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# A Filtration Problem through a Heterogeneous Porous Medium

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**Abstract.** The flow of a fluid through a heterogeneous porous medium is studied, assuming it is governed by a nonlinear Darcy's law and Dirichlet boundary conditions. Under a general condition on the permeability we prove that the free boundary is locally a continuous curve in some local coordinates. We also prove the uniqueness of the $S_3$-connected solution.

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**Key words** : Heterogeneous Porous Medium, Nonlinear Darcy's law, Free Boundary.

## Introduction

In this paper we are concerned with the study of a heterogeneous dam problem. We recall that there exists a large literature on this subject where both homogeneous and heterogeneous dams were studied. In [Ly1] a two dimensional heterogeneous dam was studied with a permeability matrix $a(X) = (a_{ij}(X))$ satisfying

$$a_{12}(X) = 0 \quad \text{a.e. in} \quad \Omega \quad \text{and} \quad \frac{\partial}{\partial y} a_{22}(X) \geq 0 \quad \text{in} \quad \mathcal{D}'(\Omega), \tag{0.1}$$

where $X = (x, y)$ denotes points of $\mathbb{R}^2$. It was then proved that the free boundary separating the wet and dry regions of the dam is a continuous curve of the form

$$y = \Phi(x). \tag{0.2}$$

It was also proved that the $S_3$-connected solution is unique.
Note that similar results where established in [FH] and [SV] for a permeability matrix of the form $a(X) = k(X)I_2$ with $\frac{\partial k}{\partial y} \geq 0$ in $\mathcal{D}'(\Omega)$. In [CF] the authors studied a rectangular dam with a permeability matrix $a(X) = k(y)I_2$ such that $\frac{\partial k}{\partial y} \leq 0$ in $\mathcal{D}'(\Omega)$. They proved that the free boundary is a curve of the form $x = \psi(y)$. This is what motivated us to consider this problem under the weaker assumption

$$div(a_{12}(X), a_{22}(X)) \geq 0 \quad \text{in} \quad \mathcal{D}'(\Omega). \tag{0.3}$$

Actually we suppose the flow obeying to a general Darcy's law that involves a nonlinear operator $\mathcal{A}(X,\xi)$ which generalizes the classical linear law where $\mathcal{A}(X,\xi) = a(X)\xi$. Then under the assumption $div(\mathcal{A}(X,e)) \geq 0$ in $\mathcal{D}'(\Omega)$, where $e = (0,1)$, and some regularity conditions on $\mathcal{A}$, we establish that the free boundary is locally a continuous graph of the form

$$\tau = \Phi(\omega), \tag{0.4}$$

where $(\tau, \omega) = SoT^{-1}(x, y)$, $S$ and $T$ are $C^1$-diffeomorphisms. Taking advantage of this result we prove the uniqueness of the $S_3$-connected solution.
We would like to point out that to our best knowledge the condition (0.3) appeared firstly in [A] and was used by the author to prove a similar inequality to (2.2). However the continuity of the free boundary and the uniqueness of solutions under this general condition are addressed here for the first time. Moreover in

all previous studies, authors considered exclusively dams that are enclosed between two curves : $y = s_-(x)$ and $y = s_+(x)$ which represent respectively the bottom and the top of the dam. This implicitly assumes that the dam is vertically convex. In this study we will free ourselves from this constraint and allow a wide variety of geometrical forms for our dam. For the existence of a solution we only require that $\Omega$ is locally Lipschits. However for the study of the free boundary we will assume that $\Omega$ is locally of class $C^1$.

## 1. Formulation of the Problem

A porous medium that we denote by $\Omega$ is supplied by several reservoirs of a fluid which infiltrates through $\Omega$. We assume that $\Omega$ is a bounded locally Lipschitz domain of $\mathbb{R}^2$ with boundary $\partial\Omega = S_1 \cup S_2 \cup S_3$, where $S_1$ is the impervious part, $S_2$ is the part in contact with air and $S_3 = \bigcup_{i=1}^{i=N} S_{3,i}$ with $S_{3,i}$ $(i = 1, ..., N)$ the part in contact with the bottom of the $i^{th}$ reservoir.
We assume that the flow in $\Omega$ has reached a steady state and we look for the fluid pressure $p$ and the saturated region $S$ of the porous medium. The boundary $\partial S$ of $S$ is divided into four parts (see figure 1)

$\Gamma_1 \subset S_1$ : the impervious part,
$\Gamma_2 \subset \Omega$ : the free boundary,
$\Gamma_3 \subset S_3$ : the part covered by fluid,
$\Gamma_4 \subset S_2$ : the part where the fluid flows outside $\Omega$.

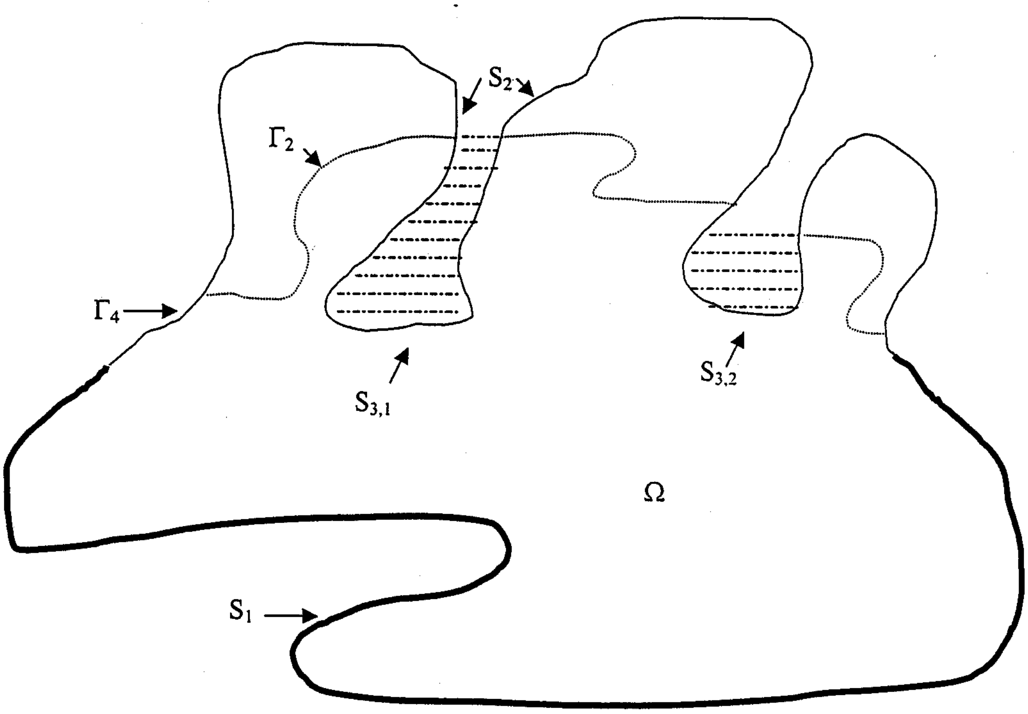


Figure 1

The flow is governed by the following nonlinear Darcy law

$$v = -\mathcal{A}(X, \nabla(p + y)) = -\mathcal{A}(X, \nabla u) \tag{1.1}$$

where $v$ is the fluid velocity, $u = p + y$ is the hydrostatic head and $\mathcal{A} : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a mapping that where $v$ is the fluid velocity, $u = p + y$ is the hydrostatic head and $\mathcal{A} : \Omega \times I\!R^2 \longrightarrow I\!R^2$ is a mapping that satisfies the following assumptions for some constants $q > 1$ and $0 < c_0 \leq c_1 < \infty$ :

$$\begin{cases} X \longmapsto \mathcal{A}(X,\xi) \text{ is measurable } \forall \xi \in I\!R^2, \\ \xi \longmapsto \mathcal{A}(X,\xi) \text{ is continuous for a.e } X \in \Omega, \\ \text{for all } \xi \in I\!R^2 \text{ and for a.e. } X \in \Omega, \quad \mathcal{A}(X,\xi).\xi \geq c_0|\xi|^q \quad \text{and} \quad |\mathcal{A}(X,\xi)| \leq c_1|\xi|^{q-1}, \\ \text{for all } \xi,\zeta \in I\!R^2 \text{ and for a.e. } X \in \Omega, \quad (\mathcal{A}(X,\xi) - \mathcal{A}(X,\zeta)).(\xi - \zeta) \geq 0. \end{cases} \tag{1.2}$$

Moreover we have the following boundary conditions

$$\begin{cases} p = 0 \text{ on } S_2, \quad p = \varphi \text{ on } S_3, \quad v.\nu = 0 \text{ on } \Gamma_1, \\ p = 0 \quad \text{and} \quad v.\nu = 0 \quad \text{on } \Gamma_2 \quad \text{and} \quad v.\nu \geq 0 \text{ on } \Gamma_4 \end{cases} \tag{1.3}$$

where $\varphi$ is a nonnegative Lipschitz continuous function which represents the fluid pressure at the bottoms of the reservoirs. For convenience we assume that $S_3$ is open relatively to $\partial\Omega$.
Assuming the flow to be incompressible and taking into account (1.1)-(1.3), we are led (see [CaL]) to the following problem

$$\begin{cases} \text{Find } (u,g) \in W^{1,q}(\Omega) \times L^\infty(\Omega) \text{ such that :} \\ (i) \quad u \geq y, \quad 0 \leq g \leq 1, \quad g(u-y) = 0 \ \text{ a.e. in } \Omega, \\ (ii) \quad u = \varphi + y \quad \text{on } S_2 \cup S_3, \\ (iii) \quad \displaystyle\int_\Omega (\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)).\nabla\xi dX \leq 0 \\ \qquad\qquad \forall \xi \in W^{1,q}(\Omega), \quad \xi = 0 \text{ on } S_3, \quad \xi \geq 0 \text{ on } S_2. \end{cases} \tag{P}$$

For the existence of a solution of (P) under the assumptions (1.2), we refer the reader to [Ly3] where an existence result is given for generalized boundary conditions. The reader can also adapt the proof in [CaL] obtained for the case $\mathcal{A}(X,\xi) = |\xi|^{q-2}\xi$. We also mention that the same problem with leaky boundary conditions is studied in [Ly4] under the assumption $\mathcal{A}(X,e) = k(X)e$ with $\dfrac{\partial k}{\partial y} \geq 0$.

## 2. Properties of the Solution

Arguing as in [Ly1], [ChiL] or [Ly4], we obtain

**Proposition 2.1.** *Let $(u,g)$ be a solution of $(P)$. Then we have*

$$div\big(\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)\big) = 0 \qquad in\ \mathcal{D}'(\Omega). \tag{2.1}$$

*Moreover if $div\big(\mathcal{A}(X,e)\big) \geq 0$ in $\mathcal{D}'(\Omega)$, we obtain*

$$div\big(\mathcal{A}(X,\nabla u)\big) = div\big(g\mathcal{A}(X,e)\big) \geq 0 \qquad in\ \mathcal{D}'(\Omega). \tag{2.2}$$

From now on, we shall assume that

- $\mathcal{A}(.,e) = (a^1(.),a^2(.)) \in C^1(\bar\Omega)$ (2.3)
- $div\big(\mathcal{A}(X,e)\big) \geq 0$ in $\mathcal{D}'(\Omega)$ (2.4)

. There exists points $P_1, P_2, ...., P_K \in \Gamma = \partial\Omega$,
local coordinates $(x_1^k, x_2^k), (k = 1, ..., K)$ in the neighborhood of each $P_k$,
and $C^1$-diffeomorphisms $\quad \psi_k : U_k = (-\alpha_k, \alpha_k) \times (-r, r) \longrightarrow V_k = \psi_k(U_k)$

$$(x_1^k, x_2^k) \longmapsto (x_1^k, x_2^k + \sigma_k(x_1^k))$$

such that if $\quad U_k^+ = (-\alpha_k, \alpha_k) \times (0, r), \quad U_k^- = (-\alpha_k, \alpha_k) \times (-r, 0), \quad U_k^0 = (-\alpha_k, \alpha_k) \times \{0\}$,

$$\text{then } \psi_k(U_k^+) \subset \Omega, \quad \psi_k(U_k^-) \subset \mathbb{R}^2 \setminus \Omega, \quad \Gamma_k = \psi_k(U_k^0) \subset \Gamma \text{ and } \bigcup_{k=1}^{k=K} \Gamma_k = \Gamma \tag{2.5}$$

$$. \ \forall k = 1, ..., K \quad \sigma_k'(x_1^k) a^1(x_1^k, \sigma_k(x_1^k)) - a^2(x_1^k, \sigma_k(x_1^k)) \neq 0 \qquad \forall x_1^k \in (-\alpha_k, \alpha_k). \tag{2.6}$$

Then we consider the following differential system

$$\begin{cases} X'(t, \omega, h) = \mathcal{A}(X(t, \omega, h), e) \\ X(0, \omega, h) = (\omega, h) \end{cases} \tag{$E(\omega, h)$}$$

where $h \in \pi_y(\Omega)$ and $\omega \in \pi_x(\Omega \cap [y = h])$.
By the classical theory of ordinary differential equations there exists a unique maximal solution $X(., \omega, h)$ of $E(\omega, h)$ which is defined on $[\alpha(\omega, h), \beta(\omega, h)]$ with $X(\alpha(\omega, h), \omega, h) \in \partial\Omega \cap ([y < h]), X(\beta(\omega, h), \omega, h) \in \partial\Omega \cap ([y > h])$ (see Figure 2). For simplicity we will denote in the sequel $X(t, \omega, h)$, $\alpha(\omega, h)$ and $\beta(\omega, h)$ respectively by $X(t, \omega)$, $\alpha(\omega)$ and $\beta(\omega)$. We note that (2.6) means that the orbits of $E(\omega, h)$ don't cross $\partial\Omega$ tangentially.

