King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 280

May 2002

Generalized Macaulay-Northcott Modules and Tor-Groups

Liu Zhongkui and Javed Ahsan

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

Generalized Macaulay-Northcott Modules and Tor-Groups

Liu Zhongkui*
Department of Mathematics, Northwest Normal University
Lanzhou, 730070, China
liuzk@nwnu.edu.cn

Javed Ahsan
Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran, 31261, Saudi Arabia
e-mail: jahsan@kfupm.edu.sa

Abstract Let $(S, \leq)$ be a strictly totally ordered monoid which is also artinian, and R a right noetherian ring. Assume that M is a finitely generated right R-module and N a left R-module. Denote by $[[M^{S,\leq}]]$ and $[N^{S,\leq}]$ the module of generalized power series over M, and the generalized Macaulay-Northcott module over N, respectively. Then we show that there exists an isomorphism of Abelian groups:

$$Tor_i^{[[R^{S,\leq}]]}([[M^{S,\leq}]], [N^{S,\leq}]) \cong \bigoplus_{s \in S} Tor_i^R(M, N).$$

Key Words: generalized Macaulay-Northcott module, ring of generalized power series, Tor-group.

2000 Mathematics Subject Classification: 16W60, 16E30

1. Rings of generalized power series

All rings considered here are associative with identity. Any concept and notation not defined here can be found in [1-3].

*Supported by National Natural Science Foundation of China (10171082) and by Foundation for University Key Teacher by the Ministry of Education(GG-110-10736-1001)

Let $(S, \leq)$ be an odered set. Recall that $(S, \leq)$ is artinian if every strictly decreasing sequence of elements of S is finite, and that $(S, \leq)$ is narrow if every subset of pairwise order-incomparable elements of S is finite. Let S be a commutative monoid. Unless stated otherwise, the operation of S shall be denoted additively, and the neutral element by 0. The following definition is due to [4].

Let $(S, \leq)$ be a strictly ordered monoid (that is, $(S, \leq)$ is an ordered monoid satisfying the condition that, if $s, s', t \in S$ and $s < s'$, then $s + t < s' + t$), and R a ring. Let $[[R^{S,\leq}]]$ be the set of all maps $f : S \longrightarrow R$ such that $supp(f) = \{s \in S | f(s) \neq 0\}$ is artinian and narrow. With pointwise addition, and the operation of convolution

$$(fg)(s) = \sum_{(u,v) \in X_s(f,g)} f(u)g(v),$$

where $X_s(f,g) = \{(u,v) \in S \times S | s = u + v, f(u) \neq 0, g(v) \neq 0\}$ is a finite set by [4, 4.1] for every $s \in S$ and $f, g \in [[R^{S,\leq}]]$, $[[R^{S,\leq}]]$ becomes a ring, which is called the ring of generalized power series. The elements of $[[R^{S,\leq}]]$ are called generalized power series with coefficients in R and exponents in S. Many examples and results of rings of generalized power series are given in [1-7].

2. Modules of generalized power series

Let M be a right R-module over a ring R and $(S, \leq)$ a strictly ordered monoid. Set $[[M^{S,\leq}]]$ is the set of all maps $\phi : S \longrightarrow M$ such that $supp(\phi) = \{s \in S | \phi(s) \neq 0\}$ is artinian and narrow. With pointwise addition, $[[M^{S,\leq}]]$ is an abelian additive group. For each $f \in [[R^{S,\leq}]]$, each $\phi \in [[M^{S,\leq}]]$, and $s \in S$, denote

$$X_s(\phi, f) = \{(u,v) \in S \times S | s = u + v, \phi(u) \neq 0, f(v) \neq 0\}.$$

Then, by [8, Lemma 1], $X_s(\phi, f)$ is finite. Now $[[M^{S,\leq}]]$ can be turned into a right $[[R^{S,\leq}]]$-module by the scalar multiplication defined as following

$$(\phi f)(s) = \sum_{(u,v) \in X_s(\phi,f)} \phi(u)f(v)$$

for each $f \in [[R^{S,\leq}]]$ and each $\phi \in [[M^{S,\leq}]]$. $[[M^{S,\leq}]]$ is called the module of generalized power series over a right R-module M. The elements of $[[M^{S,\leq}]]$ are called generalized power series with coefficients in M and exponents in S.

