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Global existence of weak solutions in nonlinear thermoelasticity with second sound

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Abstract

In this work we establish a global existence of weak solutions for a multidimensional nonlinear system of thermoelasticity with second sound.

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1 Introduction

Results concerning existence, blow up, and asymptotic behaviors of smooth, as well as weak solutions in classical thermoelasticity have been established by several authors over the past two decades. See in this regard [1 - 7], [9 - 14], and [18].

For the one-dimensional thermoelasticity with second sound, Tarabek [19] used the usual energy argument and proved global existence of smooth solutions with smooth and small initial data. He also showed that the solution decays to the rest state however, no rate of decay has been discussed. Racke [15] lately studied the asymptotic behavior in the one-dimensional situation and established exponential decay results for several initial boundary value problems. In particular he showed that, for small enough initial data, classical solutions of a certain nonlinear problem decay exponentially to the equilibrium state. It is also worthmentionning the work of Saouli [17], where he used the nonlinear semigroup theory to establish a local existence of smooth solutions to a system similar to the one discussed in [15].

Concerning the multi-dimensional case ($n = 2, 3$) Racke [16] established an existence result for the problem

$$\begin{gathered} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla div u + \beta\nabla\theta = 0 \\ \theta_t + \gamma div q + \delta div u_t = 0 \\ \tau q_t + q + \kappa\nabla\theta = 0, \quad x \in \Omega, \qquad t > 0 \\ u(.,0) = u_0, \quad u_t(.,0) = u_1, \quad \theta(.,0) = \theta_0, \quad q(.,0) = q_0, \quad x \in \Omega \\ u = \theta = 0, \qquad x \in \partial\Omega\ ,\ t \geq 0, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$, with a smooth boundary $\partial\Omega$, $u = u(x,t) \in \mathbb{R}^n$ is the displacement vector, $\theta = \theta(x,t)$ is the difference temperature, $q = q(x,t) \in \mathbb{R}^n$ is the heat flux vector, and $\mu, \lambda, \beta, \gamma, \delta, \tau, \kappa$ are positive constants, where μ, λ are Lame moduli and τ is the relaxation time, a small parameter compared to the others. In particular if $\tau = 0$, (1.1) reduces to the system of classical thermoelasticity, in which the heat flux is given by Fourier's law instead of Cattaneo's law. He also proved, under the conditions $rotu = rotq = 0$, an exponential decay result for (1.1). This result is extended to the radially symmetric solution, as it is only a special case. Messaoudi [8] looked into the following semilinear problem

$$\begin{gathered} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla divu + \beta\nabla\theta = |u|^{p-2}u \\ \theta_t + \gamma divq + \delta divu_t = 0 \\ \tau q_t + q + \kappa\nabla\theta = 0, \quad x \in \Omega, \qquad t > 0 \\ u(.,0) = u_0, \quad u_t(.,0) = u_1, \quad \theta(.,0) = \theta_0, \quad q(.,0) = q_0, \quad x \in \Omega \\ u = \theta = 0, \qquad x \in \partial\Omega \ , \ t \geq 0, \end{gathered} \tag{1.2}$$

for $p > 2$. This is a similar problem to (1.1) with a nonlinear source term competing with the damping factor. He extended Racke's local existence result to (1.2) and showed that solutions with negative energy blow up in finite time. This work generalized the one in [5 - 7] to thermoelasticity with second sound.

In this paper we are concerned (1.2). We will prove the existence of weak solutions and show that these solutions are global if the initial data are small enough. This paper is organized as follows : in section two we establish the local existence. In section three the global result is proved.

