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Abstract

Ill-posed problems arise in many branches of science and engineering. In the typical situation one is intrested in recovering a whole function, given a finite number of noisy measurements on functionals. In this paper we shall be considering Inversion of Laplace Transform which is a severely ill-posed problem. We shall convert the Laplace Transform to an integral equation of first kind of convolution type which is also an ill-posed problem and we shall be using two methods to solve the integral equation (i) Maximum Likelihood Regularization method and (ii) Cross-validation method. Both methods yield good results as shown in the tables. The methods are applied to various test examples available in the literature and the results are shown in the tables.

AMS (MOS) subject classification: 65R20, 65R30

Key Words: Ill-posed problem, inversion of Laplace transform, Convolution equation, Maximum-likelihood method, Filter function, Plancherels' theorem, Regularization parameter. Cross-validation method.

1 Introduction

Inverse problems pertain to situations where one is interested in making inferences about a phenomenon from partial or incomplete information. Accordingly, statistical estimation is an inverse problem.

In modern science there is an increasingly important class of inverse problems which are not amenable to classical statistical estimation procedures and such problems are termed ill-posed. The notion of ill-posedness is usually attributed to Hadamard [12]; a modern treatment of the concept appeared in Tikhonov and Arsenin [26].

The Laplace transform inversion is a severely ill-posed problem in the terminology of improperly posed problems. Unfortunately, many problems of physical interest lead to Laplace transforms whose inverses are not readily expressed in terms of tabulated functions. Although there exist extensive tables of transforms and their inverses, it is highly desirable, therefore, to have methods for approximate numerical inversion. However, no single method gives optimum results for all purposes and all occasions.

For detailed bibliography, the reader should consult Piessens [19], Piessens and Branchers [20], and a review and comparison is given in Davies [6], Talbot [24], Istratov [13] and Shan [23].

The problem of the reovery of a real function $f(t)$ for $t \geq 0$, given its Laplace transform

$$\int_0^\infty e^{-st} f(t)dt = g(s) \tag{1.1}$$

for real values of s, is an ill-posed problem and is therefore affected by numerical instability.

The ill-posedness of Laplace transform inversion in the case where $f \in L^2(R_+)$ and $g(s)$ is known for all real and positive values of s, can be investigated by means of Mellin transform [17], by means of maximum entropy method [18].

The above mentioned methods do not include regularization techniques. Regularization methods have been discussed by Brianzi [4], Essah and delves [9], Ang[1], Gelfat [11], Varah [27], Wahba [28], Eggermont [8], Bertero [2] and Pinkus [21]. In particular, the theory is used to tackle the Laplace transform inversion in a well-conditioned (regularized) manner. This difficult numerical problem which is frequently encountered by plysicists and engineers is still the subject of much attention in he literature.

Finally, we include some specific numerical examples to illustrate and demonstrate clearly the need to consider the information content in order to avoid obtaining meaningless results. In(1.1) given $g(s)$ for $s \geq 0$, we wish to find $f(t)$ for $t \geq 0$and $f(t) = 0$ for $t < 0$; so that (1.1) holds. Frequently $g(s)$ is measured at certain points. We assume $g(s)$ is given analytically with known $f(t)$, so that we can measure the error in the numerical solution.

2 Fredholm Equation of Convolution Type

The convolution integral equations are widely used in physics, especially in spectroscopy and the reader is referred to two special monographs by Blass et. al. [3], and Jansen [14].