Similarly, if M is a left R-module, then $[[M^{S,\leq}]]$ is a left $[[R^{S,\leq}]]$-module. Examples and results of modules of generalized power series are given in [8].

Let M be a right R-module. Define a mapping $\alpha : M \otimes_R [[R^{S,\leq}]] \longrightarrow [[M^{S,\leq}]]$ via

$$\alpha(\sum(m_i \otimes f_i))(s) = \sum m_i f_i(s), \qquad \forall m_i \in M, \forall f_i \in [[R^{S,\leq}]], \forall s \in S.$$

Proposition 2.1. *Let M be a finitely generated right R-module. Then α is an epimorphism of right $[[R^{S,\leq}]]$-modules.*

Proof. For any $\sum_1^n (m_i \otimes f_i) \in M \otimes_R [[R^{S,\leq}]]$, we have

$$\begin{aligned} supp\left(\alpha\left(\sum_1^n (m_i \otimes f_i)\right)\right) &= \left\{s \in S \,|\, \sum_1^n m_i f_i(s) \neq 0\right\} \\ &\subseteq \cup_1^n \{s \in S | f_i(s) \neq 0\} = \cup_1^n supp(f_i). \end{aligned}$$

Thus $supp(\alpha(\sum_1^n (m_i \otimes f_i)))$ is artinian and narrow. Hence $\alpha(\sum_1^n (m_i \otimes f_i)) \in [[M^{S,\leq}]]$. Now it is easy to see that the mapping α is well-defined. For any $f, g \in [[R^{S,\leq}]]$, any $m \in M$ and any $s \in S$,

$$\begin{aligned} \alpha((m \otimes f)g)(s) &= \alpha(m \otimes (fg))(s) = m(fg)(s) \\ &= m\left(\sum_{(u,v)\in X_s(f,g)} f(u)g(v)\right) \\ &= \sum_{(u,v)\in X_s(f,g)} (mf(u))g(v) \\ &= \sum_{(u,v)\in X_s(f,g)} \alpha(m \otimes f)(u)g(v) \\ &= \sum_{(u,v)\in X_1} \alpha(m \otimes f)(u)g(v) + \sum_{(u,v)\in X_2} \alpha(m \otimes f)(u)g(v) \\ &= (\alpha(m \otimes f)g)(s), \end{aligned}$$

where

$$\begin{aligned} X_1 &= \{(u,v) | f(u) \neq 0, g(v) \neq 0, u + v = s, \alpha(m \otimes f)(u) = 0\}, \\ X_2 &= \{(u,v) | f(u) \neq 0, g(v) \neq 0, u + v = s, \alpha(m \otimes f)(u) \neq 0\} \\ &= \{(u,v) | g(v) \neq 0, u + v = s, \alpha(m \otimes f)(u) \neq 0\}, \end{aligned}$$

and clearly, $X_s(f,g) = X_1 \cup X_2$. Thus $\alpha((m \otimes f)g) = \alpha(m \otimes f)g$. This means that α is an $[[R^{S,\leq}]]$-homomorphism.

Now suppose that M is generated by $x_1, \cdots, x_n$. For any $h \in [[M^{S,\leq}]]$ and any $s \in supp(h)$, we write $h(s) = \sum_{j=1}^n x_j r_{js}$, where $r_{js} \in R$. Define $f_j : S \longrightarrow R$ via

$$f_j(s) = \begin{cases} r_{js}, & s \in supp(h) \\ 0, & s \notin supp(h). \end{cases}$$

Clearly $supp(f_j) \subseteq supp(h)$, and so $supp(f_j)$ is artinian and narrow. This means that $f_j \in [[R^{S,\leq}]]$. Now for any $s \in supp(h)$, $\alpha(\sum_{j=1}^n (x_j \otimes f_j))(s) = \sum_{j=1}^n \alpha(x_j \otimes f_j)(s) = \sum_{j=1}^n x_j f_j(s) = \sum_{j=1}^n x_j r_{js} = h(s)$. If $s \notin supp(h)$, then clearly $\alpha(\sum_{j=1}^n (x_j \otimes f_j))(s) = h(s)$. Hence $\alpha(\sum_{j=1}^n (x_j \otimes f_j)) = h$. This means that α is an epimorphism.

