{M^2} = \alpha_3$, we get $J_2 \leq 0$ and then:

$$J_1 \leq \frac{1}{\sqrt{\left(\frac{\kappa}{b_{22}(\omega^{-1}of(x))}\right)^2 + f'^2(x)}}\left(\frac{Q_f}{Log(1+d)} - \frac{Q_s}{h^2+2h}\right)$$
$$.b(z) \quad {}^t\left(f'(x), \frac{-\kappa}{b_{22}(\omega^{-1}of(x))}\right)^t\left(f'(x), \frac{-\kappa}{b_{22}(\omega^{-1}of(x))}\right).$$

Note that we have:

$$\frac{Q_f}{Log(1+d)} < \frac{Q_s}{h^2+2h} \Leftrightarrow R > \frac{e^{\frac{Q_f}{Q_s}(h^2+2h)} - 1 + a\alpha}{\sqrt{1+\alpha^2}(\sqrt{1+\alpha^2}-1)}.$$

Thus if $\alpha \in (0, \alpha_* = \min(\alpha_1, \alpha_2, \alpha_3))$ and

$$R \geq R_*(\alpha) = \max\left(\frac{a}{\alpha}, \frac{2M^2}{\kappa^2}(h+1), \frac{e^{\frac{Q_f}{Q_s}(h^2+2h)} - 1 + a\alpha}{\sqrt{1+\alpha^2}(\sqrt{1+\alpha^2}-1)}\right),$$

we get $I(\xi) \leq 0$ and (3.9) holds.

4th step

$$\text{For } R > \max\left(\frac{a}{\alpha}, \frac{a\alpha}{\sqrt{1+\alpha^2}(\sqrt{1+\alpha^2}-1)}\right), \qquad \text{one has:}$$
$$v \leq \psi \quad \text{on } \partial D \quad \text{and} \quad \lim_{x\to-\infty} v(x,z) \leq \lim_{x\to-\infty} \psi(x,z). \tag{3.10}$$

Indeed we have:

$$.v(x,-h) = G(f(x) - \omega(-h)) = G(f(x)+h)$$
$$\leq G(h) = Q_s = \psi(x,-h) \quad \text{for } x < f^{-1}(-h)$$

$$.v(x,-h) = -K(\omega(-h) - f(x)) = -K(-h - f(x))$$
$$\leq 0 < Q_s = \psi(x,-h) \quad \text{for } x \in (f^{-1}(-h), f_d^{-1}(-h))$$

$$.v(x,0) = G(f(x) - \omega(0)) = G(0) = 0 = \psi(x,0) \quad \text{for } x \leq -b$$
$$.v(x,0) = -K(\omega(0) - f(x)) = -K(-f(x)) \leq 0 = \psi(x,0) \quad \text{for } x \in (-b, x_d)$$

where x_d is defined by: $f_d(x_d) = 0$ and is to be chosen such that $x_d = 0$. In fact:

$$\begin{aligned} f_d(x_d) = 0 \Leftrightarrow x_d &= (z_R + d)\frac{R}{a+b} + x_R \\ &= \frac{R}{a+b}\Big(\frac{R}{a+b}\frac{R}{\sqrt{1+(\frac{R}{a+b})^2}} - R + d + \frac{a+b}{\sqrt{1+(\frac{R}{a+b})^2}} \\ &\quad - \frac{(a+b)^2}{R} + \frac{a(a+b)}{R}\Big) \\ &= \frac{1}{\alpha}(R\sqrt{1+\alpha^2} - (1+\alpha^2)R + d + a\alpha) \end{aligned}$$

Then we can choose x_d=0 if $d = -a\alpha + R\sqrt{1+\alpha^2}(\sqrt{1+\alpha^2}-1) > 0$ which holds if:

$$R > \frac{a\alpha}{\sqrt{1+\alpha^2}(\sqrt{1+\alpha^2}-1)}.$$

$$.v(x,\omega^{-1}of(x)) = -K(f_d(x) - f(x)) = -K(d) = -Q_f \leq \psi(x,\omega^{-1}of(x))$$
$$\text{for } x \in (x_d, f_d^{-1}(-h))$$
$$.\lim_{x\to-\infty} v(x,z) = \lim_{x\to-\infty} v_1(x,z) = \lim_{x\to-\infty} G(f(x) - \omega(z))$$
$$= G(-\omega(z)) \leq \frac{Q_s}{h}\omega(z) = v_{-\infty}(z) = \lim_{x\to-\infty}\psi(x,z).$$

