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G.K. Beg

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G.K. Beg
Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran 31261 , Saudi Arabia
Email: kabir@kfupm.edu.sa

**Abstract**

Stirling number of the second kind are generated with the help of an operator applied to exponential function. Consequently some identities involving Stirling number of the second kind are obtained.

## Introduction

Stirling number of the second kind, denoted by $S(n,r), r = 1,2,\ldots,n$, are popularly defined by $x^n = \sum_{r=1}^{n} S(n,r)x_{(r)}$ where $x_{(r)} = x(x-1)\ldots(x-r+1)$ is the factorial polynomial of order $r$. The Stirling numbers of the second kind follow the following recurrence formula:

$$S(n,r) = S(n-1,r-1) + rS(n-1,r), \quad r = 1,2,......,n$$

where $S(n,0) = 0$ for all $n$. In this paper we define the following operator

$$\hat{D} \equiv xD \equiv x\frac{d}{dx}$$

and apply it to the exponential function $e^x$ to generate Stirling numbers of the second kind. An exponential type of series is obtained as a consequence. Some new identities involving Stirling numbers are obtained. Dubinski's formula is found as a special case.

## Main Results

Define the function

$$f(n,x) = \sum_{i=1}^{\infty} \frac{i^n x^i}{i!} \tag{1}$$

where $n$ is a positive integer.

Let us define the operator

$$\hat{D} \equiv xD \equiv x\frac{d}{dx}$$

Then we have the following theorem

**Theorem 2** *For* $r, n \in N$ $\hat{D}^n e^x = e^x \sum_{r=1}^{n} S(n,r)x^r$ *holds,* where $S(n,r)$ denotes the Stirling number of the second kind.

**Proof.**

$$\begin{aligned}
\hat{D}^1 e^x &= x\frac{d}{dx}e^x = e^x \sum_{r=1}^{1} S(1,r)x^r \\
\hat{D}^2 e^x &= x\frac{d}{dx}\left(xe^x\right) \\
&= e^x\left(x + x^2\right) \\
&= e^x \sum_{r=1}^{2} S(2,r)x^r
\end{aligned}$$

$$\begin{aligned}
\hat{D}^3 e^x &= x\frac{d}{dx}e^x\left(x + x^2\right) \\
&= xe^x\left(1 + 3x + x^2\right) \\
&= e^x\left(x + 3x^2 + x^3\right) \\
&= e^x \sum_{r=1}^{3} S(3,r)x^r.
\end{aligned}$$

Now the proof follows by induction:

$$\begin{aligned}
\hat{D}^{n+1} e^x &= x\frac{d}{dx}\left[\ e^x \sum_{r=1}^{n} S(n,r)x^r\ \right] \\
&= xe^x\left[\ \sum_{r=1}^{n} S(n,r)x^r + \sum_{r=1}^{n} S(n,r)rx^{r-1}\ \right] \\
&= e^x\left[\ \sum_{r=1}^{n} S(n,r)x^{r+1} + \sum_{r=1}^{n} S(n,r)x^r\ \right]
\end{aligned}$$

Putting $r + 1 = j$ in the first summation

$= e^x \left[\ \sum_{j=2}^{n+1} S(n, j-1)x^j + \sum_{j=1}^{n} jS(n,j)x^j\ \right]$

$= e^x \left[\ S(n,1)x + \sum_{j=2}^{n}\{S(n, j-1) + jS(n,j)\}x^j + S(n,n)x^{n+1}\ \right]$

$= e^x \left[\ S(n+1,1)x + \sum_{j=2}^{n} S(n+1,j)x^j + S(n+1,n+1)x^{n+1}\ \right]$

$= e^x \sum_{j=1}^{n+1} S(n+1,j)x^j$

This completes the proof.

**Theorem 2** *For $n \in N$ and $x \in R$ the following holds:* $\hat{D}^n e^x = f(n,x) = \sum_{i=1}^{\infty} \frac{i^n x^i}{i!}$.

**Proof.**

$$\begin{aligned} \hat{D}^1 e^x &= x \frac{d}{dx} e^x \\ &= x e^x \\ &= x + \frac{x^2}{1!} + \frac{x^3}{2!} + \frac{x^4}{3!} + \dots \\ &= \left[ \tfrac{x}{1!} + \tfrac{2x^2}{2!} + \tfrac{3x^3}{3!} + \dots \right] \\ &= f(1,x) \end{aligned}$$

By induction

$$\begin{aligned} \hat{D}^{n+1} e^x &= \hat{D} f(n,x) \\ &= x \frac{d}{dx} \sum_{i=1}^{\infty} \frac{i^n x^i}{i!} \\ &= x \sum_{i=1}^{\infty} \frac{i^{n+1} x^{i-1}}{i!} \\ &= \sum_{i=1}^{\infty} \frac{i^{n+1} x^i}{i!} = f(n+1,x). \end{aligned}$$

This completes the proof.

**Theorem 3** *For $n \in N$ and $x \in R$*

$$f(n,x) = e^x \sum_{r=1}^{n} S(n,r) x^r = \sum_{i=1}^{\infty} \frac{i^n x^i}{i!} \tag{2}$$

*Proof.* The proof follows from Theorem 1 and Theorem 2

For $x = 1$, (2) becomes:

$$B_n := \sum_{r=1}^{n} S(n,r) = \frac{1}{e} \sum_{i=1}^{\infty} \frac{i^n}{i!} \tag{3}$$

*which is Dobinski's formula (see [1], Formula 3, p44).*

**Theorem 4** *For $r, n \in N$*

$$\sum_{j=1}^{r} \frac{S(n,j)}{(r-j)!} = \frac{r^n}{r!}$$

*In particular, when $r = n$, we have*

$$\sum_{r=1}^{n} \frac{S(n,r)}{(n-r)!} = \frac{n^n}{n!}$$

**Proof.** Equating the coefficient of $x^r$ of both sides of (2) we get the desired result.

**Corollary** *For* $r, n \in N$

$$\sum_{j=1}^{r} S(n,j)P_{r,j} = r^n$$

*In particular when* $r = n$, *we get*

$$\sum_{r=1}^{n} S(n,r)P_{n,r} = n^n$$

*where* $P_{n,r}$ *is the permutation of* $n$ *things taken* $r$ *at a time.*

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**Reference**

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2. Eisen, M. (1969) *Elementary Combinatorial analysis,* Cordon and Breach Science Publishers, New York.