Let M, N be right R-modules and $\alpha : M \longrightarrow N$ an R-homomorphism. Define a mapping $[[\alpha^{S,\leq}]] : [[M^{S,\leq}]] \longrightarrow [[N^{S,\leq}]]$ via

$$\begin{aligned} [[\alpha^{S,\leq}]](g) : & S \longrightarrow N \\ & s \longrightarrow \alpha(g(s)) \end{aligned}$$

for any $g \in [[M^{S,\leq}]]$. Clearly $supp([[\alpha^{S,\leq}]](g)) \subseteq supp(g)$. Thus it follows that $supp([[\alpha^{S,\leq}]](g))$ is artinian and narrow. Hence $[[\alpha^{S,\leq}]](g) \in [[N^{S,\leq}]]$. This means that $[[\alpha^{S,\leq}]]$ is well-defined. The following results appeared in [9].

LEMMA 2.2.

(1) $[[\alpha^{S,\leq}]]$ *is an* $[[R^{S,\leq}]]$*-homomorphism.*

(2) *If* $M \xrightarrow{\alpha} N \xrightarrow{\beta} L$ *is a complex, then so is*

$$[[M^{S,\leq}]] \xrightarrow{[[\alpha^{S,\leq}]]} [[N^{S,\leq}]] \xrightarrow{[[\beta^{S,\leq}]]} [[L^{S,\leq}]].$$

(3) *The functor* $[[(-)^{S,\leq}]] : Mod\text{-}R \longrightarrow Mod\text{-}[[R^{S,\leq}]]$ *is exact.*

LEMMA 2.3. *If M is a finitely generated free right R-module, then α is an isomorphism of right $[[R^{S,\leq}]]$-modules.*

PROOF. Without loss of generality, we assume that $M = \oplus_1^n R$. By Lemma 2.2, there exists a natural isomorphism of right $[[R^{S,\leq}]]$-modules $[[A^{S,\leq}]] \oplus [[B^{S,\leq}]] \cong [[(A \oplus B)^{S,\leq}]]$ for any right R-modules A and B. Thus we have a natural isomorphism of right $[[R^{S,\leq}]]$-modules $\beta : M \otimes_R [[R^{S,\leq}]] \cong \oplus_1^n (R \otimes_R [[R^{S,\leq}]]) \cong \oplus_1^n [[R^{S,\leq}]] \cong [[(\oplus_1^n R)^{S,\leq}]] \cong [[M^{S,\leq}]]$. It is easy to see that $\alpha = \beta$. Thus α is an isomorphism of right $[[R^{S,\leq}]]$-modules.

Lemma 2.4. *If M is a finitely presented right R-module, then α is an isomorphism of right $[[R^{S,\leq}]]$-modules.*

Proof. Choose an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ with F finitely generated free. There is induced a commutative diagram

$$\begin{array}{ccccccc} K \otimes_R [[R^{S,\leq}]] & \longrightarrow & F \otimes_R [[R^{S,\leq}]] & \longrightarrow & M \otimes_R [[R^{S,\leq}]] & \longrightarrow & 0 \\ \gamma \downarrow & & \beta \downarrow & & \alpha \downarrow & & \\ [[K^{S,\leq}]] & \longrightarrow & [[F^{S,\leq}]] & \longrightarrow & [[M^{S,\leq}]] & \longrightarrow & 0 \end{array}$$

with exact rows by Lemma 2.2(3). By Lemma 2.3, β is an isomorphism. When M is finitely presented, K is finitely generated and hence γ is an epimorphism by Proposition 2.1. An easy diagram chase shows that α must then be a monomorphism, hence an isomorphism.