5^{th} *step*

For all $\alpha \in (0,\alpha_*)$, $\quad R > \max\Big(R_*(\alpha), \dfrac{a\alpha}{\sqrt{1+\alpha^2}(\sqrt{1+\alpha^2}-1)}\Big)$, one has:

$$\int_D \big(b(z)\nabla(v-\psi_0) + (\theta-\gamma_0)e_x\big)\nabla\zeta = 0 \qquad \forall \zeta \in \mathcal{D}(I\!R^2) \tag{3.11}$$

where $\psi_0 = \min(\psi, v)$ and $\gamma_0 = \min(\gamma, \theta)$.
Indeed let $\zeta \in \mathcal{D}(I\!R^2)$ and consider for $i = 1,2$ the function ζ_i defined as in the proof of Lemma 2.1. Then for $\epsilon > 0$, $\min\big(\zeta_i, \frac{v-\psi_0}{\epsilon}\big) \in H_0^1(D)$ by (3.10) and is nonnegative with compact support. By (3.9) and $(P)i$), we get:

$$\int_{D\cap[v-\psi_0\geq\epsilon\zeta_i]} b(z)\nabla(v-\psi_0)\nabla\zeta_i + \int_D (\theta-\gamma_0)\zeta_{ix} \leq \int_D (\theta-\gamma_0)\Big(\zeta_i - \frac{v-\psi_0}{\epsilon}\Big)^+_x$$
$$\leq (\gamma_s-\gamma_f)\int_{I_0} \Big(\zeta_i + \frac{\psi(f^{-1}o\omega(z),z)}{\epsilon}\Big)^+ - \Big(\zeta_i - \frac{v(g(z),z)}{\epsilon}\Big)^+$$

where $I_0 = \{z \in (-h^*,0)/\, g(z) < f^{-1}o\omega(z)\}$. Letting $\epsilon \to 0$, we get:

$$\int_D \big(b(z)\nabla(v-\psi_0) + (\theta-\gamma_0)e_x\big)\nabla\zeta_i \leq 0.$$

To obtain (3.11), we argue as in the proof of Lemma 2.1.

6^{th} *step* : Conclusion

Let $u = (v-\psi_0)\chi(D)$ and $\zeta \in \mathcal{D}([z > \omega^{-1}of(x)]\cap\Omega)$. Note that $\theta = \gamma_f = \gamma_0$ in D_2. We deduce from (3.11):

$$\int_{[z>\omega^{-1}of(x)]\cap D} b(z)\nabla u.\nabla\zeta = 0.$$

Since u vanishes on $[z = \omega^{-1}of_d(x)]$, then $u \in H^1_{loc}(\Omega)$ and:

$$\int_{[z>\omega^{-1}of(x)]\cap\Omega} b(z)\nabla u.\nabla\zeta = 0.$$

Moreover $u \geq 0$ and $u = 0$ in $\Omega \setminus D$. So by the strong maximum principle we get $u = 0$ in $[z > \omega^{-1}of(x)] \cap \Omega$ which leads to $v \leq \psi$ in D_2. In particular we obtain $\psi(f^{-1}o\omega(z),z) \geq v(f^{-1}o\omega(z),z) = v(x,\omega^{-1}of(x)) = 0$. So $g(z) \geq f^{-1}o\omega(z) \quad \forall z \in (-h^*,0)$ and then:

$$g(0-) \geq f^{-1}o\omega(0) = f^{-1}(0) = -b > -\infty.$$

Theorem 3.4 *If $\mathcal{B}(z,\xi) = |\xi|^{q-2}\xi$, then we have $g(0-) = \lim_{z\to-\infty} g(z) > -\infty$.*

Proof. We follow the proof of Theorem 3.3 and we use the same notations with $\omega(z) = z$.

<u>1^{st} *step*</u> For $R > R_1 = \frac{h+1}{q-1}(|q-2| + q - 1)$, one has $\Delta v_1 > 0$ in D_1.