2 Local Existence

In this section, we establish a local existence result for (1.2) under a suitable condition on p. First we state an existence result for a related linear problem.

$$\begin{gathered} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla divu + \beta\nabla\theta = f \\ \theta_t + \gamma divq + \delta divu_t = 0 \\ \tau q_t + q + \kappa\nabla\theta = 0, \quad x \in \Omega, \qquad t > 0 \\ u(.,0) = u_0, \quad u_t(.,0) = u_1, \quad \theta(.,0) = \theta_0, \quad q(.,0) = q_0, \quad x \in \Omega \\ u = \theta = 0, \qquad x \in \partial\Omega \ , \ t \geq 0. \end{gathered} \tag{2.1}$$

The proof is a direct result of Theorem 2.2 of [16]. For this purpose we introduce the following spaces

$$\begin{aligned} \Pi \ &: = \ \left[H_0^1(\Omega) \cap H^2(\Omega)\right]^n \times \left[H_0^1(\Omega)\right]^n \times H_0^1(\Omega) \times D \\ D \ &: = \{q \in \left[L^2(\Omega)\right]^n / divq \in L^2(\Omega)\} \\ H \ &: = \ \left[H_0^1(\Omega)\right]^n \times \left[L^2(\Omega)\right]^n \times L^2(\Omega) \times \left[L^2(\Omega)\right]^n \end{aligned} \tag{2.2}$$

$$\Lambda := \max_{0 \leq t \leq T} ||(u, u_t, \theta, q)(.,t)||_H^2, \qquad \Lambda_0 := ||(u_0, u_1, \theta_0, q_0)||_H^2 \tag{2.3}$$

Lemma 2.1. *Assume that $f \in (C^1([0,T];L^2(\Omega))^n$. Then given any initial data $(u_0,u_1,\theta_0,q_0) \in \Pi$, the problem (2.1) has a unique strong solution satisfying*

$$(u, u_t, \theta, q) \in C([0,T];\Pi) \cap C^1([0,T];H). \tag{2.4}$$

Theorem 2.2. *Assume that $f \in (C([0,T];L^2(\Omega))^n$. Then given any initial data $(u_0,u_1,\theta_0,q_0) \in H$, the problem (2.1) has a unique weak solution satisfying*

$$(u, u_t, \theta, q) \in C([0,T];H). \tag{2.5}$$

Moreover we have

$$\Lambda \leq \Gamma\Lambda_0 + \Gamma T \max_{0\leq t\leq T} ||f(.,t)||_2^2, \tag{2.6}$$

where Γ is a constant depending on $\mu,\lambda,\beta,\gamma,\delta,\kappa,\tau$ only.

Proof. We approximate u_0,u_1,θ_0,q_0 by sequences $(u_0^n),(u_1^n),(\theta_0^n),(q_0^n)$ in $C_0^\infty(\Omega)$, and f by a sequence (f^n) in $C^1([0,\ T];\ C_0^\infty(\Omega))$. We then consider the set of the linear problems

$$\begin{array}{c} u_{tt}^n - \mu\Delta u^n - (\mu+\lambda)\nabla div u^n + \beta\nabla\theta^n = f^{\ n} \\ \theta_t^n + \gamma div q^n + \delta div u_t^n = 0 \\ \tau q_t^n + q^n + \kappa\nabla\theta^n = 0, \quad x\in\Omega, \qquad t>0 \\ u^n(.,0) = u_0^n; \quad u_t^n(.,0) = u_1^n, \quad \theta^n(.,0) = \theta_0^n; \quad q^n(.,0) = q_0^n, \quad x \in \Omega \\ u^n = \theta^n = 0, \qquad x\in\partial\Omega\ ,\ t\geq 0. \end{array} \tag{2.7}$$

Lemma 2.1 guarantees the existence of a sequence of unique solutions

$$(u^n, u_t^n, \theta^n, q^n) \in C([0,T];\Pi) \cap C^1([0,T];H).$$

Now we proceed to show that $(u^n, u_t^n, \theta^n, q^n)$ is a Cauchy sequence. For this aim, we set

$$U := u^n - u^m, \qquad \Theta := \theta^n - \theta^m, \qquad Q := q^n - q^m$$

It is straightforward to see that (U,Θ,Q) satisfies

$$\begin{array}{c} U_{tt} - \mu\Delta U - (\mu+\lambda)\nabla div U + \beta\nabla\Theta = f^{\ n} - f^{\ m} \\ \Theta_t + \gamma div Q + \delta div U_t = 0 \\ \tau Q_t + Q + \kappa\nabla\Theta = 0, \quad x\in\Omega, \qquad t>0 \\ U(.,0) = U_0, \quad U_t(.,0) = U_1, \quad \Theta(.,0) = \Theta_0; \quad Q(.,0) = Q_0, \quad x \in \Omega \\ U = \Theta = 0, \qquad x\in\partial\Omega\ ,\ t\geq 0, \end{array} \tag{2.8}$$

