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SHEAR VELOCITY INVERSION PROCEDURE FOR LOVE WAVES

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Abstract: An inverse solution is presented to the seismic inverse problem for one-dimensional shear velocity variations. The Love waves, traveling in a layer overlying a half space, incident upon delta function potential are considered. The equation of motion for Love waves is transformed to the Schrödinger equation and then the potential is recovered by applying Gelfand-Levitan and Marchenko procedure.

1. Introduction

The determination of the properties of the medium from waves that have been reflected by or transmitted through the medium is a classical inverse scattering problem. Inverse scattering theory assumes only the general physical properties of the scattering object and then determines analytically specific model of that target using only the knowledge of the incident fields and the scattering data. The general problem can be simplified if the scattering object is assumed to be an inhomogeneous region whose material parameter has only one-dimensional spatial variations. Such inverse problems are of interest in geophysics and seismology and underground acoustics due to their various applications in determination of inhomogeneities and exploration of minerals, see e.g. Bleistein et al [2], Claerbout [3], Gray [7], Liner [9], among others. This paper presents an analytical solution to the inverse scattering problem for Love wave propagation in an inhomogeneous medium.

The analytic solution is obtained by transforming the equation of motion for Love wave propagation into a one-dimensional Schrödinger equation. The basic underlying assumption in transformation of equation of motion is that the unknown coefficient can be written as small perturbation from a known reference value, see e.g. Cohen and Bleistein [4, 5]. A further assumption for one-dimensional case is that the coefficient varies in one direction only. In this paper, we study the inverse problem arising from the Love wave propagating in a layer of uniform thickness overlying a homogeneous, isotropic half space. The layer is assumed to undergo a change in terms of its elastic properties and thus gives rise to a potential term in the Schrödinger equation corresponding to the surface wave motion. Using the formulation proposed by Gelfand-Levitan [6] and Marchenko [10], we recover the potential function which in turn determines the change in the surface layer. In the previously introduced methods, see e.g. Kay and Moses [8], the Helmholtz equation is transformed to Schrödinger equation by first transforming independent variable via Liouville transformation followed by a transformation of dependent field variable. Once this has been done, the recovered potential must be used to solve the Riccati equation, followed by the coordinate stretching process which converts the profile back to the geometric space. Each of these operations individually provide a veritable minefield of problems and pitfalls and application to Love wave problems does not seem to be apparent. As opposed to this, we use simpler and more direct transformation, which avoids the problems mentioned above.

2. Formulation of the Problem

We consider Love waves travelling from right to left in a layer overlying a half space. The geometry of the problem is shown in the figure 1.

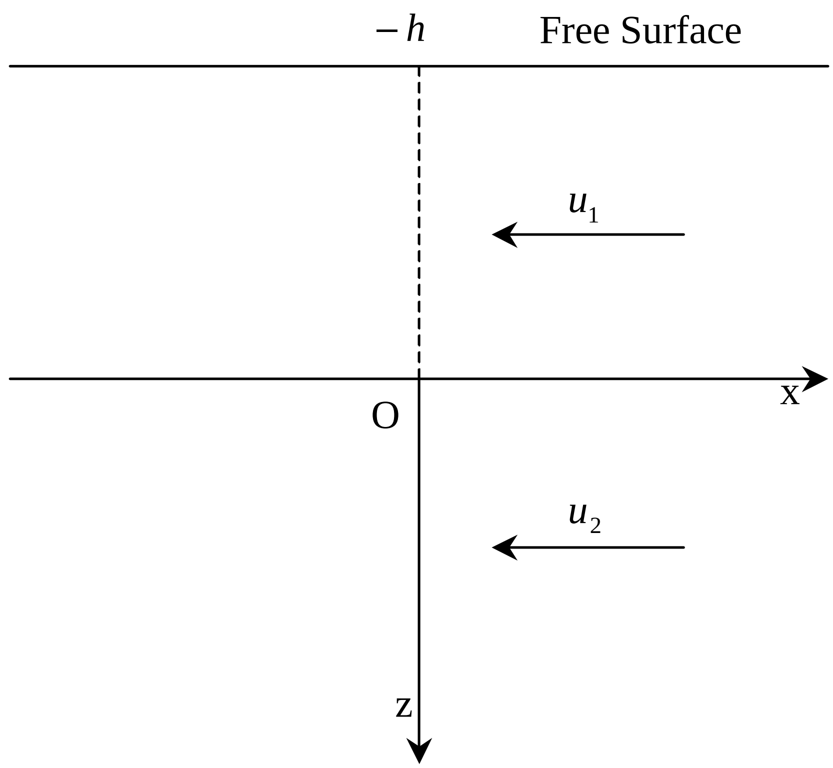

Figure 1: Geometry of the problem.