Indeed we have:

$$\begin{aligned}\Delta_q v_1 &= div(|\nabla v_1|^{q-2}\nabla v_1) = \frac{\partial}{\partial x}(|\nabla v_1|^{q-2}\frac{\partial v_1}{\partial x}) + \frac{\partial}{\partial z}(|\nabla v_1|^{q-2}\frac{\partial v_1}{\partial z})\\ &= (G'(f(x)-z))^{q-2}(1+f'^2(x))^{\frac{q-2}{2}}\Big((q-1)G''(f(x)-z)(1+f'^2(x))\\ &+ G'(f(x)-z)f''(x)\frac{1+(q-1)f'^2(x)}{1+f'^2(x)}\Big)\end{aligned}$$

. If $x < -b$ or $x > x_R$, we have $f''(x) = 0$ and then $\Delta_q v_1 > 0$ in D_1.
. If $-b < x < x_R$, we have:

$$\begin{aligned}\Delta_q v_1 &= (2\lambda(1+f(x)-z))^{q-2}(1+f'^2(x))^{\frac{q-2}{2}}\Big(2\lambda\frac{(R^2-(x+b)^2)^{3/2}}{R^2}\\ &.((q-1)(R+z-1) + (q-2)(f(x)-z+1)\frac{(R^2-(x+b)^2)}{R^2}\Big)\\ &\geq (2\lambda(1+f(x)-z))^{q-2}(1+f'^2(x))^{\frac{q-2}{2}}\Big(2\lambda\frac{(R^2-(x+b)^2)^{3/2}}{R^2}\\ &.((q-1)(R-h-1) - |q-2|(h+1)\Big)\end{aligned}$$

since $\left|(q-2)(f(x)-z+1)\frac{(R^2-(x+b)^2)}{R^2}\right| \leq |q-2|(h+1)$.
So for $R > R_1$ we get $\Delta_q v_1 > 0$ in D_1.

<u>2^{nd} *step*</u>

$$\text{For } \alpha\in(0,1) \quad \text{and } R > R_2(\alpha) = \max\Big(\frac{a}{\alpha}, \frac{(q-1)+|q-2|(h+1)}{(q-1)\big(\frac{2}{\sqrt{1+\alpha^2}}-1\big)}\Big)$$

one has $\Delta_q v_2 \geq 0$ in $\mathcal{D}'(D_2)$.
Indeed we have:

$$\begin{aligned}\Delta_q v_2 &= (K'(z-f(x)))^{q-2}(1+f'^2(x))^{\frac{q-2}{2}}\Big(-(q-1)K''(z-f(x))(1+f'^2(x))\\ &+ K'(z-f(x))f''(x)(q-1-\frac{q-2}{1+f'^2(x)})\Big).\end{aligned}$$

. For $x_R < x < f_d^{-1}(-h)$, we have $f''(x) = 0$ and then $\Delta v_2 \geq 0$ in D_2.

. If $-b < x < x_R$, we have:

$$\begin{aligned}
\Delta_q v_2 &= \mu(K'(z-f(x)))^{q-2}(1+f'^2(x))^{\frac{q-2}{2}}\Big(\frac{q-1}{(1+z-f(x))^2}\frac{R^2}{R^2-(x+b)^2} \\
&+ \frac{1}{1+z-f(x)}\Big(\frac{-R^2}{(R^2-(x+b)^2)^{3/2}}\Big)\Big((q-1)-(q-2)\frac{(R^2-(x+b)^2)}{R^2}\Big)\Big) \\
&= \mu(K'(z-f(x)))^{q-2}(1+f'^2(x))^{\frac{q-2}{2}}\frac{R^2}{(R^2-(x+b)^2)^{3/2}}\frac{1}{(1+z-f(x))^2} \\
&.\Big((q-1)(2\sqrt{R^2-(x+b)^2}-R)-(q-1)(1+z) \\
&+ (q-2)(1+z-f(x))\frac{(R^2-(x+b)^2)}{R^2}\Big) \\
&\geq \mu(K'(z-f(x)))^{q-2}(1+f'^2(x))^{\frac{q-2}{2}}\frac{R^2}{(R^2-(x+b)^2)^{3/2}}\frac{1}{(1+z-f(x))^2} \\
&.\Big((q-1)\Big(\frac{2}{\sqrt{1+\alpha^2}}-1\Big)R-(q-1)-|q-2|(h+1)\Big)
\end{aligned}$$

since $2\sqrt{R^2-(x+b)^2}-R \geq 2\sqrt{R^2-(x_R+b)^2}-R = R\Big(\frac{2}{\sqrt{1+\alpha^2}}-1\Big)$.
If we choose $\alpha \in (0,1)$ such that $\frac{2}{\sqrt{1+\alpha^2}}-1>0$, we get $\Delta_q v_2 > 0$ in D_2 for $R \geq \max(\frac{a}{\alpha}, R_2(\alpha))$.