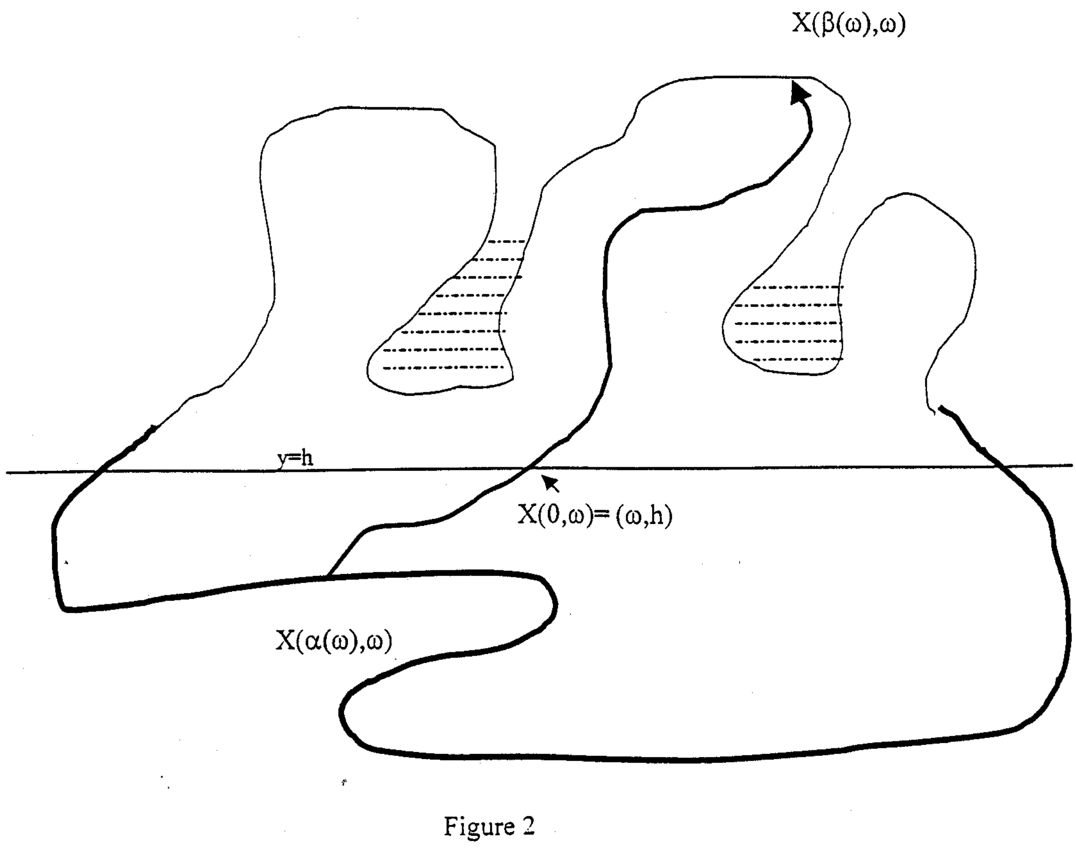


Figure 2

Moreover under the assumptions (2.3) and (2.6), one can prove

**Proposition 2.2.** $\alpha, \beta \in C^1(\pi_x(\Omega \cap [y = h]))$.

**Proof.** Let $h \in \pi_y(\Omega)$ and $\omega_0 \in \pi_x(\Omega \cap [y = h])$. There exists $k \in \{1, ..., K\}$ such that $X(\alpha(\omega_0), \omega_0) \in \Gamma_k$. By continuity there exists $\eta > 0$ small enough such that $X(\alpha(\omega), \omega) \in \Gamma_k \quad \forall \omega \in (\omega_0 - \eta, \omega_0 + \eta)$. Consequently we have $\sigma_k(X_1(\alpha(\omega), \omega)) = X_2(\alpha(\omega), \omega)$ for all $\omega \in (\omega_0 - \eta, \omega_0 + \eta)$. This means that $\alpha(\omega)$ satisfies $F(\alpha(\omega), \omega) = 0$ for all $\omega \in (\omega_0 - \eta, \omega_0 + \eta)$, where $F$ is given by : $F = \sigma_k o X_1 - X_2$.
Taking into account (2.3) there exists an open set $\Omega^*$ containing $\bar{\Omega}$ such that $\mathcal{A}(., e) \in C^1(\Omega^*)$. Then for each $\omega \in \pi_x(\Omega^* \cap [y = h])$, there exists a unique maximal solution $X^*(., \omega)$ of the differential system $(E(\omega, h))$ defined on $[\alpha^*(\omega), \beta^*(\omega)]$. Obviously we have $X^*_{|(\alpha(\omega),\beta(\omega))} = X$ when $\omega \in \pi_x(\Omega \cap [y = h])$.
Let $F^* = \sigma_k o X_1^* - X_2^*$ defined on $D^* = \{(t, \omega) \,/\, \omega \in (\omega_0 - \eta, \omega_0 + \eta),\, t \in (\alpha^*(\omega), \beta^*(\omega))\}$. We have $F^* \in C^1(D^*)$ since $X_i^* \in C^1(D^*)$ and $\sigma_k \in C^1(-\alpha_k, \alpha_k)$. In addition $F^*$ is a $C^1$ extension of $F$ to $D^*$ and we have

$$\begin{aligned}\frac{\partial F^*}{\partial t}(t, \omega) &= \sigma_k'(X_1^*(t, \omega)).\frac{\partial X_1^*}{\partial t}(t, \omega) - \frac{\partial X_2^*}{\partial t}(t, \omega)\\ &= \sigma_k'(X_1^*(t, \omega)).a^1(X_1^*(t, \omega)) - a^2(X_1^*(t, \omega))\end{aligned}$$

and

$$\frac{\partial F^*}{\partial t}(\alpha(\omega_0), \omega_0) = \sigma_k'(X_1(\alpha(\omega_0), \omega_0)).a^1(X_1(\alpha(\omega_0), \omega_0)) - a^2(X_1(\alpha(\omega_0), \omega_0)) \neq 0.$$

Therefore by the implicit function theorem, we deduce that there exists $\delta \in (0, \eta)$ and a unique function $f : (\omega_0 - \delta, \omega_0 + \delta) \to I\!R$ such that

$$\begin{aligned}&F^*(t, \omega) = 0 \Leftrightarrow t = f(\omega)\\ &f(\omega_0) = \alpha(\omega_0) \quad \text{and} \quad f \in C^1(\omega_0 - \delta, \omega_0 + \delta).\end{aligned}$$

Since $F^*(\alpha(\omega), \omega) = F(\alpha(\omega), \omega) = 0$, it follows that $\alpha(\omega) = f(\omega)$ and $\alpha \in C^1(\omega_0 - \delta, \omega_0 + \delta)$. Thus $\alpha \in C^1(\pi_x(\Omega \cap [y = h]))$.
In the same way we prove that $\beta \in C^1(\pi_x(\Omega \cap [y = h]))$. □

For each $h \in \pi_y(\Omega)$ we define the set

$$D_h = \{(t, \omega) \,/\, \omega \in \pi_x(\Omega \cap [y = h]),\, t \in (\alpha(\omega), \beta(\omega))\}$$

and consider the mappings $T_h$ and $S_h$ defined by :

$$T_h : \quad \begin{aligned} D_h &\longrightarrow T_h(D_h)\\ (t, \omega) &\longmapsto T_h(t, \omega) = X(t, \omega) = (T_h^1, T_h^2)(t, \omega)\end{aligned}$$

$$S_h : \quad \begin{aligned} D_h &\longrightarrow S_h(D_h)\\ (t, \omega) &\longmapsto S_h(t, \omega) = (\omega, L_h(t, \omega)) = (\omega, \tau)\end{aligned}$$

where $L_h(t, \omega) = \displaystyle\int_{\alpha(\omega)}^t |\mathcal{A}(X(s, \omega), e)| ds = \int_{\alpha(\omega)}^t |X'(s, \omega)| ds$ represents the arc Length of the curve $X(., \omega)$ from the point $X(\alpha(\omega), \omega)$ to the point $X(t, \omega)$. Then we have

**Proposition 2.3.**

$$\Omega = \bigsqcup_{h \in \pi_y(\Omega)} T_h(D_h), \quad T_h \quad \textit{and} \quad S_h \textit{ are } C^1 \textit{ diffeomorphisms.}$$

**Proof.** First for each $(x, y) \in \Omega$ we have $(x, y) = X(0, \omega) = T_h(0, \omega)$ with $\omega = x$ and $h = y$.
Next thanks to (2.3) we have $T_h \in C^1(D_h)$. By Proposition 2.2, $S_h$ is also in $C^1(S_h(D))$. To see that they are diffeomorphisms, it suffices to verify that $det(\mathcal{J}T_h)$ and $det(\mathcal{J}S_h)$ do not vanish. We denote by $\mathcal{J}F$ the Jacobian matrix of the transformation $F$ and by $det(\mathcal{J}F)$ the determinant of $\mathcal{J}F$.

First one can check that :

$$
\begin{aligned}
&det\mathcal{J}S_h = -|\mathcal{A}(X(t,\omega),e)| < 0\\
&Y_h(t,\omega) = det(\mathcal{J}T_h) = a^1(X(t,\omega))\frac{\partial X_2}{\partial\omega} - a^2(X(t,\omega))\frac{\partial X_1}{\partial\omega},\\
&\frac{\partial Y_h}{\partial t}(t,\omega) = Y_h(t,\omega).\{div(\mathcal{A}(.,e))\}(X(t,\omega)).
\end{aligned}
$$

Therefore

$$Y_h(t,\omega) = Y_h(0,\omega).\exp(\int_0^t \{div(\mathcal{A}(.,e))\}(X(s,\omega))ds).$$

Since $Y_h(0,\omega) = -a^2(X(0,\omega)) < 0$, we get $Y_h(t,\omega) < 0\ \forall t \in (\alpha(\omega),\beta(\omega))$, $\omega \in \pi_x(\Omega \cap [y=h])$. □

**Theorem 2.4.** *Let $(u,g)$ be a solution of $(P)$. We have for each $h \in \pi_y(\Omega)$*

$$\frac{\partial}{\partial\tau}(goT_hoS_h^{-1}.(-YoS_h^{-1})) \geq 0 \qquad in\ \mathcal{D}'(S_h(D_h)).$$

**Proof.** Let $\phi \in \mathcal{D}(S_h(D_h))$, $\phi \geq 0$. Then $\phi oS_hoT_h^{-1} \in C_0^1(T_h(D_h))$ and by (2.2), we have

$$\int_{T_h(D_h)} g\mathcal{A}(X,e)\nabla(\phi oS_hoT_h^{-1})dX \leq 0.$$

Using the change of variables $T_h(t,\omega) = (x,y)$ and the fact that

$$\mathcal{A}(X(t,\omega),e)\big(\nabla(\phi oS_hoT_h^{-1})\big)oT_h.(-Y_h(t,\omega)) = -Y_h(t,\omega)\frac{\partial}{\partial t}(\phi oS_h)$$

we get

$$\int_{D_h} goT_h(t,\omega).(-Y_h(t,\omega).\frac{\partial}{\partial t}(\phi oS_h)dtd\omega \leq 0.$$

Now if we use the change of variables $S_h^{-1}$, we get

$$\int_{S_h(D_h)} goT_hoS_h^{-1}(\omega,\tau).(-Y_hoS_h^{-1}(\omega,\tau)).\big(\frac{\partial}{\partial t}(\phi oS_h)\big)oS_h^{-1}.|det\mathcal{J}S_h^{-1}|d\omega d\tau \leq 0.$$

Taking into account that

$$\big(\frac{\partial}{\partial t}(\phi oS_h)\big)oS_h^{-1} = \frac{\partial\phi}{\partial\tau}.|\mathcal{A}(.,e)|oT_hoS_h^{-1}(\omega,\tau) = \frac{\partial\phi}{\partial\tau}.|det\mathcal{J}S_h|,$$

we obtain

$$\int_{S_h(D_h)} goT_hoS_h^{-1}(\omega,\tau).(-Y_hoS_h^{-1}(\omega,\tau)).\frac{\partial\phi}{\partial\tau}d\omega d\tau \leq 0.$$

□

In what follows we assume that there exist nonnegative constants $\kappa$, $\sigma$ and positive constants $\lambda_0$, $\lambda_1$ with $\sigma \leq 1$ and $\lambda_1 \geq \lambda_0$ such that for all $X, Y \in \Omega$, $\zeta, \xi \in \mathbb{R}^2$

$$\sum_{i,j}\frac{\partial\mathcal{A}^i}{\partial\zeta_j}(X,\zeta)\xi_i\xi_j \geq \lambda_0(\kappa + |\zeta|^{q-2})|\xi|^2, \qquad (2.7)$$

$$\left|\frac{\partial \mathcal{A}^i}{\partial \zeta_j}(X,\zeta)\right| \leq \lambda_1(\kappa + |\zeta|^{q-2}), \tag{2.8}$$

and when $q \neq 2$, we assume

$$|\mathcal{A}(X,\zeta) - \mathcal{A}(Y,\zeta)| \leq \lambda_1(1 + |\zeta|^{q-1})(|x-y|^{\sigma}). \tag{2.9}$$

**Remark 2.5** Note that assumptions (2.7)-(2.8) are satisfied in the case where $\mathcal{A}(X,\zeta) = |\zeta|^{q-2}\zeta$. Moreover under assumptions (2.7)-(2.8), we deduce from (2.1) (see [R] [Du]) that $u \in C^{0,\gamma}_{loc}(\Omega \cup S_2 \cup S_3)$ for some $\gamma \in (0,1)$. Also by (2.1) and $(P)i)$ we have $div(\mathcal{A}(X,u)) = 0$ in $\mathcal{D}'([u > x_2])$. It follows then by (2.7)-(2.9)(see [Di] [Le] for example) that $u \in C^{1,\delta}_{loc}([u > x_2])$ for some $\delta \in (0,1)$.

