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A Regularity Result for a Heterogeneous Evolution Dam Problem

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Abstract

We consider a non steady-state fluid flow through a heterogeneous porous medium governed by a nonlinear Darcy law. Under a general condition on the permeability we prove a continuity result for the function characterizing the wet part of the dam.

1 Formulation of the Problem

A porous medium that we denote by Ω is supplied by several reservoirs of a fluid which infiltrates through Ω. We assume that Ω is a bounded locally Lipschitz domain of $\mathbb{R}^n$ with boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where Γ_1 is the impervious part, Γ_2 is the part in contact with either air or fluid reservoirs. Let T be a positive number, $Q = \Omega \times (0, T)$ and ϕ a nonnegative Lipschitz function defined on $\overline{Q}$.
We are concerned with the problem of finding the pressure p and the saturation χ of the fluid. For convenience we introduce the following functions : $\psi = \phi + x_n$, $u = p + x_n$ and $g = 1 - \chi$. The flow inside Ω is incompressible and obeys to the following generalized Darcy law (see [10])

$$v = -\mathcal{A}(x, \nabla(p + x_n))$$

where $\mathcal{A}$ is a function defined in $\Omega \times \mathbb{R}^N$.
Using the mass conservation law, Darcy's law, the boundary conditions and the initial data, we obtain the following strong formulation for our problem (see [4])

$$(SF)\begin{cases} i)\quad u \geq x_n, \quad 0 \leq g \leq 1, \quad g.(u - x_n) = 0, & \text{in } Q \\ ii)\quad div(\mathcal{A}(x, \nabla u) - g\mathcal{A}(x, e)) + g_t \ = \ 0, & \text{in } Q \\ iii)\quad u = \psi, & \text{on } \Sigma_2 \\ iv)\quad g(.,0) = g_0, & \text{in } \Omega \\ v)\quad (\mathcal{A}(x, \nabla u) - g\mathcal{A}(x, e)).\nu = 0, & \text{on } \Sigma_1 \\ vi)\quad (\mathcal{A}(x, \nabla u) - g\mathcal{A}(x, e)).\nu \leq 0, & \text{on } \Sigma_4 \end{cases}$$

where $g_0 \in L^\infty(\Omega)$, $e = (0, ..., 0, 1) \in I\!\!R^n$,

- $\Sigma_1 = \Gamma_1 \times (0, T)$: the impervious part,
- $\Sigma_2 = \Gamma_2 \times (0, T)$: the pervious part,
- $\Sigma_3 = \Sigma_2 \cap \{\phi > 0\}$: the part covered by fluid,
- $\Sigma_4 = \Sigma_4 \cap \{\phi = 0\}$: the part where the fluid flows outside Ω.

For $\mathcal{A}$ we assume the following with $q > 1$ and $0 < m \leq M < \infty$:

$$\begin{cases} i) & X \longmapsto \mathcal{A}(X, \xi) \text{ is measurable } \forall \xi \in I\!\!R^2, \\ ii) & \xi \longmapsto \mathcal{A}(X, \xi) \text{ is continuous for a.e } X \in \Omega, \\ iii) & \text{for all } \xi \in I\!\!R^2 \text{ and for a.e. } X \in \Omega, \\ & \qquad \mathcal{A}(X, \xi).\xi \geq m|\xi|^q \quad \text{and} \quad |\mathcal{A}(X, \xi)| \leq M|\xi|^{q-1}, \\ iv) & \text{for all } \xi, \zeta \in I\!\!R^2 \text{ and for a.e. } X \in \Omega, \\ & \qquad (\mathcal{A}(X, \xi) - \mathcal{A}(X, \zeta)).(\xi - \zeta) \geq 0. \end{cases} \tag{1.1}$$

Using the strong formulation, we are led to the following weak formulation

$$(P)\begin{cases} \text{Find } (u, g) \in L^q(0, T, W^{1,q}(\Omega)) \times L^\infty(Q) \text{ such that } : \\ (i)\quad u \geq x_n, \quad 0 \leq g \leq 1, \quad g(u - x_n) = 0 \ \text{ a.e. in } Q, \\ (ii)\quad u = \psi \quad \text{ on } \Sigma_2, \\ (iii)\quad \displaystyle\int_Q \big(\mathcal{A}(X, \nabla u) - g\mathcal{A}(X, e)\big).\nabla\xi + g\xi_t dxdt + \int_\Omega g_0(x)\xi(x, 0)dx \ \leq \ 0 \\ \qquad \forall \xi \in W^{1,q}(Q), \quad \xi = 0 \text{ on } \Sigma_3, \quad \xi \geq 0 \text{ on } \Sigma_4, \quad \xi(x, T) = 0 \ \text{ a.e. in } \Omega. \end{cases}$$