where

$$U_0 := u_0^n - u_0^m, \qquad U_1 := u_1^n - u_1^m, \qquad \Theta_0 := \theta_0^n - \theta_0^m, \qquad Q_0 := q_0^n - q_0^m$$

We multiply equations (2.8) by $U_t, \beta\Theta/\delta\ , \beta\gamma Q/(\delta\kappa)$ respectively and integrate over $\Omega \times (0,\ t)$ to get

$$\frac{1}{2}\int_\Omega [|U_t|^2 + \mu|\nabla U|^2 + (\lambda+\mu)(divU)^2 + \frac{\beta}{\delta}|\Theta|^2 + \frac{\gamma\beta\tau}{\delta\kappa}|Q|^2](x,t)dx \leq$$

$$\frac{1}{2}\int_\Omega [|U_1|^2+\mu|\nabla U_0|^2+(\lambda+\mu)(divU_0)^2+\frac{\beta}{\delta}|\Theta_0|^2+\frac{\gamma\beta\tau}{\delta\kappa}|Q_0|^2](x)dx$$
$$+\int_0^t\int_\Omega (f^{\ n}\ -f^{\ m})\,(x,s).u_t(x,s)dxds. \tag{2.9}$$

Since (u_0^n), (u_1^n), (θ_0^n), (q_0^n) and $f^{\ n}$ are Cauchy we conclude from (2.9) that the sequence $(u^n, u_t^n, \theta^n, q^n)$ is Cauchy in $C([0,T);H)$, hence it converges to (u,v,θ,q) in $C([0,T);H)$. By using the appropriate test functions, we easily see that $v=u_t$. We now show that the limit (u,u_t,θ,q) is a weak solution in the sense of distribution. That is for each (v,ϕ,ξ) in $[C_0^\infty(\Omega)]^{2n+1}$ we must show that

$$\begin{array}{c}\frac{d}{dt}\int_\Omega u_t(x,t).v(x)dx+\mu\int_\Omega\nabla u(x,t).\nabla v(x)dx+(\mu+\lambda)\int_\Omega divudivv\\ -\beta\int_\Omega\theta(x,t)divv(x)dx=\int_\Omega f(x,t)v(x)dx\\ \frac{d}{dt}\int_\Omega\theta(x,t)\phi(x)dx-\gamma\int_\Omega q(x,t).\nabla\phi(x)dx-\delta\int_\Omega u_t(x,t).\nabla\phi(x)dx=0\\ \frac{d}{dt}\int_\Omega q(x,t).\xi(x)dx+\int_\Omega q(x,t).\xi(x)dx-\kappa\int_\Omega\theta(x,t)div\xi(x)dx=0\end{array} \tag{2.10}$$

on $[0,\ T]$. To establish this, we multiply equation $(2.7)_1$ by v and integrate over $\Omega\times(0,t)$, so we obtain

$$\begin{array}{c}\int_\Omega u_t^n(x,t).v(x)dx+\mu\int_0^t\int_\Omega\nabla u^n(x,t).\nabla v(x)dx+(\mu+\lambda)\int_0^t\int_\Omega divu^ndivv\\ -\beta\int_0^t\int_\Omega\theta^n(x,t)divv(x)dx=\int_0^t\int_\Omega f^n(x,t)v(x)dx\ +\int_\Omega u_1(x).v(x)dx\end{array} \tag{2.11}$$