<u>3^{rd} *step*</u> : There exists $\alpha_* > 0$ and $R_* > 0$ such that:

$$\forall \alpha \in (0,\alpha_*), \quad \forall R > R_*, \text{ one has } \Delta_q v \geq 0 \text{ in } \mathcal{D}'(D).$$

Indeed let $\xi \in \mathcal{D}(D)$, $\xi \geq 0$. We have:

$$\begin{aligned}
I(\xi) &= \int_D (|\nabla v|^{q-2}\nabla v + \theta e_x)\nabla\xi \\
&= \int_{D_1} (|\nabla v_1|^{q-2}\nabla v_1 + \gamma_s e_x)\nabla\xi + \int_{D_2} (|\nabla v_2|^{q-2}\nabla v_2 + \gamma_f e_x)\nabla\xi \\
&= < -\Delta_q v_1, \xi > + < -\Delta_q v_2, \xi > \\
&+ \int_{[z=f(x)]\cap D} ((|\nabla v_1|^{q-2}\nabla v_1 - |\nabla v_2|^{q-2}\nabla v_2).\nu + (\gamma_s-\gamma_f)\nu_x)\xi \\
&\leq \int_{[z=f(x)]\cap D} ((|\nabla v_1|^{q-2}\nabla v_1 - |\nabla v_2|^{q-2}\nabla v_2).\nu + (\gamma_s-\gamma_f)\nu_x)\xi \\
&= \int_{[z=f(x)]\cap D} J_1\xi
\end{aligned}$$

if we choose $\alpha \in (0,1)$ and $R \geq \max\Big(R_1, R_2(\alpha), \frac{a}{\alpha}\Big)$.

Recall that:

$$
\begin{aligned}
\nabla v_1(x, f(x)) &= \frac{2Q_s}{h^2+2h}(f'(x), -1)\\
|\nabla v_1|^{q-2}\nabla v_1 &= \Big(\frac{2Q_s}{h^2+2h}\Big)^{q-1}(1+f'^2(x))^{\frac{q-2}{2}}(f'(x), -1)\\
\nabla v_2(x, f(x)) &= \frac{Q_f}{Log(1+d)}(f'(x), -1)\\
|\nabla v_1|^{q-2}\nabla v_1 &= \Big(\frac{Q_f}{Log(1+d)}\Big)^{q-1}(1+f'^2(x))^{\frac{q-2}{2}}(f'(x), -1)\\
\nu(x, f(x)) &= \frac{1}{\sqrt{1+f'^2(x)}}(-f'(x), 1)
\end{aligned}
$$

Then :

$$
\begin{aligned}
J_1 &= (1+f'^2(x))^{\frac{q-1}{2}}\Big(\Big(\frac{Q_f}{Log(1+d)}\Big)^{q-1} - \Big(\frac{2Q_s}{h^2+2h}\Big)^{q-1}\Big)\\
&+ (\gamma_s - \gamma_f)\Big(\frac{-f'(x)}{\sqrt{1+f'^2(x)}}\Big)\\
&= (1+f'^2(x))^{\frac{q-1}{2}}\Big(\Big(\frac{Q_f}{Log(1+d)}\Big)^{q-1} - \frac{1}{2}\Big(\frac{2Q_s}{h^2+2h}\Big)^{q-1}\Big)\\
&+ (1+f'^2(x))^{\frac{q-1}{2}}\Big((\gamma_s - \gamma_f)\frac{-f'(x)}{(1+f'^2(x))^{q/2}} - \frac{1}{2}\Big(\frac{2Q_s}{h^2+h}\Big)^{q-1}\Big).
\end{aligned}
$$

Note that we have:

$$
\frac{-f'(x)}{(1+f'^2(x))^{q/2}} \leq -f'(x) \leq \alpha \qquad \forall x \in (-b, f_d^{-1}(-h))
$$

and for

$$
R > R_3(\alpha) = \frac{e^{\frac{Q_f(h^2+2h)}{2^{\frac{q-2}{q-1}}Q_s}} - 1 + a\alpha}{\sqrt{1+\alpha^2}(\sqrt{1+\alpha^2}-1)}, \qquad \text{one has :}
$$

$$
\Big(\frac{Q_f}{Log(1+d)}\Big)^{q-1} < \frac{1}{2}\Big(\frac{2Q_s}{h^2+2h}\Big)^{q-1} = \Big(\frac{Q_s}{2^{\frac{1}{q-1}}(h^2+2h)}\Big)^{q-1}.
$$

Thus if

$$
\begin{aligned}
&\alpha \in \Big(0, \alpha_* = \min\Big(1, \frac{1}{2(\gamma_s - \gamma_f)}\Big(\frac{2Q_s}{h^2+2h}\Big)^{q-1}\Big)\Big)\\
\text{and} \quad &R \geq R_* = \max\big(R_1, R_2(\alpha), R_3(\alpha), \frac{a}{\alpha}\big)
\end{aligned}
$$

we get $J_1 \leq 0$ and then $I(\xi) \leq 0$.