Under the assumptions (1.1), the existence of a solution was proved in [11] and also in [4] for generalized boundary conditions. Here we are concerned with the continuity of the function g. Assuming that $\mathcal{A}(X,e)$ is a constant vector, it was proved (see [11]) that $g \in C^0([0,T], L^p(\Omega))$ for all $p \geq 1$. The same result was proved in [3] and [2] in the case where $\mathcal{A}(X,\xi) = \xi$. Our main objective in this paper is to extend this regularity result to the case where $div(\mathcal{A}(X,e)) \geq 0$. The idea of the proof is based on a monotonicity result of g along the orbits of a certain differential equation. A similar monotonicity is proved in [5] for the stationary case.

We recall the following results from [11]

Proposition 1.1. *For each solution (u,g) of (P), we have*

$$u \in L^\infty(Q) \quad \text{and} \quad g \in C^0([0,T], W^{-1,q'}(\Omega)), \tag{1.2}$$

$$div\big(\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)\big) + g_t = 0 \qquad in\ \mathcal{D}'(Q). \tag{1.3}$$

Moreover if $div\big(\mathcal{A}(X,e)\big) \geq 0$ in $\mathcal{D}'(\Omega)$, we obtain

$$div\big(\mathcal{A}(X,\nabla u)\big) - g_t = div\big(g\mathcal{A}(X,e)\big) \geq 0 \qquad in\ \mathcal{D}'(Q). \tag{1.4}$$

2 A Monotonicity Property of g

From now on, we assume that

$$\mathcal{A}(.,e) = (a^1(.), a^2(.)) \in C^1(\bar{\Omega}) \tag{2.1}$$

$$div\big(\mathcal{A}(X,e)\big) \geq 0 \quad \text{in } \mathcal{C}^0(\Omega) \tag{2.2}$$

$$\partial\Omega \quad \text{is of class } C^1 \tag{2.3}$$

$$\mathcal{A}(X,e)\cdot\nu \neq 0 \quad \forall X \in \partial\Omega. \tag{2.4}$$

Then we consider the following differential system

$$(E(\omega,h)) \begin{cases} X'(s,\omega,h) &= \mathcal{A}(X(s,\omega,h),e) \\ X(0,\omega,h) &= (\omega,h) \end{cases}$$

where $h \in \pi_{x_n}(\Omega)$, $\omega \in \pi_{x'}(\Omega \cap [x_n = h])$ and where $\pi_{x'}$ and π_{x_n} are respectively the orthogonal projections on the hyperplane $x_n = 0$ and the x_n axis.
By the classical theory of ordinary differential equations there exists a unique maximal solution $X(.,\omega,h)$ of $E(\omega,h)$ which is defined on $[\alpha_-(\omega,h), \alpha_+(\omega,h)]$ with

$$X(\alpha_-(\omega,h),\omega,h) \in \partial\Omega \cap ([x_n < h]) \quad \text{and} \quad X(\alpha_+(\omega,h),\omega,h) \in \partial\Omega \cap ([x_n > h]).$$

Under the assumptions (2.1), (2.3) and (2.4), one can prove (see [5]) that $\alpha_-, \alpha_+ \in C^1(\pi_{x'}(\Omega \cap [x_n = h]))$. This allows us to introduce the following definition as in [5].

Definition 2.1. *For each $h \in \pi_{x_n}(\Omega)$ we define the set*

$$D_h = \{(s,\omega) \,/\, \omega \in \pi_{x'}(\Omega \cap [x_n = h]),\, s \in (\alpha_-(\omega,h), \alpha_+(\omega,h))\}$$

and consider the mappings $T_h : D_h \longrightarrow T_h(D_h)$ and $S_h : D_h \longrightarrow S_h(D_h)$ defined by

$$T_h(s,\omega) = X(s,\omega,h) \quad \textit{and} \quad S_h(s,\omega) = (\omega, L_h(s,\omega)) = (\omega,\tau),$$

where $L_h(s,\omega) = \displaystyle\int_{\alpha_-(\omega,h)}^{s} |\mathcal{A}(X(\nu,\omega,h),e)|d\nu = \int_{\alpha_-(\omega,h)}^{s} |X'(\nu,\omega,h)|d\nu$ represents the arc Length of the curve $X(.,\omega,h)$ from the point $X(\alpha_-(\omega,h),\omega,h)$ to the point $X(s,\omega,h)$.