As $n\to\infty$, it is easy to see that

$$\begin{array}{rcl}\int_\Omega\nabla u^n(x,t).\nabla v(x)dx&\to&\int_\Omega\nabla u(x,t).\nabla v(x)dx\\ \int_\Omega divu^ndivv&\to&\int_\Omega divudivv\\ \int_\Omega\theta^n(x,t)divv(x)dx&\to&\int_\Omega\theta(x,t)divv(x)dx\\ \int_\Omega f^n(x,t)v(x)dx&\to&\int_\Omega f(x,t)v(x)dx\end{array}$$

in $C([0,T])$. We therefore conclude from (2.11) that $\int_\Omega u_t(x,t).v(x)dx$ $\{=\lim\int_\Omega u_t^n(x,t).v(x)dx\}$ is a continuous function on $[0,\ T]$, so $(2.7)_1$ holds on $[0,\ T]$. By repeating the same argument, $(2.7)_2$ and $(2.7)_3$ hold on $[0,\ T]$. For the inequality (2.6), we start from the energy inequality for (u^n,θ^n,q^n) :

$$\frac{1}{2}\int_\Omega[|u_t^n|^2+\mu|\nabla u^n|^2+(\lambda+\mu)(divu^n)^2+\frac{\beta}{\delta}|\theta^n|^2+\frac{\gamma\beta\tau}{\delta\kappa}|q^n|^2](x,t)dx\le$$
$$\frac{1}{2}\int_\Omega[|u_1^n|^2+\mu|\nabla u_0^n|^2+(\lambda+\mu)(divu_0^n)^2+\frac{\beta}{\delta}|\theta_0^n|^2+\frac{\gamma\beta\tau}{\delta\kappa}|q_0^n|^2](x)dx$$
$$+\int_0^t\int_\Omega f^n(x,s).u_t^n(x,s)dxds.$$

By taking n to infinity, (2.6) is established. The uniqueness follows from the energy inequality (2.6).

Theorem 2.3. *Assume that*

$$2\le p\le\frac{2(n-1)}{n-2}. \tag{2.12}$$

Then given any initial data $(u_0, u_1, \theta_0, q_0) \in H$, the problem (1.2) has a unique weak solution

$$(u, u_t, \theta, q) \in C([0,T); H), \tag{2.13}$$

for T small.

Proof. For $M > 0$ large and $T > 0$, we define a class of functions $Z(M,T)$ which consists of all functions (w, ϕ, ξ), for which $(w, w_t, \phi, \xi) \in C([0,T); H)$, satisfying the initial conditions of (1.2), and

$$\max_{0 \le t \le T} ||(w, w_t, \phi, \xi)(., t)||_H^2 \le M^2.$$

$Z(M,T)$ is nonempty if M is large enough. This follows from the trace theorem. We also define the map h from $Z(M,T)$ into $C([0,T); H)$ by $(u, \theta, q) := h(v, \phi, \xi)$, where (u, θ, q) is the unique solution of the linear problem

$$\begin{gathered} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla divu + \beta\nabla\theta = |v|^{p-2}v \\ \theta_t + \gamma divq + \delta divu_t = 0 \\ \tau q_t + q + \kappa\nabla\theta = 0, \quad x \in \Omega, \qquad t > 0 \\ u(.,0) = u_0, \quad u_t(.,0) = u_1, \quad \theta(.,0) = \theta_0, \quad q(.,0) = q_0, \quad x \in \Omega \\ u = \theta = 0, \qquad x \in \partial\Omega \ , t \ge 0. \end{gathered} \tag{2.14}$$

This is guaranteed by theorem 2.3 and condition (2.12), which makes $|v|^{p-2}v$ in $C([0, T]; L^2(\Omega))$. We would like to show that, for M sufficiently large and T sufficiently small, h is a contraction from $Z(M,T)$ into itself. By using the energy equality (2.6) we get

$$\max_{0 \le t \le T} ||(u, u_t, \theta, q)(., t)||_H^2 \le \Gamma ||(u_0, u_1, \theta_0, q_0)||_H^2 + \Gamma T \max_{0 \le t \le T} |||v|^{p-1}(., t)||_2^2, \tag{2.15}$$