We shall convert the Laplace transform into the in first kind integral equation of convoluton type with the following substitution in equation (1.1)

$$s = a^x \text{ and } t = a^{-y} \text{ where } a > 1 \tag{2.1}$$

Then

$$g\left(a^x\right) = \int_{-\infty}^{\infty} \log a \, e^{-a^{x-y}} f\left(a^{-y}\right) dy \tag{2.2}$$

multiplying both sides of (2.2) by a^x, we obtain the convolution equation.

$$\int_{-\infty}^{\infty} K(x-y)F(y)dy = G(x), \qquad -\infty \leq x \leq \infty \tag{2.3}$$

where

$$\left.\begin{array}{l} G(x) = a^x g\left(a^x\right) = sg(s) \\ K(x) = (\log a) a^x \, e^{-a^x} = (\log a) s e^{-s} \\ F(y) = f\left(a^{-y}\right) = f(t) \end{array}\right\} \tag{2.4}$$

Equation (2.3) occurs widely in the applied sciences. K and G are known Kernel and data functions respectively, and F is to be determined. We shall assume that F, G and

K lie in suitable function spaces such as $L_2(R)$, so that their Fourier transforms (FT_s) exist. ($\wedge$ denotes FTs and $\vee$ denotes inverse FTs).

3 Description of the Method

We assume that the support of each function F, G and K is essentially finite and contained within the interval $[0, T]$, where T is the period and equal to Nh, where N is the number of data points and h is the spacing.

Let T_N be the space of Trigonometric polynomials of degree at most N with period T. We shall look for filtered solution of (2.3) within the space T_N for the following reasons:

(a) The discretization error in the Convolution may be made precisely zero at grid points.

(b) Fast FourierTransform (FFT) routines are easily employed in the solution procedure.

(c) The adoption of T_N as the approximating function space is iteslf a regularizing feature.

Let G and K be given at N equally spaced points $x_n = nh, n = 0, 1, 2, \ldots, N-1$ with spacing $h = T/N$. Then G and K are interpolated by G_N and $K_N \in T_N$ where

$$G_N(x) = \frac{1}{N}\sum_{q=0}^{N-1} \hat{G}_{N,q} \exp\ (i\omega_q x) \tag{3.1}$$

$$\hat{G}_{N,q} = \sum_{n=0}^{N-1} G_n \exp\ (-i\omega_q x_n) \tag{3.2}$$

where

$$G(x_n) = G_N(x_n) = G_n, \qquad \omega_q = \frac{2\pi q}{T} \tag{3.3}$$

Similar expresions as (3.1) and (3.2) can be obtained for K_N.

Consider (2.3) the Fredholm integral equation of the first kind of convolution type

$$\left.\begin{array}{c}(KF)(x) = \int_{-\infty}^{\infty} K(x-y)F(y)dy = G(x) \\ -\infty \leq x \leq \infty\end{array}\right\} \tag{3.4}$$

where G and K are known functions in $L_2(R)$ and $F \in H^P(R)$ is to be found. Then from the convolution theorem we have

$$\hat{K}(\omega)\hat{F}(\omega) = \hat{G}(\omega) \tag{3.5}$$

whence

$$F(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\hat{G}(\omega)}{\hat{K}(\omega)} \exp\ (i\omega y)dw \tag{3.6}$$

The ill-posedness of (3.4) is reflected by the fact that any small perturbation E in G whose transform $\hat{E}(\omega)$ does not decay faster than $\hat{K}(\omega)$ as $|\omega| \to \infty$ will result in a perturbation in $\frac{\hat{G}(\omega)}{\hat{K}(\omega)}$ which will grow without bound.

When G is in exact, therefore, we may seek a stable or filtered approximation to F given by

$$F_\lambda(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(\omega;\lambda)\frac{\hat{G}(\omega)}{\hat{K}(\omega)} \exp\ (i\omega y)d\omega \tag{3.7}$$

where $z(\omega;\lambda)$ is a filter function dependent on a parameter λ.

In this paper we restrict attention to filters generated from regularization theory. The smoothing functional

$$C(f;\lambda) = ||KF - G||_2^2 + \lambda\Omega(F) \tag{3.8}$$

is minimized in an appropriate subspace of L_2, where $\Omega(F)$ is a stabilizing functional in the form of a smoothing norm.

$$\Omega(F) = ||LF||^2 \tag{3.9}$$

and L is a linear operator. The regularization parameter λ controls the trade-off between smoothness, as imposed by Ω and the exent to which (3.4) is satisfied. We restrict attention to regularization of oder $p(p = 2$ in our case), where L in (3.9) is the pth order differential operator $LF = F^{(p)}$ and the norm in (3.9) is L_2.