Then we have

Proposition 2.1.

$$\Omega = \bigsqcup_{h\in\pi_{x_n}(\Omega)} T_h(D_h), \quad T_h \quad \textit{and} \quad S_h \textit{ are } C^1 \textit{ diffeomorphisms.}$$

Proof. We refer to [5] for the proof and we recall the following properties

$$det\mathcal{J}S_h = -|\mathcal{A}(X(t,\omega,h),e)| < 0$$
$$Y_h(t,\omega) = det(\mathcal{J}T_h) = -a^2(X(0,\omega,h)).\exp(\int_0^t \{div(\mathcal{A}(.,e))\}(X(s,\omega,h))ds) < 0,$$

where $\mathcal{J}F$ is the jacobian of F. □

The following monotonicity result generalizes the fact that $g_{x_n} - g_t \geq 0$ in $\mathcal{D}'(Q)$ when $\mathcal{A}(X,\xi) = \xi$ (see [2]). It will play a major role for the continuity proof of g.

Theorem 2.1. *Let (u,g) be a solution of (P). We have for each $h \in \pi_{x_n}(\Omega)$*

$$f_\tau - \lambda f_t \geq 0 \quad \textit{in} \quad \mathcal{D}'(S_h(D_h) \times (0,T))$$

where $f = goT_hoS_h^{-1}.(-Y_hoS_h^{-1})$ and $\lambda(\omega,\tau) = |\mathcal{A}(T_hoS_h^{-1}(\omega,\tau),e)|^{-1}$.

Proof. Let $\phi \in \mathcal{D}(S_h(D_h)\times(0,T))$, $\phi \geq 0$. Then $\widetilde{\phi}(x', x_n, t) = \phi(S_h o T_h^{-1}(x', x_n), t) \in C_0^1(T_h(D_h) \times (0,T))$, $\widetilde{\phi} \geq 0$ and by (1.4) and (2.2), we have

$$\int_{T_h(D_h)\times(0,T)} g\mathcal{A}(X,e).\nabla\widetilde{\phi} - g\widetilde{\phi}_t dxdt \leq 0$$

which can be written

$$\int_{T_h(D_h)\times(0,T)} g\mathcal{A}(X,e).\nabla(\phi o S_h o T_h^{-1}) - g\phi_t(S_h o T_h^{-1}(x', x_n), t)dx'dx_n dt \leq 0.$$

Using the change of variables $(T_h(s,\omega), t) = (x', x_n, t)$ and the fact that

$$\mathcal{A}(X(s,\omega), e)\big(\nabla(\phi o S_h o T_h^{-1})\big) o T_h.(-Y_h(s,\omega)) = -Y_h(s,\omega)\frac{\partial}{\partial s}(\phi o S_h)$$

we get

$$\int_{D_h\times(0,T)} g(T_h(s,\omega),t).(-Y_h(s,\omega)).\frac{\partial}{\partial s}\phi(S_h(s,\omega),t)dsd\omega dt$$
$$-\int_{D_h\times(0,T)} g(T_h(s,\omega),t).(-Y_h(s,\omega)).\frac{\partial}{\partial t}\phi(S_h(s,\omega),t)dsd\omega dt \leq 0$$

which becomes after using the change of variables S_h^{-1}

$$\int_{S_h(D_h)\times(0,T)} g(T_h o S_h^{-1}(\omega,\tau),t).(-Y_h o S_h^{-1}(\omega,\tau)).\big(\frac{\partial}{\partial s}(\phi o S_h(.),.)\big)(S_h^{-1}(\omega,\tau),t)$$
$$.|det\mathcal{J}S_h^{-1}(\omega,\tau)|d\omega d\tau dt$$
$$-\int_{S_h(D_h)\times(0,T)} g(T_h o S_h^{-1}(\omega,\tau),t).(-Y_h o S_h^{-1}.|det\mathcal{J}S_h^{-1}(\omega,\tau)|$$
$$.\frac{\partial\phi}{\partial t}(\omega,\tau,t).d\omega d\tau dt \leq 0.$$