Remark 2.5. The converse of Lemma 2.4 is not true in general. Let R be a ring. Suppose that the monoid S is trivially ordered. Then the artinian and narrow subsets are the finite subsets. Thus for every right R-module M, there exists an isomorphism of right R-modules $[[M^{S,\leq}]] \cong \oplus_{s \in S} M$. Similarly, there exists an isomorphism of left R-modules $[[R^{S,\leq}]] \cong \oplus_{s \in S} R$. Thus there exists an isomorphism of Abelian groups $\beta : M \otimes_R [[R^{S,\leq}]] \cong M \otimes_R (\oplus_{s \in S} R) \cong \oplus_{s \in S}(M \otimes_R R) \cong \oplus_{s \in S} M \cong [[M^{S,\leq}]]$. It is easy to see that $\alpha = \beta$. Thus, by Proposition 2.1, α is an isomorphism of right $[[R^{S,\leq}]]$-modules. But we can take M such that it is not finitely presented.

Lemma 2.6. *If P_R is finitely generated projective, then $[[P^{S,\leq}]]$ is a projective right $[[R^{S,\leq}]]$-module.*

Proof. It follows from Lemma 2.2 that for any right R-modules M and N, there exists an isomorphism of right $[[R^{S,\leq}]]$-modules $[[M^{S,\leq}]] \oplus [[N^{S,\leq}]] \cong [[(M \oplus N)^{S,\leq}]]$. Now the result follows.

3. Generalized Macaulay-Northcott modules

If M is a left R-module, we let $[M^{S,\leq}]$ be the set of all maps $\phi : S \longrightarrow M$ such that the set $supp(\phi) = \{s \in S | \phi(s) \neq 0\}$ is finite. Now $[M^{S,\leq}]$ can be turned into a left $[[R^{S,\leq}]]$-module under some additional conditions. The addition in $[M^{S,\leq}]$ is componentwise and the scalar multiplication is defined as follows

$$(f\phi)(s) = \sum_{t \in S} f(t)\phi(s+t), \qquad \text{for every } s \in S,$$

where $f \in [[R^{S,\leq}]]$, and $\phi \in [M^{S,\leq}]$. Since the set $supp(\phi)$ is finite, this multiplication is well-defined. If $(S, \leq)$ is a strictly totally ordered monoid which is also artinian, then, from [6], $[M^{S,\leq}]$ becomes a left $[[R^{S,\leq}]]$-module, which we called the generalized Macaulay-Northcott module.

For example, if $S = \mathbb{N}$ and $\leq$ is the usual order, then $[M^{\mathbb{N},\leq}] \cong M[x^{-1}]$, the usual left $R[[x]]$-module introduced in [10, 11], which is called the Macaulay-Northcott module in [12, 13].

We shall henceforth assume that $(S, \leq)$ is a strictly totally ordered monoid which is also artinian. Then it is easy to see that $(S, \leq)$ satisfies the condition that $0 \leq s$ for every $s \in S$ ([14]).

For any abelian additive group G, we denote by $[[G^{S,\leq}]]$ the set of all maps $h : S \longrightarrow G$. With pointwise addition, $[[G^{S,\leq}]]$ is an abelian additive group.

For any R-homomorphism $\alpha : M \longrightarrow N$, define $f \in [[Hom_R(M, N)^{S,\leq}]]$ via $f(0) = \alpha$ and $f(x) = 0$ for all $0 \neq x \in S$. By [6, Lemma 2.3] and its proof, there exists $[\alpha^{S,\leq}] \in Hom_{[[R^{S,\leq}]]}([M^{S,\leq}], [N^{S,\leq}])$ such that for any $\phi \in [M^{S,\leq}]$ and any $s \in S$,

$$[\alpha^{S,\leq}](\phi)(s) = \sum_{u \in S} f(u)(\phi(s+u)) = \alpha(\phi(s)).$$

LEMMA 3.1. *The functor* $[(-)^{S,\leq}] : R\text{-}Mod \longrightarrow [[R^{S,\leq}]]\text{-}Mod$ *defined as* $[(-)^{S,\leq}](M) = [M^{S,\leq}]$, $[(-)^{S,\leq}](\alpha) = [\alpha^{S,\leq}]$, *is exact.*

PROOF. It follows from [14, Lemma 5].