We need the following maximum principle (see [Da])

**Lemma 2.6.** *(Strong maximum principle)*
*Let $u_1$ and $u_2$ be two functions defined on a domain $D$ of $\mathbb{R}^2$ such that $u_1, u_2 \in C^1(D)$, $u_1 \geq u_2$ in $D$, the set $\{X \in D/\, \nabla u_1(X) = \nabla u_2(X) = 0\}$ is empty and $div\big(\mathcal{A}(X,\nabla u_1) - \mathcal{A}(X,\nabla u_2)\big) \leq 0$. Then we have either*

$$u_1 = u_2 \ \text{ in } D \quad \text{or} \quad u_1 > u_2 \ \text{ in } D.$$

The following theorem will allow us to define the free boundary $\partial([u > y]) \cap \Omega$ locally as a curve.

**Theorem 2.7.** *Let $(u,g)$ be a solution of $(P)$ and let $X_0 = T_h o S_h^{-1}(\omega_0,\tau_0) = (x_0,y_0) \in \Omega$.*
*i) If $u(X_0) = uoT_h o S_h^{-1}(\omega_0,\tau_0) > y_0$ then there exists $\epsilon > 0$ such that*

$$uoT_h o S_h^{-1}(\omega,\tau) > T_h^2 o S_h^{-1}(\omega,\tau) \qquad \forall(\omega,\tau) \in C_\epsilon$$

*where $C_\epsilon = \{(\omega,\tau) \in S_h(D_h)\,/\,|\omega - \omega_0| < \epsilon,\ \tau < \tau_0 + \epsilon\}$ i.e.*

$$u(x,y) > y \qquad \forall(x,y) \in T_h o S_h^{-1}(C_\epsilon).$$

*ii) If $u(X_0) = y_0 = uoT_h o S_h^{-1}(\omega_0,\tau_0) = T_h^2 o S_h^{-1}(\omega_0,\tau_0)$ then*

$$uoT_h o S_h^{-1}(\omega_0,\tau) = T_h^2 o S_h^{-1}(\omega_0,\tau) \qquad \forall \tau \geq \tau_0.$$

**Proof.** $i)$ By continuity, there exists $\epsilon > 0$ such that

$$uoT_h o S_h^{-1}(\omega,\tau) > T_h^2 o S_h^{-1}(\omega,\tau) \qquad \forall(\omega,\tau) \in (\omega_0 - \epsilon, \omega_0 + \epsilon) \times (\tau_0 - \epsilon, \tau_0 + \epsilon) = Q_\epsilon.$$

Then $goT_h o S_h^{-1}(\omega,\tau) = 0$ for a.e. $(\omega,\tau) \in Q_\epsilon$. By Theorem 2.4 and since $-Y_h o S_h^{-1} > 0$, $g \geq 0$, we get $goT_h o S_h^{-1} = 0$ a.e. in $C_\epsilon$, i.e. $g = 0$ a.e. in $T_h o S_h^{-1}(C_\epsilon)$ (see Figure 3).
By (2.1) we have $div(\mathcal{A}(X,\nabla u)) = div(g\mathcal{A}(X,e)) = 0$ in $\mathcal{D}'(T_h o S_h^{-1}(C_\epsilon))$. Since $div(\mathcal{A}(X,\nabla y)) \geq 0$ in $\mathcal{D}'(\Omega)$, $\nabla y = e \neq 0$, $u \geq y$ in $T_h o S_h^{-1}(C_\epsilon)$ and $u > y$ in $T_h o S_h^{-1}(Q_\epsilon)$, then by Lemma 2.6, we get

$$u > y \qquad \text{in } T_h o S_h^{-1}(C_\epsilon).$$

*ii)* Assume that there exists $\tau_1 > \tau_0$ such that $uoT_h oS_h^{-1}(\omega_0,\tau_1) > T_h^2 oS_h^{-1}(\omega_0,\tau_1)$. then by i), there exists $\epsilon > 0$ such that $uoT_h oS_h^{-1}(\omega,\tau) > T_h^2 oS_h^{-1}(\omega,\tau)$ $\forall(\omega,\tau) \in \{(\omega,\tau) \in S_h(D_h) \,/\, |\omega-\omega_0| < \epsilon,\ \tau < \tau_1+\epsilon\}$ then $uoT_h oS_h^{-1}(\omega_0,\tau_0) > T_h^2 oS_h^{-1}(\omega_0,\tau_0)$ which is a contradiction.

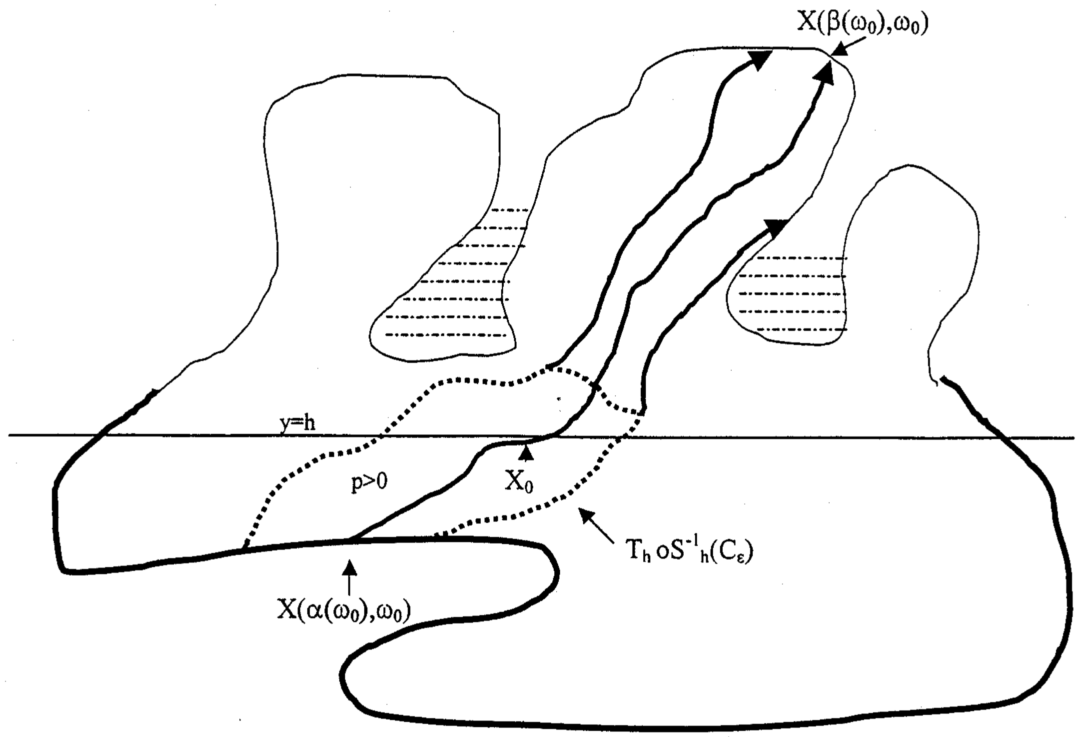


Figure 3

□

**Remark 2.8.** *i)* The result of theorem 2.7 means that if a point $X_0$ is in the wet region, then the part of the curve $X(.,\omega)$ solution of $E(\omega,h)$ and passing through $X_0$ at $t_0$ remains in the wet region for all $t \le t_0$.
*ii)* In [Ly1], [ChiL] and [Ly4] we assumed that $\mathcal{A}(X,e) = k(X)e$ which leads to $X_1'(t) = 0$ for all $t$ and the curve $X(.,\omega)$ is a vertical segment. Therefore the free boundary is represented by a curve of the form $y = \Phi(x)$.
*iii)* We have $u = p+y = \varphi+y > y$ on $S_{3,i}$ ($\varphi > 0$ on $S_3$), $i = 1,...,N$ and $u \in C^0(\Omega \cup S_3)$, then $u > y$ below $S_3$ in the following sense :

$$u(X(t,\omega)) > X_2(t,\omega) \qquad \forall t \in [\alpha(\omega),\beta(\omega)] \quad \text{such that} \quad X(\beta(\omega).\omega) \in S_3.$$

Now for each $h \in \pi_y(\Omega)$ we define the function $\Phi_h$ on $\pi_x(\Omega \cap [y=h])$ by :

$$\Phi_h(\omega) = \begin{cases} \sup\{\tau/\,(\omega,\tau) \in S_h(D_h),\ \ uoT_h oS_h^{-1}(\omega,\tau) > T_h^2 oS_h^{-1}(\omega,\tau)\} \\ \qquad\qquad\qquad\qquad\qquad \text{if this set is not empty} \\ 0 \qquad\qquad\qquad\qquad\quad \text{otherwise.} \end{cases}$$

$\Phi_h$ is well defined and we have :

**Proposition 2.9.** *$\Phi_h$ is lower semi-continuous on $\pi_x(\Omega \cap [y = h])$. Moreover*

$$[poT_h oS_h^{-1}(\omega, \tau) > 0] = [uoT_h oS_h^{-1}(\omega, \tau) > T_h^2 oS_h^{-1}(\omega, \tau)] = [\tau < \Phi_h(\omega)].$$

**Proof.**
$*$ Let $\omega_0 \in \pi_x(\Omega \cap [y = h])$
if $\Phi_h(\omega_0) = 0$ then for $\epsilon > 0$, $\Phi_h(\omega) \geq 0 > \Phi_h(\omega_0) - \epsilon \quad \forall \omega$,
if $\Phi_h(\omega_0) > 0$ then for $\epsilon > 0$, there exists $\tau_\epsilon > 0$ such that

$$\Phi_h(\omega_0) \geq \tau_\epsilon > \Phi_h(\omega_0) - \epsilon/2 \quad \text{and}$$
$$(uoT_h oS_h^{-1} - T_h^2 oS_h^{-1})(\omega_0, \tau_\epsilon) = poT_h oS_h^{-1}(\omega_0, \tau_\epsilon) > 0.$$

There exists a unique $t_\epsilon \in (\alpha(\omega_0), \beta(\omega_0))$ such that $\tau_\epsilon = L_h(t_\epsilon, \omega_0)$.
We can find (see Theorem 3.4 p 24 in [H]) $\eta_1 > 0$ such that

$$X(t, \omega) \text{ exists } \forall (t, \omega) \in [\alpha(\omega_0), \beta(\omega_0)] \times (\omega_0 - \eta_1, \omega_0 + \eta_1)$$
$$\text{and } (t, \omega) \longmapsto X(t, \omega) \text{ is continuous.}$$

Then by continuity, there exists $0 < \eta_2 < \eta_1$ such that

$$p(X(t, \omega)) > 0 \qquad \forall (t, \omega) \in (t_\epsilon - \eta_2, t_\epsilon + \eta_2) \times (\omega_0 - \eta_2, \omega_0 + \eta_2).$$

By Theorem 2.7, we deduce that

$$p(X(t, \omega)) > 0 \qquad \forall (t, \omega) \in (\alpha(\omega), t_\epsilon + \eta_2) \times (\omega_0 - \eta_2, \omega_0 + \eta_2).$$

Using the definition of $\Phi_h$ we have for $\omega \in (\omega_0 - \eta_2, \omega_0 + \eta_2)$

$$\begin{aligned}\Phi_h(\omega) &= \sup\{\tau \,/\, (\omega, \tau) \in S_h(D_h) \text{ and } poT_h oS_h^{-1}(\omega, \tau) > 0\} \\ &= \sup_{t \in (\alpha(\omega), \beta(\omega))} \{L_h(t, \omega) \,/\, (t, \omega) \in D_h \text{ and } p(X(t, \omega)) > 0\} \\ &\geq L_h(t_\epsilon, \omega).\end{aligned}$$

Since $L_h(t_\epsilon, \omega) = \displaystyle\int_{\alpha(\omega,h)}^{t_\epsilon} |\mathcal{A}(X(s, \omega), e)| ds$ is continuous with respect to $\omega$, there exists $0 < \eta < \eta_2$ such that

$$\begin{aligned}\forall \omega \in (\omega_0 - \eta, \omega_0 + \eta), \quad L_h(t_\epsilon, \omega) &= \int_{\alpha(\omega)}^{t_\epsilon} |\mathcal{A}(X(s, \omega), e)| ds \\ &\geq \int_{\alpha(\omega_0)}^{t_\epsilon} |\mathcal{A}(X(s, \omega_0), e)| ds - \frac{\epsilon}{2} = L_h(t_\epsilon, \omega_0) - \frac{\epsilon}{2} \\ &= \tau_\epsilon - \frac{\epsilon}{2} > \Phi_h(\omega_0) - \frac{\epsilon}{2} - \frac{\epsilon}{2}\end{aligned}$$

thus $\Phi_h(\omega) \geq \Phi_h(\omega_0) - \epsilon \quad \forall \omega \in (\omega_0 - \eta, \omega_0 + \eta)$.

$**$ Let $(\omega_0, \tau_0) \in [\tau < \Phi(\omega)]$. Assume that $poT_h oS_h^{-1}(\omega_0, \tau_0) = 0$ then by Theorem 2.7, $poT_h oS_h^{-1}(\omega_0, \tau) = 0$ $\forall \tau \geq \tau_0$. So $\Phi_h(\omega_0) = \sup\{\tau \,/\, (\omega_0, \tau) \in S_h(D_h)$ and $poT_h oS_h^{-1}(\omega_0, \tau) > 0\} \leq \tau_0$, which is a contradiction. Now let $(\omega_0, \tau_0) \in [poT_h oS_h^{-1}(\omega, \tau) > 0]$. By continuity, there exists $\eta > 0$ such that $poT_h oS_h^{-1}(\omega_0, \tau) > 0$ $\forall \tau \in (\tau_0 - \eta, \tau_0 + \eta)$. By Theorem 2.7, we deduce that : $poT_h oS_h^{-1}(\omega_0, \tau) > 0$ $\forall \tau < \tau_0 + \eta$ such that $(\omega_0, \tau) \in S_h(D_h)$. Then $\Phi_h(\omega_0) \geq \tau_0 + \eta > \tau_0$. Hence $(\omega_0, \tau) \in [\tau < \Phi_h(\omega)]$. □

The following theorem will play an important role in the proof of the continuity of the free boundary.