<u>4^{th} *step*</u> Clearly we have for $R > \max\left(\frac{a\alpha}{\sqrt{1+\alpha^2}(\sqrt{1+\alpha^2}-1)}, \frac{a}{\alpha}\right)$:

$$v \leq \psi \quad \text{on } \partial D \qquad \text{and} \qquad \lim_{x \to -\infty} v(x,z) \leq \lim_{x \to -\infty} \psi(x,z).$$

<u>5^{th} *step*</u> Arguing as in the proof of Lemma 2.1, we can establish that for all $\alpha \in (0, \alpha_*)$ and $R > \max\left(R_*, \frac{a\alpha}{\sqrt{1+\alpha^2}(\sqrt{1+\alpha^2}-1)}\right)$, we have:

$$\int_D \left((|\nabla v|^{q-2}\nabla v - |\nabla \psi_0|^{q-2}\nabla \psi_0) + (\gamma - \theta_0)e_x\right)\nabla\zeta = 0 \qquad \forall \zeta \in \mathcal{D}(D).$$

<u>6^{th} *step*</u> Assume that $D_1 \cap [\psi < 0] \neq \emptyset$ and let $\zeta \in \mathcal{D}(D_1 \cap [\psi < 0])$. Note that we have $\gamma = \gamma_f = \gamma_0$ and $\theta = \gamma_s$ in $D_1 \cap [\psi < 0]$. We also have $\psi = \psi_0$ in $D_1 \cap [\psi < 0]$ since $\psi < 0 < v$. So we we deduce that for all $\zeta \in \mathcal{D}(D)$:

$$\int_{D_1 \cap [\psi < 0]} (|\nabla v|^{q-2}\nabla v - |\nabla \psi_0|^{q-2}\nabla \psi_0)\nabla\zeta = -(\gamma_s - \gamma_f)\int_{D_1 \cap [\psi < 0]} \zeta_x = 0$$

from which it follows that $\Delta_q v = \Delta_q \psi$ in $D_1 \cap [\psi < 0]$. But $\Delta_q \psi = 0$ in $[\psi < 0]$ and $\Delta_q v = \Delta_q v_1 > 0$ in D_1. This leads to a contradiction. Thus $D_1 \cap [\psi < 0] = \emptyset$ i.e $\psi \geq 0$ in D_1. In particular we have $g(z) \geq f^{-1}(z) \quad \forall z \in (-h^*, 0)$ and then $g(0-) \geq f^{-1}(0) = -b > -\infty$.

Theorem 3.5 *If $\mathcal{B}(z,\xi) = \xi$, then we have:*

$$\lim_{z \to 0^-} g(z) = g(0-) \leq -\frac{h}{\pi} Log\left(\frac{e^{\frac{a\pi(Q_s+Q_f)}{hQ_f}} - 1}{e^{\frac{a\pi(Q_s+Q_f)}{hQ_f}} - e^{\frac{a\pi}{h}}}\right) < 0.$$

Proof. This result was announced in [1] without an explicit proof. In this reference the authors indicated that this result can be obtained by using hodograph techniques to get a semi-explicit expression for $\bar{\psi}$ which is the unique harmonic function satisfying the same boundary and limit conditions as ψ. However an explicit proof was not given. This is why we propose here a proof of this result.
Since we have $\Delta\psi \geq 0$ and $\Delta\bar{\psi} = 0$ in Ω, we get by taking into account the boundary conditions and the limit behavior at infinity that $\psi \leq \bar{\psi}$ in Ω. Moreover if we denote by $\bar{g}$ the function defined by $\bar{\psi}(\bar{g}(z), z) = 0 \; \forall z \in (-h^*, 0)$, we get:

$$g(z) \leq \bar{g}(z) \qquad \forall z \in (-h^*, 0) \qquad \text{which leads to} \quad g(0-) \leq \bar{g}(0-).$$

In the following we will prove that:

$$\bar{g}(0-) = -\frac{h}{\pi} Log\left(\frac{e^{\frac{a\pi(Q_s+Q_f)}{hQ_f}} - 1}{e^{\frac{a\pi(Q_s+Q_f)}{hQ_f}} - e^{\frac{a\pi}{h}}}\right) < 0.$$

For the seek of simplifying things we introduce the function:

$$\psi_0(x,z) = \bar{\psi}(hx, h(z-1)) \qquad \text{for } (x,z) \in \mathbb{R} \times [0,1]$$

which is harmonic in $\mathbb{R} \times (0,1)$ and satisfies $\psi_0(x,0) = Q_s$, $\psi_0(x,1) = \phi_0(x)$ and $\lim_{x \to \pm\infty} \psi_0(x,z) = v_{\pm\infty}(h(z-1))$.