Taking into account the fact that

$$\big(\frac{\partial}{\partial s}(\phi o S_h(.),.)\big)(S_h^{-1}(\omega,\tau),t) = \frac{\partial\phi}{\partial\tau}.|\mathcal{A}(.,e)|o T_h o S_h^{-1}(\omega,\tau) = \frac{\partial\phi}{\partial\tau}.|det\mathcal{J}S_h|,$$

we obtain

$$\int_{S_h(D_h)\times(0,T)} f.\frac{\partial\phi}{\partial\tau} - |\mathcal{A}(T_h o S_h^{-1}(\omega,\tau),e)|^{-1}f.\frac{\partial\phi}{\partial t}d\omega d\tau dt \leq 0.$$

□

As a consequence of Theorem 1.1, we have the following monotonicity result

Theorem 2.2. *Let (u, g) be a solution of (P). For each $h \in \pi_{x_n}(\Omega)$ and $\xi \in \mathcal{D}(S_h(D_h) \times (0,T))$, $\xi \geq 0$, the function*

$$F(k) = \int_{supp(\xi)} f\Big(\omega, \tau + k, t - \int_{\tau}^{\tau+k} \lambda(\omega,\sigma)d\sigma\Big)\xi(\omega,\tau,t)d\omega d\tau dt$$

is non-increasing on its domain of definition.

Proof. First of all we have with $Q_h = S_h(D_h) \times (0,T)$

$$F(k) = \int_{\vartheta_k^{-1}(Q_h)} f\Big(\omega, \tau + k, t - \int_{\tau}^{\tau+k} \lambda(\omega,\sigma)d\sigma\Big)\xi(\omega,\tau,t)d\omega d\tau dt \qquad (2.5)$$

where $\vartheta_k(\omega,\tau,t) = \Big(\omega, \tau + k, t - \int_{\tau}^{\tau+k} \lambda(\omega,\sigma)d\sigma\Big)$. Moreover ϑ_k is differentiable with $\mathcal{J}\vartheta_k(\omega,\tau,t) = 1$. Therefore we obtain from (2.5) by using the change of variables $\vartheta_k(\omega,\tau,t) = (\omega,\nu,s)$

$$F(k) = \int_{Q_h} f(\omega,\nu,s)\xi\Big(\omega, \nu - k, s + \int_{\nu-k}^{\nu} \lambda(\omega,\sigma)d\sigma\Big)d\omega d\nu ds. \qquad (2.6)$$

From (2.6) we deduce that F is differentiable with

$$\begin{aligned} F'(k) &= \int_{Q_h} f(\omega,\nu,s)\{-\frac{\partial\xi}{\partial\tau} + \lambda(\omega,\nu-k)\frac{\partial\xi}{\partial t}\}\Big(\omega,\nu-k,s+\int_{\nu-k}^{\nu}\lambda(\omega,\sigma)d\sigma\Big)d\omega d\nu ds \\ &= \int_{Q_h} f(\omega,\nu,s)\{-\frac{\partial\zeta}{\partial\nu} + \lambda(\omega,\nu-k)\frac{\partial\zeta}{\partial s}\}(\omega,\nu,s)d\omega d\nu ds, \end{aligned}$$

where $\zeta(\omega,\nu,s) = \xi(\omega,\nu-k,s+\int_{\nu-k}^{\nu}\lambda(\omega,\sigma)d\sigma)$.
Since $\zeta \in \mathcal{C}_0^1(Q_h)$ and $\zeta \geq 0$, it follows from Theorem 2.1 that $F'(k) \leq 0$. □

3 Continuity of g

The main result is the following theorem

Theorem 3.1. *Let (u, g) be a solution of (P). We have $g \in C^0([0,T], L^p(\Omega))$ for all $p \geq 1$.*

The proof of Theorem 3.1 is based on Theorem 2.2 and the following lemma.