**Theorem 2.10.** *Let $(u,g)$ be a solution of $(P)$. Let $(\omega_1,\tau_0),(\omega_2,\tau_0)\in S_h(D_h)$ with $\omega_1<\omega_2$ and*

$$poT_hoS_h^{-1}(\omega_i,\tau_0)=0 \qquad \forall(\omega_i,\tau)\in S_h(D_h), \quad \tau>\tau_0\geq 0.$$

*Set $Z_{\tau_0}=T_hoS_h^{-1}\big((\omega_1,\omega_2)\times(\tau_0,+\infty)\cap S_h(D_h)\big)$ and assume that $\bar{Z}_{\tau_0}\cap S_3=\emptyset$. Let $y_0\in I\!R$ such that $D_{y_0,\tau_0}=[y>y_0]\cap Z_{\tau_0}\neq\emptyset$. Then we have :*

$$\int_{D_{y_0,\tau_0}}\big(\mathcal{A}(X,\nabla u)-g\mathcal{A}(X,e)\big).edX\ \leq\ 0.$$

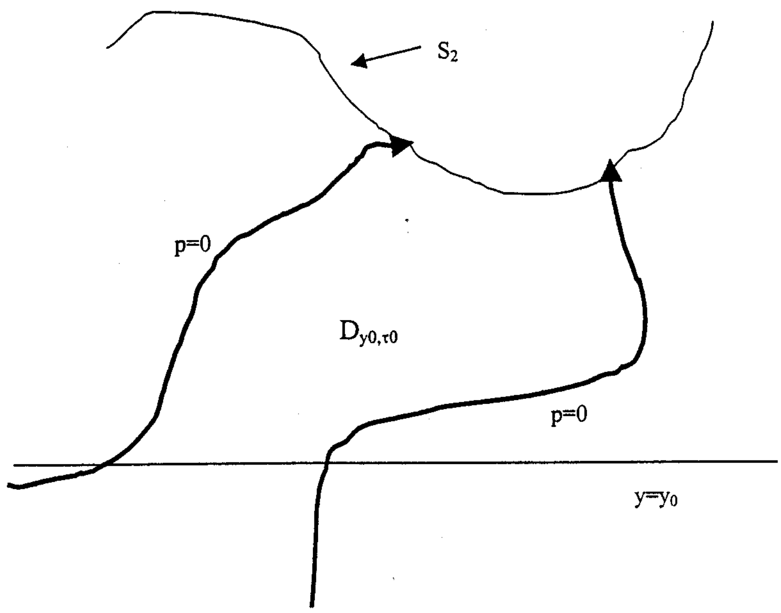


Figure 4

To prove this theorem, we need the following lemma :

**Lemma 2.11.** *Under the assumptions of Theorem 2.10, we have*

$$\int_{D_{y_0,\tau_0}}\big(\mathcal{A}(X,\nabla u)-\chi([u=y])\mathcal{A}(X,e)\big).\nabla\zeta dX$$
$$\leq\int_{\omega_1}^{\omega_2}(-Y_hoS_h^{-1})(s,\Phi_h(s)).\zeta oT_hoS_h^{-1}(s,\Phi_h(s))ds$$
$$\forall\zeta\in W^{1,q}(D_{y_0,\tau_0})\cap C^0(\bar{D}_{y_0,\tau_0}),\quad \zeta\geq 0,\quad \zeta(x,y_0)=0\quad\forall(x,y_0)\in\bar{D}_{y_0,\tau_0}.$$

**Proof.** For $\epsilon>0$, $\xi=\chi(D_{y_0,\tau_0})\min\big(\frac{u-y}{\epsilon},\zeta\big)$ is a test function for $(P)$, so we have

$$\int_{D_{y_0,\tau_0}}\mathcal{A}(X,\nabla u).\nabla\Big(\frac{u-y}{\epsilon}\wedge\zeta\Big)dX\leq\int_{D_{y_0,\tau_0}}g\mathcal{A}(X,e).\nabla\Big(\frac{u-y}{\epsilon}\wedge\zeta\Big)dX=0$$

since $g.(u-y)=0$ a.e. in $\Omega$.
Using the monotonicity of $\mathcal{A}$, we get :

$$\int_{D_{y_0,\tau_0}\cap[u-y\geq\epsilon\zeta]}(\mathcal{A}(X,\nabla u)-\mathcal{A}(X,e)).\nabla\zeta dX\ \leq -\int_{D_{y_0,\tau_0}}\mathcal{A}(X,e).\nabla(\frac{u-y}{\epsilon}\wedge\zeta)dX=-I_\epsilon.$$

Note that we have

$$\begin{aligned}I_\epsilon&=\int_{D_{y_0,\tau_0}}\chi([u>y])\mathcal{A}(X,e).\nabla(\frac{u-y}{\epsilon}\wedge\zeta)dX\\&=\int_{D_{y_0,\tau_0}}\chi([u>y])\mathcal{A}(X,e).\nabla\zeta dX-\int_{D_{y_0,\tau_0}}\mathcal{A}(X,e).\nabla(\zeta-\frac{u-y}{\epsilon})^+\chi([u>y])dX\\&=I^1-I^2_\epsilon.\end{aligned}$$

Now using the change of variables $T_h$, we obtain

$$I^2_\epsilon=\int_{T_h^{-1}(D_{y_0,\tau_0})}\chi([poT_h(t,\omega)>0])\mathcal{A}(X(t,\omega),e).(\nabla(\zeta-\frac{p}{\epsilon})^+oT_h)|Y_h(t,\omega)|dtd\omega.$$

Since we have

$$\mathcal{A}(X(t,\omega),e).(\nabla(\zeta-\frac{p}{\epsilon})^+oT_h).|Y_h(t,\omega)|=-Y_h(t,\omega).\frac{\partial}{\partial t}((\zeta-\frac{p}{\epsilon})^+(X(t,\omega,h)),$$

this leads to

$$I^2_\epsilon=-\int_{T_h^{-1}(D_{y_0,\tau_0})}\chi([poT_h(t,\omega)>0]).Y_h(t,\omega).\frac{\partial}{\partial t}((\zeta-\frac{p}{\epsilon})^+(X(t,\omega,h))dtd\omega.$$

Next using the change of variables $S_h^{-1}$, we get

$$\begin{aligned}I^2_\epsilon=&-\int_{S_hoT_h^{-1}(D_{y_0,\tau_0})}\chi([poT_hoS_h^{-1}(\omega,\tau)>0]).Y_hoS_h^{-1}(\omega,\tau)\\&\qquad.(\frac{\partial}{\partial t}((\zeta-\frac{p}{\epsilon})^+(X(t,\omega,h)))oS_h^{-1}(\omega,\tau).|det\mathcal{J}S_h^{-1}|d\omega d\tau\\=&-\int_{S_hoT_h^{-1}(D_{y_0,\tau_0})}\chi([poT_hoS_h^{-1}(\omega,\tau)>0]).Y_hoS_h^{-1}(\omega,\tau).\frac{\partial}{\partial\tau}((\zeta-\frac{p}{\epsilon})^+oT_hoS_h^{-1})(\omega,\tau)d\omega d\tau\end{aligned}$$

since we have

$$(\frac{\partial}{\partial t}((\zeta-\frac{p}{\epsilon})^+oT_h)oS_h^{-1}(\omega,\tau)=\frac{\partial}{\partial\tau}((\zeta-\frac{p}{\epsilon})^+oT_hoS_h^{-1})(\omega,\tau).|\mathcal{A}(T_hoS_h^{-1}(\omega,\tau),e)|.$$

Then

$$\begin{aligned}I^2_\epsilon=-\int_{(\omega_1,\omega_2)\times(\tau_0,+\infty)\cap S_h(D_h)\cap S_hoT_h^{-1}([y>y_0])}&\chi([\tau<\Phi_h(\omega)]).Y_hoS_h^{-1}(\omega,\tau)\\&.\frac{\partial}{\partial\tau}((\zeta-\frac{p}{\epsilon})^+oT_hoS_h^{-1})(\omega,\tau)d\omega d\tau.\end{aligned}$$

Note that $\forall\omega\in(\omega_1,\omega_2)$, $\exists!t_{y_0}(\omega)$ such that $X_2(t_{y_0}(\omega),\omega)=y_0$, $\tau_{y_0}(\omega)=\int_{\alpha(\omega)}^{t_{y_0}(\omega)}|\mathcal{A}(X(s,\omega),e)|ds$ and we can see that we have

$$\begin{aligned}&(\omega_1,\omega_2)\times(\tau_0,+\infty)\cap S_h(D_h)\cap S_hoT_h^{-1}([y>y_0])\\&\quad=\{(\omega,\tau)\in S_h(D_h),\ \omega\in(\omega_1,\omega_2)/\ \tau>\sup(\tau_0,\tau_{y_0}(\omega))\}.\end{aligned}$$

It follows that

$$I_\epsilon^2 = \int_{\omega_1}^{\omega_2} \int_{\sup(\tau_0, \tau_{y_0}(\omega))}^{\Phi_h(\omega)} (-Y_h o S_h^{-1})(\omega, \tau). \frac{\partial}{\partial \tau}((\zeta - \frac{p}{\epsilon})^+ o T_h o S_h^{-1})(\omega, \tau) d\omega d\tau.$$

By the second mean value theorem we have

$$\int_{\sup(\tau_0, \tau_{y_0}(\omega))}^{\Phi_h(\omega)} (-Y_h o S_h^{-1})(\omega, \tau). \frac{\partial}{\partial \tau}((\zeta - \frac{p}{\epsilon})^+ o T_h o S_h^{-1})(\omega, \tau) d\tau$$
$$= (-Y_h o S_h^{-1})(\omega, \Phi_h(\omega)) \int_{\tau^*(y_0,\omega)}^{\Phi_h(\omega)} \frac{\partial}{\partial \tau}((\zeta - \frac{p}{\epsilon})^+ o T_h o S_h^{-1})(\omega, \tau) d\tau$$

for almost all $\omega \in (\omega_1, \omega_2)$, where $\tau^*(y_0, \omega) \in [\tau_{y_0}(\omega), \Phi_h(\omega)]$. Moreover, we have

$$\int_{\tau^*(y_0,\omega)}^{\Phi_h(\omega)} \frac{\partial}{\partial \tau}((\zeta - \frac{p}{\epsilon})^+ o T_h o S_h^{-1})(\omega, \tau) d\tau \leq \zeta o T_h o S_h^{-1}(\omega, \Phi_h(\omega)).$$

So we deduce that

$$I_\epsilon^2 \leq \int_{\omega_1}^{\omega_2} (-Y_h o S_h^{-1})(\omega, \Phi_h(\omega)).\zeta o T_h o S_h^{-1}(\omega, \Phi_h(\omega)) d\omega.$$

Thus

$$\int_{D_{y_0,\tau_0} \cap [u - y \geq \epsilon\zeta]} (\mathcal{A}(X, \nabla u) - \mathcal{A}(X, e)).\nabla\zeta dX + \int_{D_{y_0,\tau_0}} \chi([u > y])\mathcal{A}(X, e).\nabla\zeta dX$$
$$\leq \int_{\omega_1}^{\omega_2} (-Y_h o S_h^{-1})(\omega, \Phi(\omega)).\zeta o T_h o S_h^{-1}(\omega, \Phi_h(\omega)) d\omega$$

and the lemma follows by letting $\epsilon$ go to 0. □

**Proof of Theorem 2.10.**

Let $\epsilon > 0$ and $\theta_\epsilon(\omega) = \min\big(\frac{(\omega - \omega_1)^+}{\epsilon}, 1\big).\min\big(\frac{(\omega_2 - \omega)^+}{\epsilon}, 1\big)$, $h_\epsilon = \theta_\epsilon o S_h o T_h^{-1}$. Since $\chi(D_{y_0,\tau_0}) h_\epsilon(y - y_0)$ is a test function for $(P)$ we have

$$\int_{D_{y_0,\tau_0}} (\mathcal{A}(X, \nabla u) - g\mathcal{A}(X, e)).e dX = \int_{D_{y_0,\tau_0}} (\mathcal{A}(X, \nabla u) - g\mathcal{A}(X, e)).\nabla(y - y_0) dX$$
$$= \int_{D_{y_0,\tau_0}} (\mathcal{A}(X, \nabla u) - g\mathcal{A}(X, e)).\nabla(h_\epsilon(y - y_0)) dX$$
$$+ \int_{D_{y_0,\tau_0}} (\mathcal{A}(X, \nabla u) - g\mathcal{A}(X, e)).\nabla((1 - h_\epsilon)(y - y_0)) dX$$
$$\leq \int_{D_{y_0,\tau_0}} (\mathcal{A}(X, \nabla u) - g\mathcal{A}(X, e)).\nabla((1 - h_\epsilon)(y - y_0)) dX = J_\epsilon.$$

Note that

$$J_\epsilon = \int_{D_{y_0,\tau_0}} (\mathcal{A}(X, \nabla u) - \chi([u = y])\mathcal{A}(X, e)).\nabla((1 - h_\epsilon)(y - y_0)) dX$$
$$+ \int_{D_{y_0,\tau_0}} (\chi([u = y]) - g)\mathcal{A}(X, e).\nabla((1 - h_\epsilon)(y - y_0)) dX = J_\epsilon^1 + J_\epsilon^2.$$