In [12], one can find that:

$$\psi_0(x,z) = Re\Big(\frac{i}{2}\int_{-\infty}^{+\infty} Q_s coth\frac{\pi(t-x-iz)}{2} + \phi_0(ht) th\frac{\pi(t-x-iz)}{2} dt\Big)$$

$$= -\frac{Q_s}{2}\int_{-\infty}^{+\infty} \frac{ch\pi(t-x) + cos\pi z}{sh^2\pi(t-x) + sin^2\pi z} sin\pi z dt$$

$$+\frac{1}{2}\int_{-\infty}^{+\infty} \phi_0(ht)\frac{sin\pi z}{ch\pi(t-x) + cos\pi z} dt$$

$$= -\frac{Q_s}{2}\int_{-\infty}^{+\infty} \frac{ch\pi s + cos\pi z}{sh^2\pi s + sin^2\pi z} sin\pi z ds + \frac{1}{2} I(x,z)$$

where,

$$\begin{aligned} I(x,z) = & \int_0^{\frac{a}{h}} -\frac{Q_f}{a} ht \frac{sin\pi z}{ch\pi(t-x) + cos\pi z} dt + \int_{\frac{a}{h}}^{+\infty} -Q_f \frac{sin\pi z}{ch\pi(t-x) + cos\pi z} dt \\ = & -\frac{Q_f}{a} h \int_{-x}^{\frac{a}{h}-x} t\frac{sin\pi z}{ch\pi t + cos\pi z} dt - \frac{Q_f}{a} hx \int_{-x}^{\frac{a}{h}-x} \frac{sin\pi z}{ch\pi t + cos\pi z} dt \\ & - Q_f \int_{\frac{a}{h}-x}^{+\infty} \frac{sin\pi z}{ch\pi t + cos\pi z} dt. \end{aligned}$$

Then we deduce that:

$$\begin{aligned} \frac{\partial \psi_0}{\partial x} = \frac{1}{2}\frac{\partial I}{\partial x} = & -\frac{Q_f}{2a} h \int_{-x}^{\frac{a}{h}-x} \frac{sin\pi z}{ch\pi t + cos\pi z} dt \\ = & -\frac{hQ_f}{a\pi}\Big[Arctan\Big(\frac{e^{(\frac{a}{h}-x)\pi} + cos\pi z}{sin\pi z}\Big) - Arctan\Big(\frac{e^{-\pi x} + cos\pi z}{sin\pi z}\Big)\Big] \end{aligned}$$

and for $(x,z) \in \Omega$:

$$\bar{\psi}(x,z) = v_{-\infty}(z) - \frac{hQ_f}{a\pi} \int_{-\infty}^{\frac{x}{h}} \Big(Arctan\Big(\frac{e^{-\pi t} - cos\frac{\pi z}{h}}{sin\frac{\pi z}{h}} \Big) - Arctan\Big(\frac{e^{\left(\frac{x}{h}-t\right)\pi} - cos\frac{\pi z}{h}}{sin\frac{\pi z}{h}} \Big) \Big) dt.$$

Using this formula we give an asymptotic behavior of $\bar{\psi}$ near $z = 0$. Note that for $x \leq 0$ and $t < \frac{x}{h}$, we have $e^{-\pi t} > 1$ and $e^{\left(\frac{x}{h}-t\right)\pi} > 1$. Then we obtain:

$$Arctan\Big(\frac{e^{-\pi t} - cos\frac{\pi z}{h}}{sin\frac{\pi z}{h}} \Big) = -\frac{\pi}{2} + \frac{\pi}{h} \frac{1}{1 - e^{-\pi t}} z + \frac{z^2}{2} p(z,t)$$

$$Arctan\Big(\frac{e^{\left(\frac{x}{h}-t\right)\pi} - cos\frac{\pi z}{h}}{sin\frac{\pi z}{h}} \Big) = -\frac{\pi}{2} + \frac{\pi}{h} \frac{1}{1 - e^{\left(\frac{x}{h}-t\right)\pi}} z + \frac{z^2}{2} q(z,t)$$

with $p(z,t), q(z,t) \approx e^{\pi t}$ as $t \to -\infty$.
So we get:

$$\begin{aligned}
0 = \bar{\psi}(\bar{g}(z),z) &= -\frac{Q_s}{h} z - \frac{Q_f}{a} z \int_{-\infty}^{\bar{g}(z)} \Big(\frac{1}{1 - e^{-\pi t}} - \frac{1}{1 - e^{\left(\frac{x}{h}-t\right)\pi}} \Big) dt \\
&- \frac{hQ_f}{2\pi a} z^2 \int_{-\infty}^{\bar{g}(z)} (p(z,t) - q(z,t)) dt \\
&- \frac{Q_s}{h} - \frac{Q_f}{a} \int_{-\infty}^{\bar{g}(z)} \Big(\frac{1}{1 - e^{-\pi t}} - \frac{1}{1 - e^{\left(\frac{x}{h}-t\right)\pi}} \Big) dt \\
&= z \int_{-\infty}^{\bar{g}(z)} (p(z,t) - q(z,t)) dt.
\end{aligned}$$

Letting $z \to 0$, we obtain:

$$\frac{Q_f}{a} \int_{-\infty}^{\bar{g}(0)} \Big(\frac{1}{1 - e^{-\pi t}} - \frac{1}{1 - e^{\left(\frac{x}{h}-t\right)\pi}} \Big) dt = \frac{Q_s}{h}$$

and by computing this last integral we find the result.

Remark 3.1 *As a consequence of the above theorem, we have also* $g(0-) < 0$ *when:*

$$\mathcal{B}(z,\xi) = \Big(\frac{C^2}{b(z)} \xi_1, b(z) \xi_2 \Big) \qquad \textit{with} \quad C = \frac{-h}{\int_{-h}^{0} \frac{ds}{b(s)}}.$$

Indeed, set $v(x,z) = \bar{\psi}(x, \omega(z))$ with $\omega(z) = C \int_z^0 \frac{ds}{b(s)}$. Then we remark that

$$div(\mathcal{B}(z, \nabla v)) = \frac{C^2}{b(z)} \Delta \bar{\psi}(x, \omega(z)) = 0.$$

Moreover $v = \bar{\psi}$ on $\partial\Omega$ and $\lim_{x \to \pm\infty} v = \lim_{x \to \pm\infty} \bar{\psi}$. Then we deduce that $v \leq \bar{\psi}$ in Ω which leads to:

$$\psi(\bar{g}(\omega(z)), z) \leq v(\bar{g}(\omega(z)), z) = \bar{\psi}(\bar{g}(\omega(z)), \omega(z)) = 0 \quad \forall z \in (-h^*, 0).$$

So $g(z) \leq \bar{g}(\omega(z)) \quad \forall z \in (-h^*, 0)$ and then $g(0-) \leq \bar{g}(0-) < 0$.

Appendix

This section is devoted to some technical Lemmas.

Lemma A.1 (Strong maximum principle) *Let Ω be a domain of $I\!R^2$ and let u_1, $u_2 \in W^{1,q}_{loc}(\Omega)$ such that:*

$$div(\mathcal{B}(X, \nabla u_1)) = div(\mathcal{B}(X, \nabla u_2)) = 0 \qquad in\ \mathcal{D}'(\Omega) \tag{A.1}$$

$$u_1 \leq u_2 \qquad in\ \Omega \tag{A.2}$$

then we have either $u_1 = u_2$ in Ω or $u_1 = u_2$ in Ω.

Proof. See [8].

Lemma A.2 *Under the same hypothesis of Lemma A.1 we assume that there exists $\Gamma_0 \subset \partial\Omega$ of class $C^{1,\alpha}$ such that:*

$$u_1 = u_2 \text{ on } \Gamma_0, \qquad u_1, u_2 \in C^1(\Omega \cup \Gamma_0) \tag{A.3}$$

$$\mathcal{B}(X, \nabla u_1).\nu = \mathcal{B}(X, \nabla u_2).\nu \quad on\ \Gamma_0 \tag{A.4}$$

$$\nabla u_1(X) \neq 0 \quad \forall X \in \Gamma_0 \qquad or \qquad \nabla u_2(X) \neq 0 \quad \forall X \in \Gamma_0 \tag{A.5}$$

then $u_1 = u_2$ in Ω.

Proof. We first prove that $u_1 = u_2$ in a small open set near Γ_0. We then conclude by Lemma A.1.