Lemma 3.1. *Let (u, g) be a solution of (P). We have $f \in C^0([0,T], L^2(S_h(D_h)))$.*

Proof. First of all we deduce from Proposition 1.1 and the fact that g is bounded

$$f(\omega,\tau,t+\epsilon) \underset{\epsilon \to 0}{\rightharpoonup} f(\omega,\tau,t) \quad \text{weakly in } L^2(S_h(D_h)). \tag{3.1}$$

Let $K = S_h(D_h)$, $\Lambda(\omega,\tau) = \int_{\tau_0}^{\tau} \lambda(\omega,\sigma)d\sigma$, ($\tau_0$ fixed) and consider for $\epsilon > 0$ small enough

$$K_\epsilon = \{(\omega,\tau) \in \Omega \,/\, (\omega, \tau + \eta(\omega,\tau)) \in K\}$$

where η is defined by $\epsilon = \Lambda(\omega, \tau+\eta(\omega,\tau)) - \Lambda(\omega,\tau) = \int_{\tau}^{\tau+\eta} \lambda(\omega,\sigma)d\sigma$.
Note that it is enough to show that

$$lim_{\epsilon\to 0} \int_K f^2(\omega,\tau,t+\epsilon) - f^2(\omega,\tau,t)d\omega d\tau. \tag{3.2}$$

To do this we first remark that

$$\Big|\int_K f^2(\omega,\tau,t+\epsilon) - f^2(\omega,\tau,t)d\omega d\tau\Big| \leq \Big|\int_{K\cap K_\epsilon} f^2(\omega,\tau,t+\epsilon) - f^2(\omega,\tau,t)d\omega d\tau\Big|$$
$$+\Big|\int_{(K\setminus K_\epsilon)\cup(K_\epsilon\setminus K)} f^2(\omega,\tau,t+\epsilon) - f^2(\omega,\tau,t)d\omega d\tau\Big| = I_{\epsilon 1} + I_{\epsilon 2}. \tag{3.3}$$

Note that $I_{\epsilon 2} \leq C(|K \setminus K_\epsilon| + |K_\epsilon \setminus K|)$ (here and after we denote by C any positive constant) and therefore

$$lim_{\epsilon\to 0} I_{\epsilon 2} = 0. \tag{3.4}$$

Moreover

$$\begin{aligned} I_{\epsilon 1} &\leq \Big|\int_{K\cap K_\epsilon} f^2(\omega,\tau,t+\epsilon) - f^2(\omega,\tau+\eta,t)d\omega d\tau\Big| \\ &+ \Big|\int_{K\cap K_\epsilon} f^2(\omega,\tau+\eta,t) - f^2(\omega,\tau,t)d\omega d\tau\Big| \\ &= I_{\epsilon 3} + I_{\epsilon 4} \end{aligned} \tag{3.5}$$

Note that by Theorem 2.2 we have

$$f(\omega,\tau,t+\epsilon) \geq f(\omega,\tau+\eta,t) \ a.e.\ (\omega,\tau,t) \in Q_h.$$

Therefore, after using the fact that for a.e. $(\omega,\tau,t) \in Q_h$

$$|f^2(\omega,\tau,t+\epsilon) - f^2(\omega,\tau+\eta,t)| \leq C|f(\omega,\tau,t+\epsilon) - f(\omega,\tau+\eta,t)|,$$

we get

$$\begin{aligned} I_{\epsilon 3} &\leq C\int_{K\cap K_\epsilon} |f(\omega,\tau,t+\epsilon) - f(\omega,\tau+\eta,t)|d\omega d\tau \\ &= C\int_{K\cap K_\epsilon} (f(\omega,\tau,t+\epsilon) - f(\omega,\tau+\eta,t))d\omega d\tau \\ &= C\int_{K\cap K_\epsilon} (f(\omega,\tau,t+\epsilon) - f(\omega,\tau,t))d\omega d\tau \\ &+ C\int_{K\cap K_\epsilon} (f(\omega,\tau,t) - f(\omega,\tau+\eta,t))d\omega d\tau = CI_{\epsilon 5} + CI_{\epsilon 6} \end{aligned} \tag{3.6}$$

Now

$$\begin{aligned} |I_{\epsilon 5}| &\leq \int_K (f(\omega,\tau,t+\epsilon) - f(\omega,\tau,t))d\omega d\tau \\ &- \int_{K\setminus K_\epsilon} (f(\omega,\tau,t+\epsilon) - f(\omega,\tau,t))d\omega d\tau \\ &\leq \int_K (f(\omega,\tau,t+\epsilon) - f(\omega,\tau,t))d\omega d\tau + C|K\setminus K_\epsilon|. \end{aligned} \tag{3.7}$$