By Lemma 2.11 with $\zeta = (1 - h_\epsilon).(y - y_0) = (1 - \theta_\epsilon o S_h o T_h^{-1}).(y - y_0)$, we have

$$J_\epsilon^1 \leq \int_{\omega_1}^{\omega_2} (-Y_h o S_h^{-1})(\omega, \Phi_h(\omega)).(1 - \theta_\epsilon(\omega)).(y - y_0) o T_h o S_h^{-1}(\omega, \Phi_h(\omega)) d\omega.$$

Moreover

$$
\begin{aligned}
J_\epsilon^2 &= \int_{S_h o T_h^{-1}(D_{y_0,\tau_0})} (\chi([p o T_h o S_h^{-1}(\omega,\tau)=0]) - g o T_h o S_h^{-1}(\omega,\tau)).(-Y_h o S_h^{-1})(\omega,\tau) \\
&\qquad .\frac{\partial}{\partial\tau}\big((1-\theta_\epsilon(\omega)).(y-y_0) o T_h o S_h^{-1}(\omega,\tau)\big)d\omega d\tau \\
&= \int_{S_h o T_h^{-1}(D_{y_0,\tau_0})} (\chi([p o T_h o H_h^{-1}(\omega,\tau)=0]) - g o T_h o S_h^{-1}(\omega,\tau)).(-Y_h o S_h^{-1})(\omega,\tau) \\
&\qquad .(1-\theta_\epsilon(\omega))\frac{\partial}{\partial\tau}\big(T_h^2 o S_h^{-1}(\omega,\tau)\big)d\omega d\tau.
\end{aligned}
$$

Since $\theta_\epsilon \to 1$ when $\epsilon \to 0$, we conclude that $J_\epsilon^1 + J_\epsilon^2 \to 0$. This achieves the proof. □

**Theorem 2.12.** *Let $(u,g)$ be a solution of $(P)$. Let $\mathcal{C}_h$ be the connected component of $[\tau < \Phi_h(\omega)]$ such that $T_h o S_h^{-1}(\mathcal{C}_h) \cap S_3 = \emptyset$. Set $C_h = T_h o S_h^{-1}(\mathcal{C}_h)$, then we have*

$$\int_{C_h} (\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)).edX \le 0.$$

**Proof.**

1st step : Arguing as in the proof of Lemma 2.11, we get :

$$
\begin{aligned}
&\int_{C_h} (\mathcal{A}(X,\nabla u) - \chi([u=y])\mathcal{A}(X,e)).\nabla\zeta dX \le \int_{\pi_\omega(\mathcal{C}_h)} \big(-Y_h o S_h^{-1}(s,\Phi_h(s))\big).\zeta o T_h o S_h^{-1}(s,\Phi_h(s))ds \\
&\qquad \forall\zeta \in W^{1,q}(C_h)\cap C^0(\bar{C}_h), \quad \zeta \ge 0.
\end{aligned}
$$

2nd step : Let $\epsilon > 0$ and $A = I\!R \setminus \pi_\omega(\mathcal{C}_h)$. Set $\alpha_\epsilon(\omega) = \min\big(1, \dfrac{d(\omega,A)}{\epsilon}\big)$ and $h_\epsilon = \alpha_\epsilon o S_h o T_h^{-1}$. Then we have

$$
\begin{aligned}
\int_{C_h} (\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)).edX &= \int_{C_h} (\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)).\nabla(h_\epsilon y)dX \\
&+ \int_{C_h} (\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)).\nabla((1-h_\epsilon)y)dX \\
&\le \int_{C_h} (\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)).\nabla((1-h_\epsilon)y)dX = J_\epsilon.
\end{aligned}
$$

since $\chi(C_h)h_\epsilon y$ is a test function for $(P)$. Moreover, one has

$$
\begin{aligned}
J_\epsilon &= \int_{C_h} (\mathcal{A}(X,\nabla u) - \chi([u=y])\mathcal{A}(X,e)).\nabla((1-h_\epsilon)y)dX \\
&+ \int_{C_h} (\chi([u=y]) - g)\mathcal{A}(X,e).\nabla((1-h_\epsilon)y)dX
\end{aligned}
$$

and one can argue as at the end of the proof of Theorem 2.10. □

**Theorem 2.13.** *Let $(u,g)$ be a solution of $(P)$. Let $X_0 = (x_0,y_0) = T_h o S_h^{-1}(\omega_0,\tau_0)$ be a point in $\Omega$, $(\omega_0,\tau_0) \in S_h(D_h)$. We denote by $B_r(\omega_0,\tau_0)$ a ball with center $(\omega_0,\tau_0)$ and radius $r$ contained in $S_h(D_h)$. If $p o T_h o S_h^{-1} = 0$ in $B_r(\omega_0,\tau_0)$ then we have*

$$p o T_h o S_h^{-1} = 0 \quad \text{in } D_r, \qquad g o T_h o S_h^{-1} = 1 \quad \text{a.e. in } D_r$$

where $D_r = \{(\omega,\tau) \in S_h(D_h), \quad |\omega-\omega_0| < r, \quad \tau > \tau_0\} \cup B_r(\omega_0,\tau_0)$ *i.e.*

$$\textit{if } p=0 \textit{ in } T_h o S_h^{-1}(B_r(\omega_0,\tau_0)) \quad \textit{then} \quad p=0,\ g=1 \text{ a.e. } \textit{in } T_h o S_h^{-1}(D_r).$$

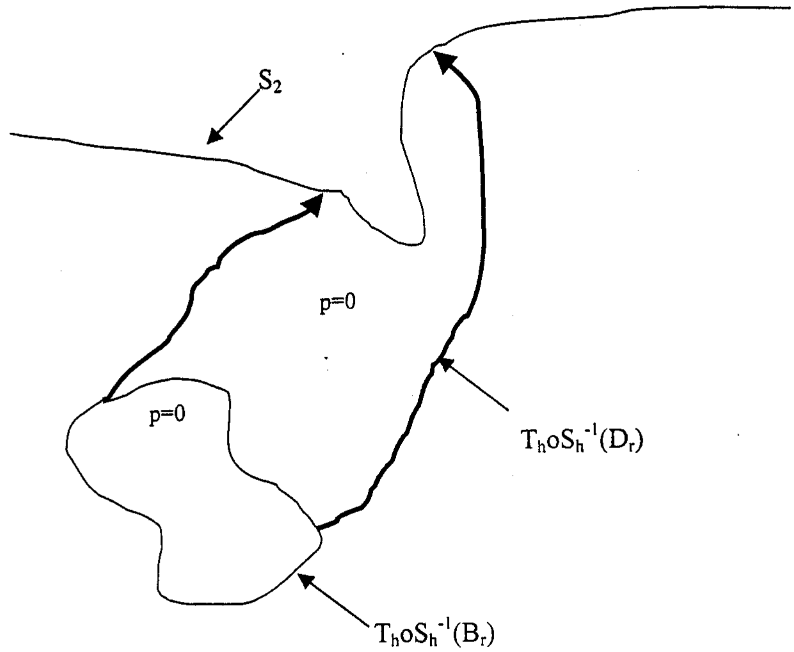


Figure 5

**Proof.** Note that by Remark 2.8, we necessarily have

$$X(\beta(\omega),\omega) \in S_2 \qquad \forall \omega \in (\omega_0 - r, \omega_0 + r).$$

By Theorem 2.7 ii), we have $poT_h oS_h^{-1} = 0$ in $D_r$.
Applying Theorem 2.10 with domains $Z_{\tau_0} = T_h oS_h^{-1}\big((\omega_1,\omega_2)\times(\tau_0,+\infty)\cap S_h(D_h)\big) \subset T_h oS_h^{-1}(D_r)$ we obtain

$$\int_{[y>y_0]\cap Z_{\tau_0}} (1-g)\mathcal{A}(X,e).edX \le 0 \qquad \forall y_0 \in I\!R \text{ such that } [y>y_0]\cap Z_{\tau_0} \neq \emptyset.$$

So $g=1$ a.e. in $Z_{\tau_0}$. This holds for all domains $Z_{\tau_0}$ in $D_r$ and we get $g=1$ a.e. in $D_r$. □

The following result is a sort of maximum principle.

**Theorem 2.14.** *Let $(u,g)$ be a solution of $(P)$, $X_0=(x_0,y_0)=T_hoS_h^{-1}(\omega_0,\tau_0)$ be a point of $\Omega$ and $B_r$ be the open ball in $S_h(D_h)$ with center $(\omega_0,\tau_0)$ and radius $r$. Then we cannot have the following occurrences*

$$
(i)\begin{cases} poT_hoS_h^{-1}(\omega_0,\tau)=0 & \forall\tau\in(\tau_0-r,\tau_0+r) \\ poT_hoS_h^{-1}(\omega,\tau)>0 & \forall(\omega,\tau)\in B_r\setminus S, \qquad S=\{\omega_0\}\times(\tau_0-r,\tau_0+r), \end{cases}
$$

$$
(ii)\begin{cases} poT_hoS_h^{-1}(\omega,\tau)=0 & \forall(\omega,\tau)\in B_r\cap[\omega\le\omega_0] \\ poT_hoS_h^{-1}(\omega,\tau)>0 & \forall(\omega,\tau)\in B_r\cap[\omega>\omega_0], \end{cases}
$$

$$
(iii)\begin{cases} poT_hoS_h^{-1}(\omega,\tau)=0 & \forall(\omega,\tau)\in B_r\cap[\omega\ge\omega_0] \\ poT_hoS_h^{-1}(\omega,\tau)>0 & \forall(\omega,\tau)\in B_r\cap[\omega<\omega_0]. \end{cases}
$$

**Proof.** *i)* Let $\xi\in\mathcal{D}(T_hoS_h^{-1}(B_r))$. Since $\pm\xi$ are test functions for $(P)$, we have

$$\int_{T_hoS_h^{-1}(B_r)}\big(\mathcal{A}(X,\nabla u)-g\mathcal{A}(X,e)\big).\nabla\xi dX=0.$$

But under assumption *i)*, $g=0$ a.e. in $T_hoS_h^{-1}(B_r)$ and so

$$\int_{T_hoS_h^{-1}(B_r)}\mathcal{A}(X,\nabla u).\nabla\xi dX=0.$$

It follows then by Lemma 2.6 that $u>y$ or $u=y$ in $T_hoS_h^{-1}(B_r)$ which is in contradiction with *i)*.

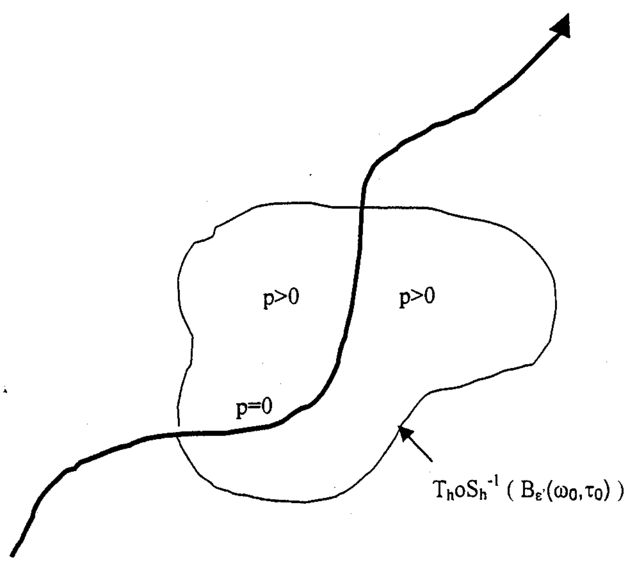


Figure 6

*ii)* Let $\xi \in \mathcal{D}(T_h o S_h^{-1}(B_r))$, $\xi \geq 0$. Then we have since $\pm\xi$ are test functions for $(P)$

$$\int_{T_h o S_h^{-1}(B_r)} \mathcal{A}(X, \nabla u).\nabla\xi dX = \int_{T_h o S_h^{-1}(B_r)} g\mathcal{A}(X, e).\nabla\xi dX$$
$$= \int_{S_h^{-1}(B_r)} goT_h\mathcal{A}(T_h(t,\omega), e).(\nabla\xi)oT_h.|Y_h(t,u)|dtd\omega$$
$$= \int_{S_h^{-1}(B_r)} goT_h.(-Y_h(t,\omega)).\frac{\partial}{\partial t}(\xi oT_h)dtd\omega$$
$$= \int_{B_r} goT_hoS_h^{-1}(\omega,\tau).(-Y_hoS_h^{-1}(\omega,\tau)).(\frac{\partial}{\partial t}(\xi oT_h))oS_h^{-1}.|det\mathcal{J}S_h^{-1}|d\omega d\tau$$
$$= \int_{B_r} goT_hoS_h^{-1}(\omega,\tau).(-Y_hoS_h^{-1}(\omega,\tau)).\frac{\partial}{\partial\tau}(\xi oT_hoS_h^{-1}))oS_h^{-1}d\omega d\tau.$$

Now by Theorem 2.13, we have $goT_hoS_h^{-1} = 1$ a.e. in $B_r \cap [\omega < \omega_0]$ and then

$$\int_{T_hoS_h^{-1}(B_r)} \mathcal{A}(X, \nabla u).\nabla\xi dX = \int_{B_r\cap[\omega<\omega_0]} (-Y_hoS_h^{-1}(\omega,\tau)).(\frac{\partial}{\partial\tau}(\xi oT_hoS_h^{-1}))oS_h^{-1}d\omega d\tau.$$
$$= \int_{B_r\cap[\omega<\omega_0]} \frac{\partial}{\partial\tau}(Y_hoS_h^{-1})(\omega,\tau).\xi oT_hoS_h^{-1}d\omega d\tau + \int_S (-Y_hoS_h^{-1})\xi oT_hoS_h^{-1}\nu_\tau$$
$$= \int_{B_r\cap[\omega<\omega_0]} \frac{\partial}{\partial\tau}(Y_hoS_h^{-1})(\omega,\tau).\xi oT_hoS_h^{-1}d\omega d\tau.$$