Now we estimate $|||v(., t)|^{p-1}||_2^2$ as follows

$$|||v(., t)|^{p-1}||_2^2 \le C||v(., t)||_{2n/(n-2)}^{2p-2} \le C||\nabla v(., t)||_2^{2p-2} \le C||(v, v_t, \phi, \xi)(., t)||_H^{2p-2}. \tag{2.16}$$

where C is the embedding constant. Therefore (2.15) and (2.16) yield

$$\max_{0 \le t \le T} ||(u, u_t, \theta, q)(., t)||_H^2 \le \Gamma ||(u_0, u_1, \theta_0, q_0)||_H^2 + \Gamma T \max_{0 \le t \le T} ||(v, v_t, \phi, \xi)(., t)||_H^{2p-2}.$$

By choosing M large enough and T sufficiently small we arrive at

$$\max_{0 \le t \le T} ||(u, u_t, \theta, q)(., t)||_H^2 \le M^2;$$

hence $(u, \theta, q) \in Z(M,T)$. This shows that h maps $Z(M,T)$ into itself. Next we verify that h is a contraction. For this aim we set

$$U = u_1 - u_2, \qquad \Theta = \theta_1 - \theta_2, \qquad Q = q_1 - q_2$$

and

$$V = v_1 - v_2, \qquad \Phi = \phi_1 - \phi_2, \qquad \zeta = \xi_1 - \xi_2$$

where $(u_1, \theta_1, q_1) = h(v_1, \phi_1, \xi_1)$ and $(u_2, \theta_2, q_2) = h(v_2, \phi_2, \xi_2)$. It is straightforward to see that (U, Θ, Q) satisfies

$$\begin{gathered} U_{tt} - \mu\Delta U - (\mu+\lambda)\nabla divU + \beta\nabla\Theta = |v_1|^{p-2}v_1 - \ |v_2|^{p-2}v_2 \\ \Theta_t + \gamma divQ + \delta divU_t = 0 \\ \tau Q_t + Q + \kappa\nabla\Theta = 0, \quad x\in\Omega, \qquad t>0 \\ U(x,0) = U_t(x,0) = \Theta(x,0) = Q(x,0) = 0, \quad x\in\Omega \\ U = \Theta = 0, \qquad x\in\partial\Omega\ ,\ t\ge 0, \end{gathered} \tag{2.17}$$

By multiplying equations (2.17) by U_t, $\beta\Theta/\delta$, $\beta\gamma Q/(\delta\kappa)$ respectively and integrating over $\Omega\times(0,\,t)$ we get

$$\begin{aligned} &\frac{1}{2}\int_\Omega \left[|U_t|^2 + \mu|\nabla U|^2 + (\lambda+\mu)(divU)^2 + \frac{\beta}{\delta}|\Theta|^2 + \frac{\gamma\beta\tau}{\delta\kappa}|Q|^2\right](x,t)dx \\ \le\ & \int_0^t\int_\Omega \left[|v_1|^{p-2}v_1 - \ |v_2|^{p-2}v_2\right]U_t(x,s)dxds. \end{aligned} \tag{2.18}$$

We then estimate the last term in (2.18) as follows

$$\begin{aligned} &\int_\Omega \left|[|v_1|^{p-2}v_1 - \ |v_2|^{p-2}v_2]U_t(x,s)\right|dx \\ \le\ & C||U_t||_2||V||_{2n/(n-2)}\left[||v_1||_{n(p-2)}^{p-2} + ||v_2||_{n(p-2)}^{p-2}\right]. \end{aligned} \tag{2.19}$$

The Sobolev embedding and condition (2.12) give

$$\begin{aligned} ||V||_{2n/(n-2)} &\le C||\nabla V||_2, \\ \left[||v_1||_{n(p-2)}^{p-2} + ||v_2||_{n(p-2)}^{p-2}\right] &\le C\left[||\nabla v_1||_2^{p-2} + ||\nabla v_2||_2^{p-2}\right], \end{aligned} \tag{2.20}$$

where C is a constant depending on Ω only. Therefore a combination of (2.18) - (2.20) leads to