Assume for example that we have $\nabla u_1(X) \neq 0\ \forall X \in \Gamma_0$. Since $u_1 \in C^1(\Omega \cup \Gamma_0)$, there exists a small ball $B(X_0, \epsilon_0)$ centered on Γ_0 such that:

$$\nabla u_1(X) \neq 0 \quad \forall X \in B(X_0, \epsilon_0) \cap (\Omega \cup \Gamma_0)$$

and then there exists two positive constants c_0 and C_1 such that:

$$c_0 \leq |\nabla u_1(X)| \leq c_1 \quad \forall X \in K_1 = \overline{B(X_0, \epsilon_0) \cap (\Omega \cup \Gamma_0)}. \tag{A.6}$$

Let $u = u_1 - u_2$ and $u_t = tu_2 + (1-t)u_1$ for any $t \in [0,1]$. Then we deduce from (A.1) and (A.4):

$$\int_{B(X_0,\epsilon_0)\cap\Omega} (\mathcal{B}(X, \nabla u_2) - \mathcal{B}(X, \nabla u_1)).\nabla\zeta = 0 \qquad \forall \zeta \in \mathcal{D}(B(X_0, \epsilon_0)). \tag{A.7}$$

Remark that:

$$\mathcal{B}(X, \nabla u_2) - \mathcal{B}(X, \nabla u_1) = \int_0^1 \frac{d}{dt}(\mathcal{B}(X, \nabla u_t))dt,$$

then (A.7) becomes:

$$\int_{B(X_0,\epsilon_0)\cap\Omega} \mathcal{C}(X)\nabla u.\nabla\zeta = 0 \qquad \forall \zeta \in \mathcal{D}(B(X_0, \epsilon_0)) \tag{A.8}$$

where:

$$\mathcal{C}(X) = \int_0^1 < \mathcal{B}(X, \nabla u_t), \nabla u_t >^{\frac{q-2}{2}} \Big(\mathcal{B}(X, \nabla u_t)$$
$$+(q-2)\frac{\mathcal{B}(X, \nabla u_t).^t(\mathcal{B}(X, \nabla u_t))}{< \mathcal{B}(X, \nabla u_t), \nabla u_t >}\Big)dt$$

from which we deduce:

$$c_2|Y|^2\lambda(X) \leq \mathcal{C}(X).Y.Y \leq c_3|Y|^2\lambda(X) \quad \forall (X,Y) \in (B(X_0, \epsilon_0) \cap \Omega) \times I\!R^2 \tag{A.9}$$

with c_2, c_3 two positive constants and $\lambda(X) = \int_0^1 |\nabla u_t|^{q-2}dt$. Using (A.6) and arguing as in [11], we can prove that $\lambda(X)$ is bounded from both sides by two positive constants λ_0, λ_1 in $B(X_0, \epsilon_0') \cap \Omega$ for some $\epsilon_0' \in (0, \epsilon_0)$. So $\mathcal{C}(X)$ is strictly elliptic in $B(X_0, \epsilon_0') \cap \Omega$. By (A.3) we can extend u by 0 to $B(X_0, \epsilon_0') \setminus \Omega$ so that $u \in W^{1,q}(B(X_0, \epsilon_0'))$. One can also extend $\mathcal{C}(X)$ by $c_2\lambda_0 I_2$ to $B(X_0, \epsilon_0') \setminus \Omega$ so that it is strictly elliptic in $B(X_0, \epsilon_0')$. Then we get from (A.8):

$$\int_{B(X_0,\epsilon_0')} \mathcal{C}(X)\nabla u.\nabla\zeta = 0 \qquad \forall \zeta \in \mathcal{D}(B(X_0, \epsilon_0')). \tag{A.10}$$

Now since $u \geq 0$ and $u = 0$ in $B(X_0, \epsilon_0') \setminus \Omega$, we get from (A.10) and the strong maximum principle for linear elliptic equations $u = 0$ in $B(X_0, \epsilon_0')$ which leads to $u_1 = u_2$ in $B(X_0, \epsilon_0') \cap \Omega$.

Lemma A.3 (Non-oscillation Lemma) *Let $z_0 \in (-h^*, 0)$, $x_0 \in I\!R$, $r > 0$ and assume that $S_{eg} = \{(x, z_0)/ \, |x - x_0| \leq r\} \subset \Gamma$, then we cannot have:*

$$\forall (x,z) \in B_r(x_0, z_0) \setminus S_{eg} \quad \psi(x,z) \neq 0$$

where $B_r(x_0, z_0)$ is the open ball of center (x_0, z_0) and radius r.

Proof. The proof follows the one of Lemma 5.1 in [4] and uses Lemma A.1.

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