It follows then that

$$\int_{T_hoS_h^{-1}(B_r)} (\mathcal{A}(X,\nabla u) - \mathcal{A}(X,\nabla y)).\nabla\xi dX = -\int_{B_r} \frac{\partial}{\partial\tau}(Y_hoS_h^{-1})(\omega,\tau).\xi oT_hoS_h^{-1}d\omega d\tau$$
$$+ \int_{B_r\cap[\omega<\omega_0]} \frac{\partial}{\partial\tau}(Y_hoS_h^{-1})(\omega,\tau).\xi oT_hoS_h^{-1}d\omega d\tau$$
$$= -\int_{B_r\cap[\omega>\omega_0]} \frac{\partial}{\partial\tau}(Y_hoS_h^{-1})(\omega,\tau).\xi oT_hoS_h^{-1}d\omega d\tau \geq 0.$$

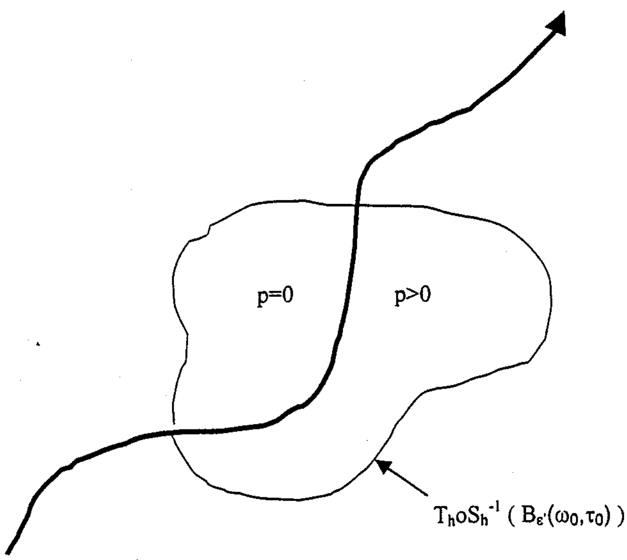


Figure 7

Since $u \geq y$ in $T_hoS_h^{-1}(B_r)$, $u = y$ in $T_hoS_h^{-1}(B_r \cap [\omega \leq \omega_0])$, $\nabla y = e \neq 0$, we deduce from Lemma 2.6 that $u = y$ in $T_hoS_h^{-1}(B_r)$ which contradicts the fact that $u > y$ in $T_hoS_h^{-1}(B_r \cap [\omega > \omega_0])$.

*iii)* We argue as in *ii)*.

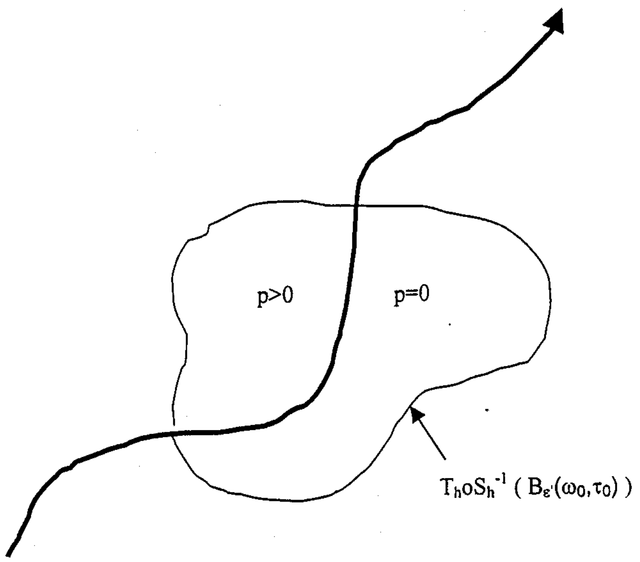


Figure 8

□

The following result expresses that if the free boundary is of Lebesgue's measure zero, then $g$ is the characteristic function of the dry region.

**Theorem 2.15.** *Let $(u, g)$ be a solution of $(P)$. If the free boundary i.e. the set $\partial\,[p > 0] \cap \Omega$ is of Lebesgue's measure zero, then we have :*

$$g = \chi([p = 0]) = \chi([u = y]).$$

**Proof.** First by $(P)i)$, we have $g = 0$ in $[p > 0]$. Next if $(x_0, y_0) \in \Omega \setminus \overline{[p > 0]}$, then since $\Omega \setminus \overline{[p > 0]}$ is an open set there exists $\epsilon_0 > 0$ small enough such that $B_{\epsilon_0}(x_0, y_0) \subset \Omega \setminus \overline{[p > 0]}$. Moreover for $\epsilon \in (0, \epsilon_0)$ small enough there exists $h \in \pi_y(\Omega)$ such that $B_\epsilon(x_0, y_0) \subset T_h(D_h) \cap [p = 0]$.
For each $\eta > 0$ such that $B_\eta(\omega, \tau) \subset S_h o T_h^{-1}(B_\epsilon(x_0, y_0))$, we have by Theorem 2.13, $g o T_h o S_h^{-1} = 1$ a.e. in $B_\eta(\omega, \tau)$. This means that $g o T_h o S_h^{-1} = 1$ a.e. in $S_h o T_h^{-1}(B_\epsilon(x_0, y_0))$ or equivalently $g = 1$ in $B_\epsilon(x_0, y_0)$. Therefore $g = 1$ in $\Omega \setminus \overline{[p > 0]}$.
Since the set $\partial[p > 0] \cap \Omega$ is of Lebesgue's measure zero, we conclude that $g = \chi([p = 0]) = \chi([u = y])$. □

## 3. Continuity of the Free Boundary

In this section we assume that $\mathcal{A}$ is strictly monotone in the following sense

$$(\mathcal{A}(X, \xi) - \mathcal{A}(X, \zeta)).(\xi - \zeta) > 0 \qquad \forall \xi \neq \zeta, \quad \forall X \in \Omega. \tag{3.1}$$

**Theorem 3.1.** *For each $h \in \pi_y(\Omega)$ the function $\Phi_h$ is continuous on $\pi_x(\Omega \cap [y = h])$.*

**Proof.** It suffices to prove that $\Phi_h$ is upper semi-continuous. Let $X_0 = T_h o S_h^{-1}(\omega_0, \tau_0) \in \Omega \cap \partial[p > 0]$. Thanks to Theorem 2.7 we have necessarily $X(\beta(\omega_0), \omega_0) \notin S_3$. Let $\epsilon > 0$ small enough.

*i)* We first assume that $X(\beta(\omega_0), \omega_0) \notin \overline{S_3}$, where $\overline{S_3}$ denotes the closure of $S_3$ relative to $\partial\Omega$.
Since $p(X_0) = poT_h o S_h^{-1}(\omega_0, \tau_0) = 0$ and $p$, $\beta$ are continuous, there exists a ball $B_{\epsilon'}(\omega_0, \tau_0)$ $(0 < \epsilon' < \epsilon)$ such that :

$$\begin{cases} poT_h o S_h^{-1}(\omega, \tau) \leq \epsilon & \forall(\omega, \tau) \in B_{\epsilon'}(\omega_0, \tau_0) \\ X(\beta(\omega), \omega) \notin \overline{S_3} & \forall \omega \in (\omega_0 - \epsilon', \omega_0 + \epsilon'). \end{cases}$$

Using Theorem 2.14, one of the following situations occurs

$$\begin{cases} a) \quad \exists(\omega_1, \tau_1) \in B_{\epsilon'}(\omega_0, \tau_0) \quad \text{such that } \omega_1 < \omega_0 \quad \text{and} \quad poT_h o S_h^{-1}(\omega_1, \tau_1) = 0 \\ b) \quad \exists(\omega_2, \tau_2) \in B_{\epsilon'}(\omega_0, \tau_0) \quad \text{such that } \omega_2 > \omega_0 \quad \text{and} \quad poT_h o S_h^{-1}(\omega_2, \tau_2) = 0. \end{cases}$$

Assume that for example $a)$ is satisfied and set $X_1 = T_h o S_h^{-1}(\omega_1, \tau_1)$ and $\tau_M = \max(\tau_1, \tau_2)$ (see Figure 9). Then we have by Theorem 2.7

$$poT_h o S_h^{-1}(\omega_i, \tau_i) = 0 \qquad \forall(\omega_i, \tau) \in S_h(D_h) \quad \text{such that } \tau > \tau_M \ (i = 1, 2).$$

Set $Z_{\tau_M} = T_h o S_h^{-1}\big((\omega_1, \omega_0) \times (\tau_M, +\infty) \cap S_h(D_h)\big)$ and let $y_0 \in I\!R$ such that $(T_h o S_h^{-1})^{-1}([y = y_0]) \cap B_{\epsilon'}(\omega_0, \tau_0) \cap [\tau > \tau_M] \neq \emptyset$.

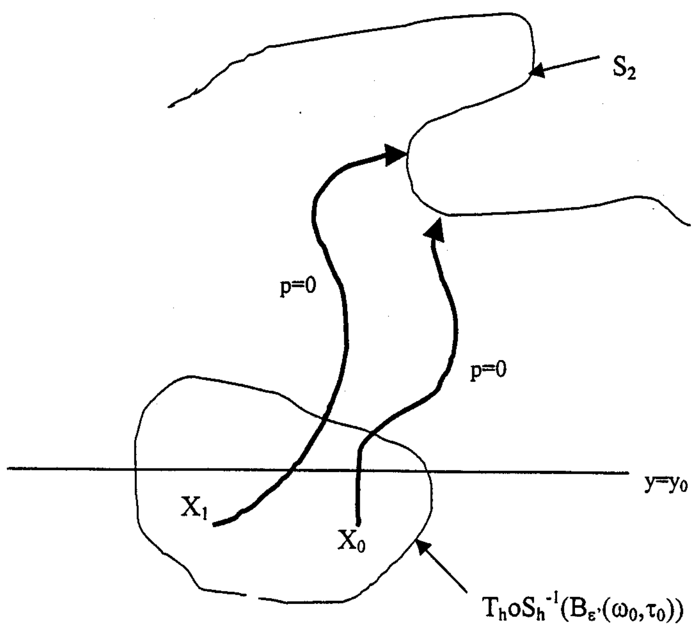


Figure 9

Note that

$$S_h o T_h^{-1}([y = y_0]) = \{(\omega, \tau_{y_0}(\omega)) \in S_h(D_h) \,/\, \tau_{y_0}(\omega) = \int_{\alpha(\omega)}^{t_{y_0}(\omega)} |\mathcal{A}(X(s, \omega), e)| ds, \ X_2(t_{y_0}(\omega), \omega) = y_0\}.$$

Set $D_{y_0,\tau_M} = [y > y_0] \cap Z_{\tau_M} \neq \emptyset$, $v(y) = (\epsilon + y_0 - y)^+ + y$, $\xi(x,y) = \chi(D_{y_0,\tau_M})(u-v)^+$.
Since $v \geq y = u$ on $\partial D_{y_0,\tau_M} \setminus ([y = y_0])$, we have $\xi = 0$ on $\partial D_{y_0,\tau_M} \setminus ([y = y_0])$. Moreover $v(y_0) = \epsilon + y \geq u(x, y_0)$ and then $\xi(x, y_0) = 0$. It follows that $\xi = 0$ on $\partial D_{y_0,\tau_M}$. So $\pm\xi$ are test functions for $(P)$ and we have

$$\int_{D_{y_0,\tau_M}} (\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)).\nabla(u-v)^+ dX = 0. \tag{3.2}$$

We also have

$$\int_{D_{y_0,\tau_M}} (\mathcal{A}(X,\nabla v) - \chi([v=y])\mathcal{A}(X,e)).\nabla(u-v)^+ dX = 0. \tag{3.3}$$

Subtracting (3.3) from (3.2), we obtain

$$\int_{D_{y_0,\tau_M}} ((\mathcal{A}(X,\nabla u) - \mathcal{A}(X,\nabla v)) + (\chi([v=y]) - g)\mathcal{A}(X,e)).\nabla(u-v)^+ dX = 0$$

which can be written

$$\int_{D_{y_0,\tau_M}\cap[v>y]} (\mathcal{A}(X,\nabla u) - \mathcal{A}(X,\nabla v)).\nabla(u-v)^+ dX \\ + \int_{D_{y_0,\tau_M}\cap[v=y]} (\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)).\nabla(u-y) dX = 0. \tag{3.4}$$

By Theorem 2.10, we have for $D_{y_0+\epsilon,\tau_M} = [y > y_0 + \epsilon] \cap Z_{\tau_M} = D_{y_0,\tau_M} \cap [v = y]$

$$\int_{D_{y_0,\tau_M}\cap[v=y]} (\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)).e dX \leq 0. \tag{3.5}$$

Adding (3.4) and (3.5), we get :

$$\int_{D_{y_0,\tau_M}\cap[v>y]} (\mathcal{A}(X,\nabla u) - \mathcal{A}(X,\nabla v)).\nabla(u-v)^+ dX \\ + \int_{D_{y_0,\tau_M}\cap[v=y]} (\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)).\nabla u dX \leq 0$$

which can be written by taking into account $(P)i)$

$$\int_{D_{y_0,\tau_M}\cap[v>y]} (\mathcal{A}(X,\nabla u) - \mathcal{A}(X,\nabla v)).\nabla(u-v)^+ dX + \int_{D_{y_0,\tau_M}\cap[v=y]\cap[u>y]} \mathcal{A}(X,\nabla u).\nabla u dX \\ + \int_{D_{y_0,\tau_M}\cap[v=y]\cap[u=y]} (1-g)\mathcal{A}(X,e).e dX \leq 0.$$

Since the three integrals in the left hand side of the above inequality are all nonnegative, we obtain by (3.1) that $\nabla(u-v)^+$ a.e. in $D_{y_0,\tau_M} \cap [v > y]$ and then since $(u-v)^+ = 0$ on $\partial D_{y_0,\tau_M}$, we get $u \leq v$ in $D_{y_0,\tau_M} \cap [v > y]$. This leads to $p(x, y_0 + \epsilon) = 0$ $\forall x \in \pi_x(D_{y_0,\tau_M})$.
Now for each $\omega \in (\omega_1, \omega_2)$, there exists a unique $t_{y_0,\epsilon}(\omega) \in (\alpha(\omega), \beta(\omega))$ such that

$$X_2(t_{y_0,\epsilon}(\omega), \omega) = y_0 + \epsilon \qquad \text{and} \qquad p(X_1(t_{y_0,\epsilon}(\omega), \omega), y_0 + \epsilon) = 0$$

and if

$$\tau_{y_0,\epsilon}(\omega) = \int_{\alpha(\omega)}^{t_{y_0,\epsilon}(\omega)} |\mathcal{A}(X(s,\omega), e)| ds$$

we obtain $poT_h oS_h^{-1}(\omega, \tau_{y_0,\epsilon}(\omega)) = poT_h(t_{y_0,\epsilon}(\omega), \omega) = p(X_1(t_{y_0,\epsilon}(\omega), \omega), y_0 + \epsilon) = 0$ $\forall \omega \in (\omega_1, \omega_2)$. Therefore $\Phi_h(\omega) \leq \tau_{y_0,\epsilon}(\omega)$.