The minimizer of (3.8) in H^p is then given by where

$$Z(\omega;\lambda) = \frac{|\hat{K}(\omega)|^2}{|\hat{K}(\omega)|^2 + \lambda\omega^{2p}} \tag{3.10}$$

If (3.4) is now replaced by

$$(K_N F_N)(x) = \int_0^T K_N(x-y)F_N(y)dy = G_N(x) \tag{3.11}$$

where K_N is periodically continued outside $(0,T)$. Then we may prove (a) above.

Lemma 3.1: Let $F \in T_N$ and

$$\underline{F} = (F(x_0), \ldots F(x_{N-1}))^T \in R^N \text{ then } N \times N \text{ matrix } K = \psi \text{ diag } \left(\hat{K}_{\omega,q}\right) \psi^H \tag{3.12}$$

where ψ is the unitary matrix with elements

$$\psi_{rs} = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi}{N} irs\right), \quad r,s = 0,1,\ldots,N-1 \tag{3.13}$$

has the property

$$(K\underline{F})_n = (K_N F)(x_n) \tag{3.14}$$

Thus from the infinite support hypothesis and (3.3), it follows that at $\{x_n\}$, (2.3) is exactly equivalent to the discrete system

$$(K\underline{F})_n = G_n \tag{3.15}$$

where K is given in (3.12) and

$$\underline{F} = (F_N(x_o), F_N(x_1), \ldots F_N(x_{N-1}))^T.$$

In T_N it is easily shown tht F_λ in (3.7) is approximated by

$$F_{N,\lambda}(x) = \sum_{q=0}^{N-1} Z_{q;\lambda} \frac{\hat{G}_{N,q}}{\hat{K}_{N,q}} \exp(i\omega_q x) \tag{3.16}$$

where the discrete pth order filter is

$$Z_{q,\lambda} = \frac{|\hat{K}_{N,q}|^2}{|\hat{K}_{N,q}|^2 + N^2\lambda\tilde{\omega}_q^{2p}} \tag{3.17}$$

and

$$\tilde{\omega}_q = \begin{cases} \omega_q, & 0 \le q < \frac{1}{2}N \\ \omega_{N-q}, & \frac{1}{2}N \le q < N-1 \end{cases} \tag{3.18}$$

To show (b) above, we note that $\sqrt{N}\psi^H$ is the discrete FT matrix representing (3.2), and so (3.15) is equivalent to the diagonal system

$$\hat{K}_{N,q}\hat{F}_{N,q} - \hat{G}_{N,q} \tag{3.19}$$

After regularizing (3.19), we get

$$\hat{K}_{N,q}\hat{F}_{N,q,\lambda} = Z_{q;\lambda}\hat{G}_{N,q} \tag{3.20}$$

so that $F_{N,q,\lambda}(x)$ may be found by multiplying the FFT of $\{G_N\}$ by the filter function, dividing by the FFT of $\{K_N\}$ and then taking the inverse FFT (fast fourier transform).

4 The Filter in a Stochastic Setting

In this section we relate the pth order convolution filter (3.17) to certain spectral densities which play a role in the maximum likelihood (ML) optimization of λ (the regularization parameter) in the next section.

Assume that the data $\{G_n\}$ are noisy and that there is an underlying function $U_N \in T_N$ such that

$$G_n = U_n\,(x_n) + \triangle_n \tag{4.1}$$

In the limit as $N \longrightarrow \infty$, $h \longrightarrow 0$ for any discrete process Y_n, we may write (see [8])

$$Y_n = \int_0^T \exp\,(2\pi i\omega n)d\,S_x(\omega) \tag{4.2}$$

where $S_x(\omega)$ s a stochastic process defined on $[0, T]$.