LEMMA 3.2. *Let* $N \leq M$ *be left* R*-modules. Then*

$$[M^{S,\leq}]/[N^{S,\leq}] \cong [(M/N)^{S,\leq}]$$

as left $[[R^{S,\leq}]]$*-modules.*

PROOF. It follows from Lemma 3.1.

LEMMA 3.3. *Let* M *be a finitely presented right* R*-module and* N *a left* R*-module. Then there is a natural isomorphism* $[[M^{S,\leq}]] \otimes_{[[R^{S,\leq}]]} [N^{S,\leq}] \cong [(M \otimes_R N)^{S,\leq}]$.

PROOF. It is easy to see that there exists an isomorphism of left R-modules $[N^{S,\leq}] \cong \oplus_{s \in S} N$. By Lemma 2.4, there exists a natural isomorphism of right $[[R^{S,\leq}]]$-modules $M \otimes_R [[R^{S,\leq}]] \cong [[M^{S,\leq}]]$ since M is finitely presented. Now, we have

$$\begin{aligned}
[[M^{S,\leq}]] \otimes_{[[R^{S,\leq}]]} [N^{S,\leq}] &\cong (M \otimes_R [[R^{S,\leq}]]) \otimes_{[[R^{S,\leq}]]} [N^{S,\leq}] \\
&\cong M \otimes_R ([[R^{S,\leq}]] \otimes_{[[R^{S,\leq}]]} [N^{S,\leq}]) \\
&\cong M \otimes_R [N^{S,\leq}] \cong M \otimes_R (\oplus_{s\in S} N) \\
&\cong \oplus_{s\in S}(M \otimes_R N) \\
&\cong [(M \otimes_R N)^{S,\leq}].
\end{aligned}$$

Clearly all isomorphisms mentioned above are natural.

4. Tor-groups

THEOREM 4.1. *Let S be a strictly totally ordered monoid which is also artinian and R a right noetherian ring. If M is a finitely generated right R-module and N a left R-module, then there exist isomorphisms of Abelian groups:*

$$Tor_i^{[[R^{S,\leq}]]}([[M^{S,\leq}]], [N^{S,\leq}]) \cong [Tor_i^R(M,N)^{S,\leq}] \cong \bigoplus_{s\in S} Tor_i^R(M,N).$$

PROOF. Since R is right noetherian, there exists a projective resolution

$$\cdots \longrightarrow P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} M \longrightarrow 0$$

of M such that $P_0, P_1, \cdots$ are finitely generated and projective. Then, by Lemma 2.2 and 2.6,

$$\cdots \longrightarrow [[P_2^{S,\leq}]] \longrightarrow [[P_1^{S,\leq}]] \longrightarrow [[P_0^{S,\leq}]] \longrightarrow [[M^{S,\leq}]] \longrightarrow 0$$

is a projective resolution of right $[[R^{S,\leq}]]$-module $[[M^{S,\leq}]]$. Consider the deleted projective resolution

$$\cdots \longrightarrow [[P_2^{S,\leq}]] \xrightarrow{[[\delta_2^{S,\leq}]]} [[P_1^{S,\leq}]] \xrightarrow{[[\delta_1^{S,\leq}]]} [[P_0^{S,\leq}]] \longrightarrow 0,$$

we have the complex

$$\cdots \longrightarrow [[P_2^{S,\leq}]] \otimes_{[[R^{S,\leq}]]} [N^{S,\leq}] \xrightarrow{[[\delta_2^{S,\leq}]](*)} [[P_1^{S,\leq}]] \otimes_{[[R^{S,\leq}]]} [N^{S,\leq}]$$

$$\xrightarrow{[[\delta_1^{S,\leq}]](*)} [[P_0^{S,\leq}]] \otimes_{[[R^{S,\leq}]]} [N^{S,\leq}] \xrightarrow{[[\delta_0^{S,\leq}]](*)} 0,$$