But since $X_2$ is increasing with respect to $t$, $X_2(t_{y_0}(\omega),\omega) = y_0$ and $X_2(t_{y_0,\epsilon}(\omega),\omega) = y_0+\epsilon$, it follows that $t_{y_0}(\omega) < t_{y_0,\epsilon}(\omega)$ and then

$$\epsilon = X_2(t_{y_0,\epsilon}(\omega),\omega) - X_2(t_{y_0}(\omega),\omega) = \int_{t_{y_0}(\omega)}^{t_{y_0,\epsilon}(\omega)} a^2(X(s,\omega),e)ds \geq \lambda(t_{y_0,\epsilon}(\omega) - t_{y_0}(\omega))$$

and

$$\tau_{y_0,\epsilon}(\omega) = \tau_{y_0}(\omega) + \int_{t_{y_0}(\omega)}^{t_{y_0,\epsilon}(\omega)} |\mathcal{A}(X(s,\omega),e)|ds \leq \tau_{y_0}(\omega) + M(t_{y_0,\epsilon}(\omega) - t_{y_0}(\omega)) \leq \tau_{y_0}(\omega) + \frac{M}{\lambda}\epsilon.$$

So

$$\Phi_h(\omega) \leq \tau_{y_0}(\omega) + \frac{M}{\lambda}\epsilon < \tau_0 + \epsilon' + \frac{M}{\lambda}\epsilon = \Phi_h(\omega_0) + \big(1+\frac{M}{\lambda}\big)\epsilon \quad \forall\omega \in (\omega_1,\omega_0).$$

Hence $\Phi_h$ is u.s.c at $\omega_0$ from the left.

Using Theorem 2.14 and arguing as above, one can prove that $\Phi_h$ is u.s.c at $\omega_0$ from the right. Thus $\Phi_h$ is continuous at $\omega_0$.

*ii)* Now we assume that $X(\beta(\omega_0),\omega_0) \in \overline{S_3} \setminus S_3$. Then one of the following situations holds :

$$\begin{cases} a) \quad \exists\eta > 0 \quad \text{such that } \forall\omega \in (\omega_0-\eta,\omega_0+\eta) \quad X(\beta(\omega),\omega) \in S_3 \quad \Leftrightarrow \quad \omega \in (\omega_0,\omega_0+\eta) \\ b) \quad \exists\eta > 0 \quad \text{such that } \forall\omega \in (\omega_0-\eta,\omega_0+\eta) \quad X(\beta(\omega),\omega) \in S_3 \quad \Leftrightarrow \quad \omega \in (\omega_0-\eta,\omega_0). \end{cases}$$

Assume for example that $a)$ holds. Then it is easy to see that

$$\Phi_h(\omega) = L_h(\beta(\omega),\omega) \quad \forall\omega \in (\omega_0,\omega_0+\eta). \tag{3.6}$$

Arguing as in $i)$ one can prove that $\Phi_h$ is continuous at $\omega_0$ from the left. On the other hand we deduce from (3.6) that $u > y$ in a right neighborhood of the curve $X(.,\omega_0)$. Using the continuity from the left and Theorem 2.14, we have necessarily $\Phi_h(\omega_0) = L_h(\beta(\omega_0),\omega_0)$. Therefore we have now

$$\Phi_h(\omega) = L_h(\beta(\omega),\omega) \quad \forall\omega \in [\omega_0,\omega_0+\eta)$$

which leads to the continuity from the right of $\Phi_h$ at $\omega_0$.

We argue similarly if $b)$ occurs. □

**Remark 3.2.** *i)* Since for each $X_0 \in \Omega \cap \partial[p>0]$, there exists $h \in \pi_y(\Omega)$ such that $X_0 \in T_h(D_h) \cap \partial[p>0]$ and then $X_0 = T_h o S_h^{-1}(\omega_0,\tau_0)$ with $\tau_0 = \Phi_h(\omega_0)$. It follows from Theorem 3.1 that the free boundary is represented near $X_0$ by the graph of the continuous function $\Phi_h$. $g = \chi([p=0])$.

*ii)* Since the free boundary is now represented locally by graphs of continuous functions, it follows by Theorem 2.5 that.

### 4. Uniqueness of the $S_3$-Connected Solutions

**Definition 4.1.** *A solution $(u,g)$ of $(P)$ is a $S_3$-connected solution if for all connected component $C$ of $[u>y]$ and each $h \in \pi_y(\Omega)$, we have :*

$$\overline{T_h o S_h^{-1}\big(\pi_x(C_h) \times I\!R \cap S_h(D_h)\big)} \cap S_3 \neq \emptyset, \qquad \textit{where} \quad C_h = (T_h o S_h^{-1})^{-1}(C \cap T_h(D_h)).$$

**Remark 4.2.** Thanks to Remark 2.8, $C$ contains the strip of $\Omega$ below $S_{3,i}$ and $S_{3,i}$ on its boundary in the following sense :

$$\forall h \in \pi_y(\Omega) \quad T_h o S_h^{-1}\big((\pi_\omega(S_{3,i}^h) \times I\!R) \cap S_h(D_h)\big) \subset C, \qquad \text{where} \quad S_{3,i}^h = (T_h o S_h^{-1})^{-1}(S_{3,i}).$$

**Theorem 4.3.** *Let $(u,g)$ be a solution of $(P)$ and $C$ a connected component of $[u > y]$ such that $\overline{C} \cap S_3 = \emptyset$. Set for each $h \in \pi_y(\Omega)$, $\mathcal{C}_h = (T_h o S_h^{-1})^{-1}(C \cap T_h(D_h))$ and $h_C = \sup\{y \,/\, (x,y) \in C\}$. Then we have*

$$\begin{aligned} &u(x,y) = (h_C - y)^+\chi(C) + y \quad \text{and} \quad g = 1 - \chi(C) \\ &\text{for} \quad (x,y) \in T_h o S_h^{-1}\big(\pi_\omega(\mathcal{C}_h) \times I\!R \cap S_h(D_h)\big) \quad \text{and} \quad h \in \pi_y(\Omega). \end{aligned}$$

**Proof.** Since $\pm\chi(C)(u-y)$ are test functions for $(P)$, we have :

$$\int_C \big(\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)\big).\nabla(u-y)dX = 0. \tag{4.1}$$

By Theorem 2.12, we have :

$$\int_C \big(\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)\big).edX \le 0. \tag{4.2}$$

Adding (4.1) and (4.2), we obtain

$$\int_C \big(\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)\big).\nabla udX \le 0$$

which can be written

$$\int_{C\cap[u>y]} \mathcal{A}(X,\nabla u).\nabla udX + \int_{C\cap[u=y]=\emptyset} (1-g)\mathcal{A}(X,e).edX \le 0.$$

It follows then that $\nabla u = 0$ a.e. in $C$ and then $u$ is equal to a some positive constant $k$ in $C$. But we can easily verify that $k = h_C$.
Using Theorem 2.7 we deduce that $u = (h_C - y)^+\chi(C) + y$ in $T_h o S_h^{-1}\big(\pi_\omega(\mathcal{C}_h) \times I\!R \cap S_h(D_h)\big)$. Since $\Phi_h$ is continuous, we deduce by Theorem 1.15 that $g = 1 - \chi(C)$ in $T_h o S_h^{-1}\big(\pi_\omega(\mathcal{C}_h) \times I\!R \cap S_h(D_h)\big)$. □

**Definition 4.4.** *We call a pool in $\Omega$ a pair $(\bar{u}, \bar{g})$ of functions defined in $\Omega$ by :*

$$\begin{aligned} &\bar{u}(x,y) = (h-y)^+\chi(C) + y \quad \text{and} \quad \bar{g}(x,y) = 1 - \chi(C)(x,y) \\ &\text{for} \quad (x,y) \in Z = T_h o S_h^{-1}\big(\pi_\omega(\mathcal{C}_h) \times I\!R \cap S_h(D_h)\big), \end{aligned}$$

*where $C = T_h o S_h^{-1}(\mathcal{C}_h)$ is a sub-domain of $\Omega$ and $h = \max\{y/(x,y) \in C\}$.*

**Theorem 4.5.** *Each solution $(u,g)$ of $(P)$ can be written as the sum of a $S_3-$connected solution and pools.*

**Proof.** See [CaL] or [Ly2]. □

In order to prove the uniqueness of the $S_3$-Connected Solution we shall assume that $\Omega$ is covered by a finite number of sets $T_h(D_h)$ that is

$$\exists h_1, ..., h_n \in \pi_y(\Omega) \quad \text{such that} \quad \Omega = \bigcup_{k=1}^{k=n} T_{h_k}(D_{h_k}). \tag{4.3}$$

Then we can state our uniqueness theorem

**Theorem 4.6.** *Under assumption (4.3), the $S_3$-connected solution is unique.*

The proof of Theorem 4.6 requires three lemmas.

**Lemma 4.7.** *Let $(u_1, g_1)$ and $(u_2, g_2)$ be two solutions of $(P)$. Then we have*

$$\int_\Omega \big((\mathcal{A}(X, \nabla u_i) - \mathcal{A}(X, \nabla u_m)) - (g_i - g_M)\mathcal{A}(X, e)\big).\nabla\zeta dX = 0 \qquad i = 1, 2 \qquad \forall\zeta \in W^{1,q}(\Omega),$$

*where $u_m = \min(u_1, u_2)$ and $g_M = \max(g_1, g_2)$.*

To prove Lemma 4.7 we need the following Lemma :

**Lemma 4.8.** *Let $(u_1, g_1)$ and $(u_2, g_2)$ be two solutions of $(P)$ and $\Phi^i_{h_k}$ $(i = 1, 2$ and $k = 1, ..., n)$ be the corresponding free boundary functions. Then we have for $i = 1, 2$ and $\zeta \in W^{1,q}(\Omega) \cap C^0(\bar\Omega)$*

$$\int_\Omega \big((\mathcal{A}(X, \nabla u_i) - \mathcal{A}(X, \nabla u_m)) - (g_i - g_M)\mathcal{A}(X, e)\big).\nabla\zeta dX$$
$$\leq \sum_{k=1}^{k=n} \int_{\mathcal{D}^i_{h_k}} (-Y_{h_k} o S^{-1}_{h_k})(\omega, \Phi^i_{h_k}(\omega)).\zeta o T_{h_k} o S^{-1}_{h_k}(\omega, \Phi^i_{h_k}(\omega)) d\omega,$$

*where $\mathcal{D}^i_{h_k} = \{\omega \in \pi_\omega(S_{h_k}(D_{h_k})) \,/\, \Phi^0_{h_k}(\omega) = \min(\Phi^1_{h_k}(\omega), \Phi^2_{h_k}(\omega)) < \Phi^i_{h_k}(\omega)\}$.*

**Proof.** First thanks to (4.3) there exits a partition of unity $(\theta_k)_{k=1}^{k=n}$ corresponding to the open covering $\big(T_{h_k}(D_{h_k})\big)_{k=1}^{k=n}$ of $\Omega$ that is

$$\begin{aligned} &\theta_k \in \mathcal{D}(T_{h_k}(D_{h_k})) \quad \forall k = 1, ..., n \\ &0 \leq \theta_k \leq 1 \quad \text{in} \quad \Omega \quad \forall k = 1, ..., n \\ &\sum_{k=1}^{k=n} \theta_k = 1 \quad \text{in} \quad \Omega. \end{aligned} \tag{4.4}$$

Then we have by taking into account (4.4)

$$\begin{aligned} &\int_\Omega \big((\mathcal{A}(X, \nabla u_i) - \mathcal{A}(X, \nabla u_m)) - (g_i - g_M)\mathcal{A}(X, e)\big).\nabla\zeta dX \\ &= \sum_{k=1}^{k=n} \int_\Omega \big((\mathcal{A}(X, \nabla u_i) - \mathcal{A}(X, \nabla u_m)) - (g_i - g_M)\mathcal{A}(X, e)\big).\nabla\zeta_k dX \\ &= \sum_{k=1}^{k=n} \int_{T_{h_k}(D_{h_k})} \big((\mathcal{A}(X, \nabla u_i) - \mathcal{A}(X, \nabla u_m)) - (g_i - g_M)\mathcal{A}(X, e)\big).\nabla\zeta_k dX \end{aligned} \tag{4.5}$$

where $\zeta_k = \theta_k\zeta$.
Now let $\epsilon > 0$ and $\xi_k = \min\big(\zeta_k, \dfrac{(u_1 - u_2)^+}{\epsilon}\big)$. Clearly $\pm\xi_k$ are test functions for $(P)$ and then we have

$$\int_\Omega \big((\mathcal{A}(X, \nabla u_1) - \mathcal{A}(X, \nabla u_2)) - (g_1 - g_2)\mathcal{A}(X, e)\big).\nabla\xi_k dX = 0. \tag{4.6}$$