<u>Lemma 4.1</u>: The variance of any integral $\int \theta(\omega)ds_x(\omega)$ is given by $\int |\theta(\omega)^2|$ where $dG_x(\omega) = E(|ds_x(\omega)|^2)$. Let $G_x(\omega)$ may be interpolated as a spectral distribution function and accordingly we shall write $dG_x(\omega) = P_x(\omega)d\omega$ where $P_x(\omega)$ is a spectral density. Now consider $F_N \in T_N$ with $\underline{F} = (F_N) \equiv F_N((x_n)$ defined by $(K\underline{F})_n = U_n, n = 0, 1, \ldots, N-1$ with K given by (3.12).

From (4.2) we have

$$F_n = \int_{m=0}^{N-1} \left\{ \left(K^{-1}\right)_{mn} \int_0^T \exp\,(w\pi i\omega_n)d, s_u(\omega) \right\} \tag{4.3}$$

$$= \int_0^T [K_N(\omega)]^{-1} \ \exp\,(2\pi i\omega n)ds_{\cup}(\omega)$$

where

$$\hat{K}_N(\omega) = \frac{1}{N}\sum_{n=0}^{N-1} K_n \exp\,(-2\pi i\omega n) \tag{4.4}$$

Assume that F_n is estmated by $\sum_{m=0}^{N-1} l_m G_{n-m}$, where $\{l_m\}$ is a filter which we shall relate to $Z_{q;\lambda}$ and $\{G_n\}$ is periodically continued for $n \notin [0, N)$.

Then the error

$$F_n - \sum_{m=0}^{N-1} l_m G_{n-m} \tag{4.5}$$

is given by

$$\int_0^T \exp\,(2\pi i\omega n)\left(\hat{K}_N(\omega)\right)^{-1} \hat{l}_N(\omega)ds_{\cup}(\omega) - \int_0^T \exp\,(2\pi i\omega n)\hat{l}_N(\omega)ds_{\triangle}(\omega) \tag{4.6}$$

where $\hat{l}_N(\omega)$ is defined as in (4.4).

From Lemma 4.1 the variance of this error is clearly

$$\int_0^T \left|\left[\hat{K}_N\omega)\right]^{-1} \ - \ \hat{l}_N(\omega)\right|^2 P_{\cup}(\omega)dw + \int_0^T \left|\hat{l}_N(\omega)\right|^2 P_{\triangle}(\omega)d\omega \tag{4.7}$$

which is minimized when

$$\hat{l}_N(\omega)\hat{K}_N(\omega) = \frac{P_{\cup}(\omega)}{P_{\cup}(\omega) + P_{\triangle}(\omega)} \tag{4.8}$$

since the Fourier coefficients of the filtered solution must satisfy

$$\hat{F}_{N,q};\lambda = \hat{l}_{Nl,q}\hat{G}_{N,q} = Z_{q;\lambda}\hat{G}_{N,q}\left[\hat{K}_{N,q}\right]^{-1},$$

we find from (4.8)

$$Z_{q;\lambda} = \hat{l}_{N,q}\hat{K}_{N,q} = \frac{P_{\cup}(qh)}{P_{\cup}(qh) + P_{\triangle}(qh)} \tag{4.9}$$

where $\hat{l}_{N,q} = \hat{l}_N(qh)$ and $\hat{K}_{Nq} = \hat{K}_N(qh)$.

5 Detrermination of λ, the Regularization Parameter by Maximum Likelihood (ML) Method

We now simply relate the filter (4.9) to the pth order filter(3.17). Assuming that errors are uncorrelated, $P_\triangle(\omega)$ has the form

$$P_\triangle(\omega) = \sigma^2 = \text{ constant,} \tag{5.1}$$

where σ^2 is the unknown variance of the noise in the data.