$$\begin{aligned} &\frac{1}{2}\int_\Omega \left[|U_t|^2 + \mu|\nabla U|^2 + (\lambda+\mu)(divU)^2 + \frac{\beta}{\delta}|\Theta|^2 + \frac{\gamma\beta\tau}{\delta\kappa}|Q|^2\right](x,t)dx \\ \le\ & C||U_t||_2||\nabla V||_2\left[||\nabla v_1||_2^{p-2} + ||\nabla v_2||_2^{p-2}\right], \end{aligned} \tag{2.21}$$

which gives, in turn,

$$\max_{0\le t\le T}||(U,U_t,\Theta,Q)(.,t)||_H \le \Gamma T M^{p-2}\max_{0\le t\le T}||(V,V_t,\Phi,\zeta)(.,t)||_H. \tag{2.22}$$

By choosing T so small that $\Gamma T M^{p-2} < 1$, (2.22) shows that h is a contraction. The contraction mapping theorem then guarantees the existence of a unique (u,θ,q) satisfying $(u,\theta,q) = h\,(u,\theta,q)$. Obviously it is a solution of (1.2). The uniqueness of this solution follows from (2.22). The proof is completed.

3 Global Existence

In this section we show, for appropriate initial data and under condition (2.12), that solution (2.13) is global ' in time'. To achieve this goal, we introduce the following functionals and class of functions.

$$\begin{aligned} I(t) &= I((u,\theta,q)(t)) = \int_\Omega \left(\mu|\nabla u|^2 + (\mu+\lambda)(divu)^2 + \frac{\beta}{\delta}\theta^2 + \frac{\gamma\beta\tau}{\delta\kappa}|q|^2\right)(x,t)dx \\ &\quad -\int_\Omega |u(x,t)|^p dx \\ J(t) &= J((u,\theta,q)(t)) = \frac{1}{2}\int_\Omega \left(\mu|\nabla u|^2 + (\mu+\lambda)(divu)^2 + \frac{\beta}{\delta}\theta^2 + \frac{\gamma\beta\tau}{\delta\kappa}|q|^2\right)(x,t)dx \\ &\quad -\frac{1}{p}\int_\Omega |u(x,t)|^p dx \qquad (3.1) \\ E(t) &= E((u,u_t,\theta,q)(t)) = J(t) + \frac{1}{2}||u_t(t)||_2^2 \\ \mathcal{H} &= \{(w,\phi,\xi) \in \left[H_0^1(\Omega)\right]^n \times L^2(\Omega) \times \left[L^2(\Omega)\right]^n \quad / \quad I(w,\phi,\xi) > 0\} \cup \{0\}, \end{aligned}$$

where

Remark 3.1 By multiplying equation (1.2) by u_t, $\beta\theta/\delta$, $\beta\gamma q/\delta\tau$ respectively and integrating over Ω,we get

$$E^{'}(t) = -\frac{\gamma\beta}{\delta\kappa}||q||_2^2 \leq 0, \qquad (3.2)$$

Lemma 2.1 *Suppose that* (2.12) *holds. and* $(u_0,\theta_0,q_0) \in \mathcal{H}$, $u_1 \in [L^2(\Omega)]^n$ *satisfying*

$$\beta = \frac{C_*^p}{\mu}\left(\frac{2p}{(p-2)\mu}E(u_0,u_1,\theta_0,q_0)\right)^{(p-2)/2} < 1, \qquad (3.3)$$

where C_* *is the embedding constant. Then* $(u,\theta,q)(t) \in \mathcal{H}$, *for each* $t \in [0,T)$.