where $[[\delta_i^{S,\leq}]](*) = [[\delta_i^{S,\leq}]] \otimes_{[[R^{S,\leq}]]} 1_{[N^{S,\leq}]}$ for every $i = 0, 1, \cdots$. On the other hand, we have the complex

$$\cdots \longrightarrow P_2 \otimes_R N \xrightarrow{\delta_2(*)} P_1 \otimes_R N \xrightarrow{\delta_1(*)} P_0 \otimes_R N \xrightarrow{\delta_0(*)} 0$$

where $\delta_i(*) = \delta_i \otimes_R 1_N$ for every $i = 0, 1, \cdots$. Thus, by Lemma 3.1, we have the complex

$$\cdots \longrightarrow [(P_2 \otimes_R N)^{S,\leq}] \xrightarrow{[\delta_2(*)^{S,\leq}]} [(P_1 \otimes_R N)^{S,\leq}]$$
$$\xrightarrow{[\delta_1(*)^{S,\leq}]} [(P_0 \otimes_R N)^{S,\leq}] \xrightarrow{[\delta_0(*)^{S,\leq}]} 0.$$

Clearly $P_0, P_1, \cdots$ are finitely presented. Thus by Lemma 3.3, there exists a natural isomorphism

$$[[P_i^{S,\leq}]] \otimes_{[[R^{S,\leq}]]} [N^{S,\leq}] \cong [(P_i \otimes_R N)^{S,\leq}].$$

Thus, by Lammas 3.1, 3.2 and 3.3, we have

$$\begin{aligned} Tor_i^{[[R^{S,\leq}]]}([[M^{S,\leq}]], [N^{S,\leq}]) &= Ker([[\delta_i^{S,\leq}]](*))/Im([[\delta_{i-1}^{S,\leq}]](*)) \\ &\cong Ker([\delta_i(*)^{S,\leq}])/Im([\delta_{i-1}(*)^{S,\leq}]) \\ &\cong [Ker(\delta_i(*))^{S,\leq}]/[Im(\delta_{i-1}(*))^{S,\leq}] \\ &\cong [(Ker(\delta_i(*))/Im(\delta_{i-1}(*)))^{S,\leq}] \\ &= [Tor_i^R(M,N)^{S,\leq}]. \end{aligned}$$

The isomorphism $[Tor_i^R(M,N)^{S,\leq}] \cong \oplus_{s\in S} Tor_i^R(M,N)$ is clear.

COROLLARY 4.2. *If R is a right noetherian ring, M a finitely generated right R-module and N a left R-module, then there exist isomorphisms of Abelian groups*

$$Tor_i^{R[[x]]}(M[[x]], N[x^{-1}]) \cong \bigoplus_{n=0}^{\infty} Tor_i^R(M,N) \cong Tor_i^R(M,N)[x^{-1}].$$

COROLLARY 4.3. *Let S be a torsion-free and cancellative monoid, and $(S, \leq)$ be artinian and narrow. If R is a right noetherian ring, M a finitely generated right R-module and N a left R-module, then*

$$Tor_i^{[[R^{S,\leq}]]}([[M^{S,\leq}]], [N^{S,\leq}]) \cong \bigoplus_S Tor_i^R(M,N).$$

PROOF. If $(S, \leq)$ is torsion-free and cancellative, then by [1, 3.3], there exists a compatible strict total order $\leq'$ on S, which is finer than $\leq$, that is, for any $s, t \in S$, $s \leq t$ implies $s \leq' t$. Since $(S, \leq)$ is artinian and narrow, by [1, 2.5] it follows that $(S, \leq')$ is artinian and narrow. Thus, by Theorem 4.1, $Tor_i^{[[R^{S,\leq'}]]}([[M^{S,\leq'}]], [N^{S,\leq'}]) \cong \oplus_S Tor_i^R(M, N)$.

On the other hand, since $(S, \leq)$ is narrow, by [1, 4.4], $[[R^{S,\leq}]] = [[R^{S,\leq'}]]$. Clearly $[[M^{S,\leq}]] = [[M^{S,\leq'}]]$ and $[N^{S,\leq}] = [N^{S,\leq'}]$. Now the result follows.