Choosing

$$P_U(\omega) = \frac{\sigma^2 \left|\hat{K}_N(\omega)\right|^2}{\lambda\tilde{\omega}^{2p}} \tag{5.2}$$

where

$$\tilde{\omega} = \begin{cases} 2\pi N\omega, & 0 \le \omega < \frac{1}{2}T \\ 2\pi N(T-\omega), & \frac{1}{2}T \le \omega < T \end{cases}$$

Then yields (3.17) from (4.9). Moreover, the spectral density for $\{G_n\}$ is then

$$P_G(\omega) = P_\cup(\omega) + P_\triangle(\omega) = \sigma^2 \left[1 + \frac{\left|\hat{K}_N(\omega)\right|^2}{\lambda\tilde{\omega}^{2p}}\right]$$

whence

$$P_G(\omega) = \sigma^2 \left(1 - Z_{q;\lambda}\right)^{-1} \tag{5.3}$$

The statisitcal likelihood of any suggested values of σ^2 and λ may now be estimated from the data.

Following Whittle [29], the Logarithm of the likelihood function of P_G is given approximately by

$$\text{constant } - \frac{1}{2}\sum_{q=0}^{N-1}[logP_G(qh) + I(qh)]/P_G(qh) \tag{5.4}$$

(using the definition of joint distribution of the random variables)

where

$$I(\omega) = \left|\sum_{n=0}^{N-1} G_n \text{ exp } (-2\pi i\omega n)\right|^2$$

is the periodogram of the data with $I(qh) = \left|\hat{G}_{N,q}\right|^2$.

We now maximize(5.4) with respectto σ^2 and λ.

The partial maximum with respect to σ^2 may be found exactly (in terms of λ) with the maximizing value of σ^2 is given by

$$\sigma^2 = \frac{1}{N}\sum_{q=1}^{N-1} \left|\hat{G}_{N,q}\right|^2 (1 - Z_{q;\lambda}) \tag{5.5}$$

The maximum w.r.t λ may then be found by minimizing

$$V_{ML}(\lambda) = \frac{1}{2} N \log \left[\sum_{q=0}^{N-1} \left|\hat{G}_{N,q}\right|^2 (1 - Z_{q;\lambda})\right] - \frac{1}{2}\sum_{q=1}^{N-1} \log (1 - Z_{q;\lambda}) . \tag{5.6}$$

Thus the optimal regularization parameter is given by the minimizer of a simple function of λ, depending on the known Fourier coefficients $\hat{G}_{N,q}$ and $\hat{K}_{N,q}$. No prior knowledge of σ^2 is assumed but an a posteriori estimate is given by (5.5). (5.6) can be minimized with respect to λ and in order to minimize $V_{ML}(\lambda)$ in equation (5.6), we have used a subroutine which uses quadratic interpolatin technique to obtain a minimum.

6 Cross Validation Method (CV).

From (3.16) we know that the filtered solution

$$F_{N,\lambda}(x) \in T_N \text{ which minimizes}$$

$$\sum_{n=0}^{N-1} \left[(K_N * F)(x_n) - G_n\right]^2 + \lambda |F^{(P)}(x)|_2^2$$

is $F_{N,\lambda}(x) = \frac{1}{N}\sum_{q=0}^{N-1} \hat{F}_{N,\lambda,q} \exp((2\pi i q x)$, where $\hat{F}_{N,\lambda,q} = Z_{q;\lambda}\frac{\hat{G}_{N},q}{\hat{K}_{N,q}}$ with $Z_{q,\lambda}$ as given in equation (3.17).