It follows from (3.1) and (3.7) that

$$lim_{\epsilon\to 0} I_{\epsilon 5} = 0. \tag{3.8}$$

Regarding $I_{\epsilon 6}$, we obtain by using the change of variables $(\omega, \tau) \to G(\omega, \tau) = (\omega, \tau + \eta(\omega, \tau)) = (\omega', \tau')$

$$I_{\epsilon 6} = \int_{K\cap K_\epsilon} f(\omega, \tau, t) d\omega d\tau - \int_{G(K\cap K_\epsilon)} f(\omega', \tau', t) \frac{\lambda(\omega', \tau')}{\lambda(\omega', \tau' - \eta(\omega, \tau))} d\omega' d\tau'.$$

Therefore it is clear that

$$lim_{\epsilon\to 0} I_{\epsilon 6} = 0. \tag{3.9}$$

In the same way we prove that

$$lim_{\epsilon\to 0} I_{\epsilon 4} = 0. \tag{3.10}$$

Taking into account (3.3)-(3.10), we get (3.2) and therefore the lemma is proved. □

Proof of Theorem 3.1. Since $-Y_h o S_h^{-1}$ is positive, uniformly bounded and independent of t, we deduce from Lemma 3.1 that $g o T_h o S_h^{-1} = \frac{f}{(-Y_h o S_h^{-1})} \in C^0([0,T], L^2(S_h(D_h)))$. Moreover by using the change of variables $T_h o S_h^{-1}$ it follows that

$$g \in C^0([0,T], L^2(T_h(D_h))). \tag{3.11}$$

Now let $(\Omega_k)_{k\geq 1}$ be a monotone covering of Ω with open subsets satisfying $\overline{\Omega_k} \subset \Omega$. Since $(T_h(D_h)_{h\in\pi_{x_n}(\Omega)}$ is an open covering of Ω, we can cover each Ω_k by a finite number of sets $T_{h_i}(D_{h_i})$ and then it is clear that

$$g \in C^0([0,T], L^2(\Omega_k)), \ \forall k \geq 1. \tag{3.12}$$

Now let $\epsilon > 0$ small enough and set $v_\epsilon(\omega, \tau, t) = |g(\omega, \tau, t+\epsilon) - g(\omega, \tau, t)|^2$. Let $\Omega'_1 = \Omega_1$, $\Omega'_{k+1} = \Omega_{k+1} \setminus \Omega_k$ for $k \geq 1$. Then $(\Omega'_k)_{k\geq 1}$ is a partition of Ω and we have

$$\int_\Omega v_\epsilon(\omega, \tau, t) d\omega d\tau = \sum_{k=1}^{\infty} \int_{\Omega'_k} v_\epsilon(\omega, \tau, t) d\omega d\tau. \tag{3.13}$$

Since $0 \leq \int_{\Omega'_k} v_\epsilon(\omega, \tau, t) d\omega d\tau \leq C|\Omega'_k|$ and $\sum_{k=1}^{\infty} C|\Omega'_k| = C|\Omega|$, it follows that

$$\lim_{\epsilon\to 0}\int_{\Omega} v_\epsilon(\omega,\tau,t)d\omega d\tau = \sum_{k=1}^{\infty}\lim_{\epsilon\to 0}\int_{\Omega'_k} v_\epsilon(\omega,\tau,t)d\omega d\tau = 0. \tag{3.14}$$

Thus $g \in C^0([0,T], L^2(\Omega))$. Using the imbedding $L^2(\Omega) \subset L^p(\Omega)$, we obtain $g \in C^0([0,T], L^p(\Omega))$ for $p \in [1,2]$.
Now for $p > 2$, we obtain the result since we have

$$\begin{aligned} |g(\omega,\tau,t+\epsilon) - g(\omega,\tau,t)|^p &= |g(\omega,\tau,t+\epsilon) - g(\omega,\tau,t)|^{p-2}|g(\omega,\tau,t+\epsilon) - g(\omega,\tau,t)|^2 \\ &\leq C|g(\omega,\tau,t+\epsilon) - g(\omega,\tau,t)|^2. \end{aligned}$$

□

Remark 3.1. *All results of this paper are clearly valid for the evolution dam problem with leaky boundary conditions ([12]).*

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