Any submonoid of the additive monoid $\mathbb{N} \cup \{0\}$ is called a numerical monoid. We have

COROLLARY 4.4. *Let S be a numerical monoid and $\leq$ the usual natural order of $\mathbb{N} \cup \{0\}$. If R is a right noetherian ring, M a finitely generated right R-module and N a left R-module, then*

$$Tor_i^{[[R^{S,\leq}]]}([[M^{S,\leq}]], [N^{S,\leq}]) \cong \bigoplus_S Tor_i^R(M, N).$$

COROLLARY 4.5. *Suppose that $(S_1, \leq_1), \ldots, (S_n, \leq_n)$ are strictly totally ordered monoids which are artinian. Denote by $(lex \leq)$ and $(revlex \leq)$ the lexicographic order, the reverse lexicographic order, respectively, on the monoid $S_1 \times \cdots \times S_n$. If R is a right noetherian ring, M a finitely generated right R-module and N a left R-module, then there exist isomorphisms of Abelian groups*

$$\begin{aligned}
&Tor_i^{[[R^{S_1 \times \cdots \times S_n, (lex \leq)}]]}([[M^{S_1 \times \cdots \times S_n, (lex \leq)}]], [N^{S_1 \times \cdots \times S_n, (lex \leq)}]) \\
&\cong Tor_i^{[[R^{S_1 \times \cdots \times S_n, (revlex \leq)}]]}([[M^{S_1 \times \cdots \times S_n, (revlex \leq)}]], [N^{S_1 \times \cdots \times S_n, (revlex \leq)}]) \\
&\cong \bigoplus_{S_1 \times \cdots \times S_n} Tor_i^R(M, N).
\end{aligned}$$

PROOF. It is easy to see that $(S_1 \times \cdots \times S_n, (lex \leq))$ and $(S_1 \times \cdots \times S_n, (revlex \leq))$ are strictly totally ordered monoids which are artinian. Thus the result follows from Theorem 4.1.

References

[1] Ribenboim, P. Noetherian Rings of Generalized Power Series. J. Pure Appl. Algebra **1992**, *79*, 293-312.

[2] Ribenboim, P. Rings of Generalized Power Series II: Units and Zero-Divisors. J. Algebra **1994**, *168*, 71-89.

[3] Ribenboim, P. Special Properties of Generalized Power Series. J. Algebra **1995**, *173*, 566-586.

[4] Ribenboim, P. Semisimple Rings and von Neumann Regular Rings of Generalized Power Series. J. Algebra **1997**, *198*, 327-338.

[5] Varadarajan, K. Noetherian Generalized Power Series Rings and Modules. Comm. Algebra **2001**, *29*(10), 245-251.

[6] Liu, Z.; Cheng, H. Quasi-Duality for the Rings of Generalized Power Series. Comm. Algebra **2000**, *28*(3), 1175-1188.

[7] Liu, Z. Morita Duality for the Rings of Generalized Power Series. Acta Mathematica Sinica, English Series, to appear.

[8] Liu, Z. A Note on Hopfian Modules. Comm. Algebra **2000**, *28*(6), 3031-3040.

[9] Liu, Z. The Ext-Groups of Generalized Macaulay-Northcott Modules. submitted.

[10] McKerrow, A. S. On the Injective Dimension of Modules of Power Series. Quart. J. Math. Oxford **1974**, *25*, 359-368.

[11] Northcott, D. G. Injective Envelopes and Inverse Polynomials. London Math. Soc. **1974**, *3*(2), 290-296.

[12] Park, S. The Macaulay-Northcott Functor. Arch. Math. **1994**, *63*, 225-230.

[13] Park, S. Inverse Polynomials and Injective Covers. Comm. Algebra **1993**, *21*(12), 4599-4613.

[14] Liu, Z. Injectivity of Modules of Generalized Inverse Polynomials. Comm. Algebra **2001**, *29*(2), 583-592.