Suppose we ignore the jth data point G_j; and define the filtered solution $F_{N,\lambda}^{[j]}(x) \in T_N$ as the minimizer of

$$\sum_{\substack{n=0 \\ n \neq j}}^{N-1} \left[(K_N * F)(x_n) - G_n\right]^2 + \lambda ||F^{(P)}(x)||_2^2$$

Then we get a vector $G^{[j]}_{N,\lambda} \in \mathcal{R}$ defined by

$$\underline{G}^{[j]}_{N,\lambda} = K\underline{F}^{[j]}_{N,\lambda} \tag{6.1}$$

Clearly the jth element $G^{[j]}_{N,\lambda,j}$ of equation (6.1) should "predict" the missing value G_j. We may thus construct the weighted mean square prediction error all j :

$$V_{cv}(\lambda) = \frac{1}{N}\sum_{j=0}^{N-1} w_j(\lambda)\left(G^{[j]}_{N,\lambda,j} - G_j\right)^2 \tag{6.2}$$

Thus the optimal λ minimizes $V_{cv}(\lambda)$ in (6.2) for a given p and does not require a knowledge of σ^2.

To minimize $V_{cv}(\lambda)$ in (6.2) is a time consuming problem, we have an alternate expression which depends on a particular choice of weights, resulting in considerable simplification. Let

$$\underline{F}_{-N,\lambda} = \left(F_{N,\lambda}(x_0), F_{N,\lambda}(x_1), \ldots, F_{N,\lambda}(x_{n-1})\right)^T \tag{6.3}$$

and define

$$\underline{G}_{-N,\lambda} = K\underline{F}_{N,\lambda} \tag{6.4}$$

Then there exist a matrix $A(\lambda)$, called an influence matrix, such that

$$\underline{G}_{N,\lambda} = A(\lambda)\underline{G}_N. \tag{6.5}$$

Let

$$\begin{aligned} \hat{K} &= diag\left(hK_{N,q}\right) \text{ and} \\ \hat{Z} &= diag\left(Z_{q;\lambda}\right). \end{aligned}$$

Then from (3.16 we see that

$$\underline{F}_{N,\lambda} = \psi(\hat{K})^{-1}\hat{Z}\hat{G}_N \tag{6.6}$$

where

$$\hat{G}_N = \psi^H\underline{G}_N \tag{6.7}$$

and so

$$A(\lambda) = \psi\hat{Z}\psi^H \tag{6.8}$$

since

$$K = \psi \hat{K} \psi^H \tag{6.9}$$

Wahba [28] has shown that the choice of weights

$$w_j(\lambda) = \left[\frac{1 - a_{jj}(\lambda)}{\frac{1}{N} Trace(I - A(\lambda))} \right], j = 0, 1, \ldots, N-1 \tag{6.10}$$

enables the expression (6.2) to be written in the simpler form

$$V_{cv}(\lambda) = \frac{\frac{1}{N} ||(I - A(\lambda))\underline{G}_N||_2^2}{[\frac{1}{N} Trace(I - A(\lambda))]^2} \tag{6.11}$$

Using (6.8) it follows that

$$V_{cv}(\lambda) = \frac{\frac{1}{N} ||(1 - \hat{Z})\hat{G}_N||_2^2}{\left[\frac{1}{N}\mathrm{Trace}(1 - \hat{Z})\right]^2} \, .$$

i.e.

$$V_{cv}(\lambda) = \frac{\frac{1}{N} \sum_{q=0}^{N-1} (1 - z_{q;\lambda})^2 \left| \hat{G}_{N,q} \right|^2}{\left[\frac{1}{N} \left(\sum_{q=0}^{N-1} (1 - z_{q;\lambda}) \right) \right]^2} \tag{6.12}$$

Since the matrix $A(\lambda)$ in (6.8) is circulant, the weights in (6.10) are all unity.

In order to minimize $V_{cv}(\lambda)$ in equation (6.12) we have used quadratic interpolation technique to obtain a minimum value of λ, (the regularization parameter).

7 Numerical Examples

In this section we tabulate the results of maximum likelihood and cross validation methods, applied to the test examples taken from the literature. Only optimal results have been quoted in Tables 1 and 2. In each of the test examples we have considered 64 data points in order to calculate the discrete Fourier coefficients of G and K. We have used double precision arithmetic to obtain the better accuracy.