Since we integrate only on the set $[u_1 > u_2]$, where $u_1 = u_m$, (4.6) becomes

$$\int_\Omega \big((\mathcal{A}(X, \nabla u_1) - \mathcal{A}(X, \nabla u_m)) - (g_1 - g_M)\mathcal{A}(X, e)\big).\nabla\xi_k dX = 0$$

which can be written

$$\int_{\Omega\cap[(u_1-u_2)^+>\epsilon\zeta_k]}((\mathcal{A}(X,\nabla u_i)-\mathcal{A}(X,\nabla u_m)).\nabla\zeta_k dX - \int_\Omega (g_1-g_M)\mathcal{A}(X,e).\nabla\zeta_k dX$$
$$\leq -\int_\Omega (g_1-g_M)\mathcal{A}(X,e).\nabla\big(\zeta_k - \frac{(u_1-u_2)^+}{\epsilon}\big)^+ dX$$
$$= -\int_{T_{h_k}(D_{h_k})} (g_1-g_M)\mathcal{A}(X,e).\nabla\big(\zeta_k - \frac{(u_1-u_2)^+}{\epsilon}\big)^+ dX = I_\epsilon^k.$$

Using the $C^1$-diffeomorphisms $T_{h_k}$ and $S_{h_k}$ we obtain

$$I_\epsilon^k = -\int_{S_{h_k}(D_{h_k})} (g_1 o T_{h_k} o S_{h_k}^{-1} - g_M o T_{h_k} o S_{h_k}^{-1})(\omega,\tau)$$
$$.(-Y_{h_k} o S_{h_k}^{-1}(\omega,\tau))\frac{\partial}{\partial\tau}\Big(\big(\zeta_k - \frac{(u_1-u_2)^+}{\epsilon}\big)^+ o T_{h_k} o S_{h_k}^{-1}\Big) d\omega d\tau$$
$$= \int_{[p_1 o T_{h_k} o S_{h_k}^{-1}>0=p_m=p_2]} (-Y_{h_k} o S_{h_k}^{-1}(\omega,\tau))\frac{\partial}{\partial\tau}\Big(\big(\zeta_k - \frac{p_1}{\epsilon}\big)^+ o T_{h_k} o S_{h_k}^{-1}\Big) d\omega d\tau$$
$$= \int_{[\tau<\Phi_{h_k}^1(\omega)]\cap[\tau>\Phi_{h_k}^2(\omega)]} (-Y_{h_k} o S_{h_k}^{-1}(\omega,\tau))\frac{\partial}{\partial\tau}\Big(\big(\zeta_k - \frac{p_1}{\epsilon}\big)^+ o T_{h_k} o S_{h_k}^{-1}\Big) d\omega d\tau$$
$$= \int_{\mathcal{D}_{h_k}^1}\int_{\Phi_{h_k}^2(\omega)}^{\Phi_{h_k}^1(\omega)} (-Y_{h_k} o S_{h_k}^{-1}(\omega,\tau))\frac{\partial}{\partial\tau}\Big(\big(\zeta_k - \frac{p_1}{\epsilon}\big)^+ o T_{h_k} o S_{h_k}^{-1}\Big) d\omega d\tau.$$

By the second mean value theorem there exists $\Phi_k^*(\omega) \in [\Phi_{h_k}^2(\omega), \Phi_{h_k}^1(\omega)]$ such that

$$I_\epsilon^k = \int_{\mathcal{D}_{h_k}^1} (-Y_{h_k} o S_{h_k}^{-1}(\omega, \Phi_k^*(\omega))) \int_{\Phi_k^*(\omega)}^{\Phi_{h_k}^1(\omega)} \frac{\partial}{\partial\tau}\Big(\big(\zeta_k - \frac{p_1}{\epsilon}\big)^+ o T_{h_k} o S_{h_k}^{-1}\Big) d\omega d\tau$$
$$\leq \int_{\mathcal{D}_{h_k}^1} (-Y_{h_k} o S_{h_k}^{-1}(\omega, \Phi_{h_k}^1(\omega))).\zeta_k o T_{h_k} o S_{h_k}^{-1}(\omega, \Phi_{h_k}^1(\omega)) d\omega.$$

It follows that

$$\int_{\Omega\cap[(u_1-u_2)^+>\epsilon\zeta_k]}(\mathcal{A}(X,\nabla u_i)-\mathcal{A}(X,\nabla u_m)).\nabla\zeta_k dX - \int_\Omega (g_1-g_M)\mathcal{A}(X,e).\nabla\zeta_k dX$$
$$\leq \int_{\mathcal{D}_{h_k}^1} (-Y_{h_k} o S_{h_k}^{-1}(\omega, \Phi_{h_k}^1(\omega))).\zeta_k o T_{h_k} o S_{h_k}^{-1}(\omega, \Phi_{h_k}^1(\omega)) d\omega.$$

Letting $\epsilon$ go to 0, we get

$$\begin{aligned}\int_\Omega(\mathcal{A}(X,\nabla u_i)-\mathcal{A}(X,\nabla u_m)).\nabla\zeta_k - (g_1-g_M)\mathcal{A}(X,e).\nabla\zeta_k dX\\ \leq \int_{\mathcal{D}_{h_k}^1} (-Y_{h_k} o S_{h_k}^{-1}(\omega, \Phi_{h_k}^1(\omega))).\zeta_k o T_{h_k} o S_{h_k}^{-1}(\omega, \Phi_{h_k}^1(\omega)) d\omega.\end{aligned} \tag{4.7}$$

Combining (4.5) and (4.7) the lemma follows.

**Proof of Lemma 4.7.** Let $\zeta \in C^1(\bar\Omega)$, $\zeta \geq 0$. For $\delta > 0$ we set $\alpha_\delta(\omega,\tau) = \Big(1 - \frac{d((\omega,\tau), A_0)}{\delta}\Big)^+$ where

$A_0 = \bigcup_{k=1}^{k=n} [p_0 o T_{h_k} o S_{h_k}^{-1}(\omega, \tau) > 0]$. Then for $h_\delta = \alpha_\delta o S_{h_k} o T_{h_k}^{-1}$, we have

$$\begin{aligned} I(\zeta) &= \int_\Omega \big((\mathcal{A}(X, \nabla u_1) - \mathcal{A}(X, \nabla u_m)) - (g_1 - g_M)\mathcal{A}(X, e)\big).\nabla\zeta dX \\ &= \int_\Omega \big((\mathcal{A}(X, \nabla u_1) - \mathcal{A}(X, \nabla u_m)) - (g_1 - g_M)\mathcal{A}(X, e)\big).\nabla(h_\delta\zeta) dX \\ &+ \int_\Omega \big((\mathcal{A}(X, \nabla u_1) - \mathcal{A}(X, \nabla u_m)) - (g_1 - g_M)\mathcal{A}(X, e)\big).\nabla((1 - h_\delta)\zeta) dX \\ &= I_\delta^1 + I_\delta^2. \end{aligned}$$

By the previous lemma we have :

$$I_\delta^1 \leq \sum_{k=1}^{k=n} \int_{\mathcal{D}_{h_k}^1} (-Y_{h_k} o S_{h_k}^{-1}(\omega, \Phi_{h_k}^1(\omega)))\zeta o T_{h_k} o S_{h_k}^{-1}(\omega, \Phi_{h_k}^1(\omega))\alpha_\delta(\omega, \Phi_{h_k}^1(\omega)) d\omega.$$

Since $(1 - h_\delta)\zeta$ is a test function for $(P)$ we have

$$\int_\Omega (\mathcal{A}(X, \nabla u_1) - g_1\mathcal{A}(X, e)).\nabla((1 - h_\delta)\zeta) dX \leq 0.$$

Moreover the function $(1 - \alpha_\delta)$ vanishes on $\overline{A_0}$ and $g_M = 1$ in $\Omega \setminus \overline{A_0}$. So :

$$\int_\Omega (\mathcal{A}(X, \nabla u_m) - g_M\mathcal{A}(X, e)).\nabla((1 - h_\delta)\zeta) dX = 0.$$

This leads to $I_\delta^2 \leq 0$ and

$$I(\zeta) \leq \sum_{k=1}^{k=n} \int_{\mathcal{D}_{h_k}^1} (-Y_{h_k} o S_{h_k}^{-1}(\omega, \Phi_{h_k}^1(\omega)))\zeta o T_{h_k} o S_{h_k}^{-1}(\omega, \Phi_{h_k}^1(\omega))\alpha_\delta(\omega, \Phi_{h_k}^1(\omega)) d\omega.$$

Letting $\delta \longrightarrow 0$, we deduce $I(\zeta) \leq 0$.
Let $M = \sup_{\overline{\Omega}} \zeta$. Then since $(M - \zeta) \in C^1(\bar{\Omega})$ and $\zeta \geq 0$ we also have $I(M - \zeta) \leq 0$. Therefore we have proved that $I(\zeta) = 0$, $\forall \zeta \in C^1(\bar{\Omega})$, $\zeta \geq 0$. This also holds by density for all $\zeta \in W^{1,q}(\Omega)$, $\zeta \geq 0$. Finally since for $\zeta \in W^{1,q}(\Omega)$ we have $\zeta = \zeta^+ - \zeta^-$ and $\zeta^+, \zeta^- \geq 0$, we obtain $I(\zeta) = I(\zeta^+) - I(\zeta^-) = 0$. □

**Lemma 4.9** *Let $\Omega_0$ be a domain of $I\!R^2$, $\Gamma_0 \subset \partial\Omega_0$ of class $C^{1,\alpha}$ and let $u_1, u_2 \in W_{loc}^{1,q}(\Omega_0)$ such that:*

$$\begin{cases} i)\ div(\mathcal{A}(X, \nabla u_1)) = div(\mathcal{A}(X, \nabla u_2)) = 0 & in\ \mathcal{D}'(\Omega_0) \\ ii)\ \ u_1 \leq u_2 \quad in\ \Omega \\ iii)\ \ u_1 = u_2\ on\ \Gamma_0, \quad u_1, u_2 \in C^1(\Omega_0 \cup \Gamma_0) \\ iv)\ \ \mathcal{A}(X, \nabla u_1).\nu = \mathcal{A}(X, \nabla u_2).\nu \quad on\ \Gamma_0 \\ v)\ \ \nabla u_1(X) \neq 0 \quad \forall X \in \Gamma_0 \quad or \quad \nabla u_2(X) \neq 0 \quad \forall X \in \Gamma_0. \end{cases}$$

*Then $u_1 = u_2$ in $\Omega$.*

**Proof.** See [ChaL]

**Proof of Theorem 4.6 .** When $\mathcal{A}(X, \xi) = a(X)\xi$ and $q = 2$ one can argue as in [Ly1]. For the general case we use Lemma 4.9.

Let $(u_1, g_1)$, $(u_2, g_2)$ be two solutions of $(P)$. By Lemma 4.1 one can see that $(u_m, g_M)$ is also a solution of $(P)$. Let $C_{m,i}$ be the connected component of the set $[u_m > y]$ that contains $S_{3,i}$ on its boundary. From Lemma 4.1 we deduce easily that *i)* and *vi)* are satisfied for $\Omega_0 = C_{m,i}$ and $\Gamma_0 = S_{3,i}$.
*ii)* is obviously satisfied and the first part of *iii)* is true since $u_1 = u_2 = \varphi + y$ on $S_3$. The second part of *iii)* is also true (see [Li]). So if *v)* is satisfied we will get $u_1 = u_m$ in $C_{m,i}$. Assume that *v)* is not true, then $\exists x_1, x_m \in \Gamma_0$ such that $\nabla u_j(x_j) = 0$, $j = 1, m$. We distinguish two cases :

If $\nabla u_1 \neq 0$ on $\Gamma_0$ or $\nabla u_m \neq 0$ on $\Gamma_0$ then by *iii)* $\exists \Gamma_0' \subset \Gamma_0$ such that $\nabla u_1 \neq 0$ on $\Gamma_0'$ or $\nabla u_m \neq 0$ on $\Gamma_0'$. From the Lemma we conclude that $u_1 = u_m$ in $C_{m,i}$.

If $\nabla u_1 = 0$ on $\Gamma_0$ and $\nabla u_m = 0$ on $\Gamma_0$, then $u_1$ and $u_m$ are both constant along $\Gamma_0$. Since $u_1 = u_m$ on $S_3$, it follows that $u_1 = u_m = h_i$ on $\Gamma_0$ for some constant $h_i$. Therefore one can extend $u_1$ and $u_m$ into $B \setminus \Omega$ by $h_i$ where $B$ is a ball centered at $S_{3,i} = \Gamma_0$ in such a way that $u_j \in C^1(C_{m,i} \cup B)$ and $div(\mathcal{A}(X, \nabla u_j)) = 0$ in $\mathcal{D}'(C_{m,i} \cup B)$. So $\nabla u_j$ has non-isolated zeros and then (see [AS]) $u_j = h_i$ in $C_{m,i}$ $(j = 1, m)$. Then $u_1 = u_m$ in $C_{m,i}$.
Now $C_{m,i}$ is a nonempty open set of the domain $C_{1,i}$. Moreover it is easy to show that $C_{m,i}$ is also closed in $C_{1,i}$. Thus $C_{1,i} = C_{m,i}$ and $u_1 = u_m$ in $C_{1,i}$. In the same way we prove that $u_2 = u_m$ in $C_{2,i} = C_{m,i}$. We conclude that $u_1 = u_2$ in $C_{1,i} = C_{2,i}$ for all $i = 1, ..., N$. This means that $u_1 = u_2$ in $[u_1 > y] = [u_2 > y]$ which leads to $u_1 = u_2$ in $\Omega$. Finally we deduce from Theorem 2.15 that $g_1 = g_2$ in $\Omega$. □

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