Proof. Since $I(u_0,\theta_0,q_0) > 0$ then there exists $T_m \leq T$ such that $I((u,\theta,q)(t)) \geq 0$ for all $t \in [0,T_m)$. This implies

$$\begin{aligned} J(t) &= \frac{1}{2}\int_\Omega \left(\mu|\nabla u|^2 + (\mu+\lambda)(divu)^2 + \frac{\beta}{\delta}\theta^2 + \frac{\gamma\beta\tau}{\delta\kappa}|q|^2\right)(x,t)dx \\ &\quad -\frac{1}{p}\int_\Omega |u(x,t)|^p dx \\ &= \frac{p-2}{2p}\int_\Omega \left(\mu|\nabla u|^2 + (\mu+\lambda)(divu)^2 + \frac{\beta}{\delta}\theta^2 + \frac{\gamma\beta\tau}{\delta\kappa}|q|^2\right)(x,t)dx \qquad (3.4) \\ &\quad +\frac{1}{p}I(u(t)) \\ &\geq \frac{p-2}{2p}\int_\Omega \left(\mu|\nabla u|^2 + (\mu+\lambda)(divu)^2 + \frac{\beta}{\delta}\theta^2 + \frac{\gamma\beta\tau}{\delta\kappa}|q|^2\right)(x,t)dx, \end{aligned}$$

$\forall t \in [0, T_m)$; hence

$$\int_\Omega |\nabla u(x,t)|^2 dx \leq \frac{2p}{(p-2)\,\mu} J(t) \leq \frac{2p}{(p-2)\,\mu} E(t). \tag{3.5}$$

By exploiting (2.12), (3.3) and (3.5), we have

$$\begin{aligned} ||u(t)||_p^p &\leq C_*^p ||\nabla u(t)||_2^p = C_*^p ||\nabla u(t)||_2^{p-2} ||\nabla u(t)||_2^2 \\ &\leq \frac{C_*^p}{\mu}\left(\frac{2p}{(p-2)\,\mu} E(u_0, u_1, \theta_0, q_0)\right)^{(p-2)/2} \mu ||\nabla u(t)||_2^2 \qquad (3.6) \\ &< \mu ||\nabla u(t)||_2^2, \qquad \forall t \in [0, T_m); \end{aligned}$$

thus

$$\begin{aligned} I(t) &= \int_\Omega \left(\mu|\nabla u|^2 + (\mu+\lambda)(divu)^2 + \frac{\beta}{\delta}\theta^2 + \frac{\gamma\beta\tau}{\delta\kappa}|q|^2\right)(x,t)dx \\ -\int_\Omega |u(x,t)|^p dx &> 0, \qquad \forall t \in [0, T_m). \end{aligned}$$

This shows that $u(t) \in H, \forall t \in [0, T_m)$. By repeating the procedure, T_m is extended to T.

Theorem 3.2 *Suppose that* (2.12) *holds. If* $(u_0, \theta_0, q_0) \in \mathcal{H}$ *and* $u_1 \in [L^2(\Omega)]^n$ *satisfying (3.3) then the solution (2.13) is global.*

Proof. It suffices to show that $||\nabla u(t)||_2^2 + ||u_t(t)||_2^2 + ||\theta(t)||_2^2 + ||q(t)||_2^2$ is bounded independently of t. To achieve this we use (3.1) and (3.2) to conclude that

$$\begin{aligned} E(u_0, u_1) &\geq E(t) = J(t) + \frac{1}{2}||u_t(t)||_2^2 \\ &\geq \frac{p-2}{2p}\int_\Omega \left(\mu|\nabla u|^2 + (\mu+\lambda)(divu)^2 + \frac{\beta}{\delta}\theta^2 + \frac{\gamma\beta\tau}{\delta\kappa}|q|^2\right)(x,t)dx \qquad (3.7) \\ &\quad + \frac{1}{p}I(t) + \frac{1}{2}||u_t(t)||_2^2 \\ &\geq \frac{p-2}{2p}\int_\Omega \left(\mu|\nabla u|^2 + (\mu+\lambda)(divu)^2 + \frac{\beta}{\delta}\theta^2 + \frac{\gamma\beta\tau}{\delta\kappa}|q|^2\right)(x,t)dx \\ &\quad + \frac{1}{2}||u_t(t)||_2^2 \qquad (3.8) \end{aligned}$$

since $I(u(t)) \geq 0$. Therefore

$$||\nabla u(t)||_2^2 + ||u_t(t)||_2^2 + ||\theta(t)||_2^2 + ||q(t)||_2^2 \leq CE(u_0, u_1, \theta_0, q_0)$$

This completes the proof.

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