Example 1: This example is takne from ([25], case 5, page 79).

$$\begin{aligned} g(s) &= \frac{1}{s - a} \\ f(t) &= f(t) = e^{-at}, \; a = 1.0 \end{aligned}$$

The optimal results are shown in Table 1 and Table 2.

Example 2: This example has been taken from ([5] and [25], case 4, page 79).

$$
\begin{aligned}
g(s) &= \frac{u \sin v}{s^2 + 2us\cos v + u^2}, \quad u = \left(a^2+b^2\right)^{-\frac{1}{2}}, \quad V = \tan^{-1}(b/a) \\
f(t) &= e^{-at}\sin(bt), \quad a = 5.0, \quad b = 2.2.
\end{aligned}
$$

The optimal results are shown in Table 1and Table 2.

Exampoe 3: This example has been taken from ([1], [4] and [17]).

$$
\begin{aligned}
g(s) &= \frac{\Gamma(a)}{(s+b)^{a+1}} \\
ft) &= t^a e^{-bt} \text{ for } a = 1.0,\, b = 1.0
\end{aligned}
$$

The optimal results are shown in Table 1 and Table 2.

Example 4: This example has been taken from ([22] and [27]).

$$
\begin{aligned}
g(s) &= \frac{6}{(s+1)^4} \\
f(t) &= t^3 e^{-t}
\end{aligned}
$$

The optimal results are shown in Table 1 and Table.2

Table-I

Results of Maximum Likelihood Method

Example	a	T	h	λ	$V_{ML}(\lambda)$	$\|\|f - f_\lambda\|\|_\infty$
1	10.0	12.50	0.19532	0.97×10^{-8}	0.777×10^{3}	0.005
2	05.0	09.00	0.14064	0.11×10^{-10}	0.839×10^{3}	0.012
3	10.0	12.10	0.18908	0.21×10^{-10}	0.141×10^{2}	0.002
4	10.0	14.50	0.22656	0.13×10^{-12}	0.1892×10^{1}	0.003

Table-2

Results of Cross Validation Method

Example	a	T	h	λ	$V_{cv}(\lambda)$	$\|\|f - f_\lambda\|\|_\infty$
1	10.0	12.50	0.19532	0.67×10^{-9}	0.631×10^{2}	0.0035
2	05.0	09.00	0.14064	0.02×10^{-11}	0.541×10^{2}	0.009
3	10.0	12.10	0.18908	0.32×10^{-11}	0.242×10^{3}	0.001
4	10.0	14.50	0.22656	0.432×10^{-12}	0.342×10^{1}	0.003

8 Concluding Remarks

Both maximum likelihood and cross validation, methods worked very well over all the four test examples which are severely ill-posed and the results are shown in Table 1 and Table 2 respectively. For the comparison purposes cross validation method has slight edge over the maximum likelihood method and it yields slightly letter results and is little less time consuming.

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References

1. Ang, D.D. et. al., 'Complex variable and regularization methods of inversion of the Laplace transform', Journal of Mathematics of Computation, vol. 53, no.188, (1989) pp. 589-608.

2. Bertero, M. and Pike, E.R., 'Exponential sampling methods for Laplace and other diagonally invariant transforms', Inverse Problem, vol. 7 (1991), pp.21-32.

3. Blass, W.e. et. al., 'Deconvolution of absorption spectra: Academic Press', New York 1981.

4. Brianzi, P. and Frontini, M. 'On the regularized inversion of the Laplace transform', inverse problems 7 (1991), pp. 355-368.

5. Cristina Cunha et. al. 'The Laguerre functions in the inversion of Laplace transform', inverse problems 9 (1993), pp. 57-68.

6. Davies, B. and Martin, B. 'Numerical inversion of the laplae transform', J. Comput. Physics, vol. 33, no.2 (1979), pp. 1-32.

7. Dong, C.W., 'A regularization method for the numerical inversion of the Laplace transform', SIAM. J. Numer. Anal. vol. 30 (1993), 759-773.

8. Eggermont, P.P.B., 'Maximum Penalized likelihood estimation and smooth EM Algorithms for positive integral equations of the first kind', Numer. Funct. Anal and optimiz. vol. 7, no.7 (1996), pp. 737-754.

9. Essah, W.A. and Delves, L.M., 'On the numerical inversion of Laplace transform', Inverse problems 4 (1988), pp. 705-724.

10. Frolov, G.A. and Kitaev, M.Y, 'Improvement of accuracy in numerical methods for inverting Laplace transform based on the post widder method', Comput. Math. Appl. vol.38 (1998), pp.23-34.

11. Gelfat, V.I. et. al., 'Programs for signal recovery from noisy data using the maximum likelihood principle', Computer Physics Communications, vol. 7 (1993) , pp. 335-348.

12. Hadamard, J., 'Lectures on the Cauchy problem in linear partial differential equations', Yale University Press, New Haven, (1923).

13. Istratov, A.A. and Vyvneko, O.F., 'Exponential analysis in physical phenomena', published in Review of Scientific Instruments, vol. 10, no.2, (1998), pp. 1233-1257.

14. Jensen, P.A. 'Deconvolution with applications in spectroscopy', Academic Press, New York, (1984).

15. Lucas, M.A., 'Assessing regularized solution', J. Austral. Math. Soc. Ser. B., vol. 30 (1988), pp.24-42.

16. Mair, B.A. and Ruyngart, F.H., 'A cross validation method for First Kind Integral Equations', J. Comp. and Control., vol. 20 (1995), pp.259-267.

17. McWhirter, J.G. and Pike, E.R., 'On the numerical inversion of Laplace transform and similar F.I. equations of the first kind', J. Phys. A., vol.11, no.9, (1978), pp.1729-1745.

18. Mead, L.R. et. al., 'Maximum Entropy in the problem of moments', J. Math. Physics vol. 25, no.8, (1984), pp. 2404-2417.

19. Piessens, R., 'Laplace transform inversion', J. Comp. Appl. Maths. vol.1, (1975), pp.115, vol.2, (1976), pp.225.

20. Piessens,R. and Btranders, M. 'Numerical Inversion of the Laplace transform using generalized Laguerre polynomials', Proc. IEE, vol.118, (1971), pp.1517-1522.

21. Pinkus, A. et. al., 'Fourier series and Integral transform', Cambridge University Press. Cambridge, (1997), pp.98-112.

22. Rodriguez, G. and Seatzu, S., 'On the numerical inversion of the Laplace transform in reproducing Kernel Hilbert Space', IMA Journal of Numerical Analysis, (1993), vol 13, pp.463-475.

23. Shan Hua Ning, 'A Method for the Numerical Inversion of Laplace Transform', J. Numer. Method. Comput. Math., vol.20, no.3, (1999), pp.231-236.

24. Talbot, A., 'The accurate Numerical Inversion of Laplace Transform', J. Inst. Maths. Applics., vol. 23, (1979), pp.97-120.

25. Theocaris, P. and Chrysakis, A.C., 'Numerical Inversion of Mellin Transform', J. Math. and Appl. vol.20, (1977), pp.73-83.

26. Tikhonov, A. and Arsenin, V., 'Solutions of ill-posed Problems', John Wiley, (1977), New York.

27. Varah, J.M., 'Pitfalls in the Numerical solutions of linear ill-posed problems', SIAM. J. Sci. Statist. Comput., vol.4, no.2 (1983), pp.164-176.

28. Wahba, G., 'Practical approximate solutions to linear operator equations, when the data are noisy', SIAM. J. Numer. Anal., vol.14 (1977), pp.651-677.

29. Whittle, P. 'Some results in time series analysis', Skand. Actuarietidekt., vol.35, (1952), pp.48-60.