The incident Love wave of the nth mode has the displacements

$$u_1 = A\cos\left[(z+h)\,\sigma_{1n}\right]V(x)\,, \tag{2.1}$$

$$u_2 = A\cos\left[\sigma_{1n}h\right]\exp\left[-\sigma_{2n}z\right]V(x)\,, \tag{2.2}$$

where A is undetermined constant and

$$\sigma_{1n} = \sqrt{\frac{\omega^2}{\beta_{10}^2} - k_n^2}, \qquad \sigma_{2n} = \sqrt{k_n^2 - \frac{\omega^2}{\beta_{20}^2}}, \tag{2.3}$$

with β_{10}, β_{20} as background profiles. The coefficient k_n is the nth root of the Love wave dispersion relation

$$\tan\left[\left(\sqrt{\frac{\omega^2}{\beta_{10}^2}-k^2}\right)h\right]=\nu\frac{\sqrt{k^2-\frac{\omega^2}{\beta_{20}^2}}}{\sqrt{\frac{\omega^2}{\beta_{10}^2}-k^2}},\qquad \nu=\frac{\mu_1}{\mu_2}, \tag{2.4}$$

corresponding to the layer of thickness h. The displacements, u_i, $i=1,2$, satisfy the following governing equations (Aki and Richards [1])

$$\frac{\partial^2 u_i}{\partial x^2}+\frac{\partial^2 u_i}{\partial z^2}=\frac{1}{\beta_i^2}\frac{\partial^2 u_i}{\partial t^2},\qquad i=1,2. \tag{2.5}$$

We use equations (2.1) and (2.2) in equation (2.5) to get

$$\frac{\partial^2 V}{\partial x^2}-\sigma_{1n}^2 V=-\frac{\omega^2}{\beta_1^2}V,\qquad -h\leq z\leq 0, \tag{2.6}$$

$$\frac{\partial^2 V}{\partial x^2}+\sigma_{2n}^2 V=-\frac{\omega^2}{\beta_2^2}V,\qquad z\geq 0. \tag{2.7}$$

Suppose there is some inhomogeneity in the x-direction in the layer. So we consider equation (2.6) and assume that β_1 is small variation in the background profile β_{10} across the inhomogeneity, i.e.

$$\beta_1=\beta_{10}\left[1+\alpha(x)\right],\quad \alpha(x)\ll 1, \tag{2.8}$$

$$\frac{1}{\beta_1^2}=\frac{1}{\beta_{10}^2}\left[1-2\alpha(x)\right], \tag{2.9}$$

where in equation (2.9), we have neglected higher powers of $\alpha(x)$. Therefore we can write

$$\begin{aligned}\frac{\omega^2}{\beta_1^2}-\sigma_{1n}^2 &= \frac{1}{\beta_{10}^2}\left[1-2\alpha(x)\right]-\left(\frac{\omega^2}{\beta_{10}^2}-k_n^2\right)=k_n^2-\frac{2\omega^2}{\beta_{10}^2}\alpha(x)\\ &= k_n^2-q(x).\end{aligned} \tag{2.10}$$

We use equation (2.10) in equation (2.6) to get

$$\frac{\partial^2 V}{\partial x^2} + \left[k_n^2 - q(x)\right] V = 0. \tag{2.11}$$

So equation (2.6) is transformed to the Schrödinger equation (2.11) with potential $q(x)$. Now we proceed to solve the inverse problem of recovering the potential by Gelfand-Levitan and Marchenko method.

3. Solution of the Inverse Problem

The general form of the potential function $q(x)$ is related to the pole-zero configuration of the reflection coefficient $R(k)$. If $R(k)$ has N poles and no zeroes, then $(N-2)\,th$ derivative of $q(x)$ and all lower derivatives will be continuous at $x=0$. If $R(k)$ has one pole and no zero then $q(x)$ will be discontinuous at $x=0$. If $R(k)$ has two poles and no zero then $q(x)$ will be finite at $x=0$, but will have an infinite slope. We are concerned here only to find discontinuities in the shear velocity, i.e., to seek the reflectors of the unknown medium. In order to demonstrate the power of the method we model discontinuities of the medium by the delta function, peaking at the point of discontinuity. So we consider the delta function potential

$$q(x) = q_0 \delta(x), \tag{3.1}$$

where q_0 is the strength of the potential. According to the inverse scattering technique, the first step is to solve the direct problem. In this case, the direct problem satisfies the equation

$$\frac{\partial^2 V}{\partial x^2} + \left[k_n^2 - q_0 \delta(x)\right] V = 0. \tag{3.2}$$

It is reasonable to assume that $\frac{\partial^2 V}{\partial x^2}$ behave like a delta function at $x = 0$. Therefore, the first derivative must undergo a jump discontinuity at that point. Integrating equation (3.2) over the interval $[-\epsilon, \epsilon]$ and then letting $\epsilon \to 0$, we obtain the jump as

$$\left.\frac{\partial V}{\partial x}\right|_{-\epsilon}^{\epsilon} = q_0 V(0), \qquad \text{as } \epsilon \to 0. \tag{3.3}$$

Let a plane wave $\exp(-ik_n x)$ be incident on the delta function potential from the right. For the potential under consideration, the fundamental solutions of the scattering equation (3.2) are

$$\phi(x) = \exp(-ik_n x) + \frac{q_0}{k_n}\sin(k_n x), \qquad x \geq 0, \tag{3.4}$$

$$= \exp(-ik_n x), \qquad x \leq 0, \tag{3.5}$$

$$\psi(x) = \exp(ik_n x) - \frac{q_0}{k_n}\sin(k_n x), \qquad x \leq 0, \tag{3.6}$$

$$= \exp(ik_n x), \qquad x \geq 0. \tag{3.7}$$

The reflection coefficient $R(k_n)$ can calculated as follows

$$R(k_n) = \frac{r(k_n)}{s(k_n)}, \tag{3.8}$$

where

$$s(k_n) = \frac{W(\phi, \psi)}{2ik_n}, \qquad r(k_n) = \frac{W\left(\phi, \overline{\psi}\right)}{-2ik_n}, \tag{3.9}$$

and $W(\phi, \psi)$ denote the Wronskian and $\overline{\psi}$ is complex conjugate of ψ. From equation (3.8) the reflection coefficient is given by

$$R\left(k_n\right)=\frac{-iq_0}{\left[2k_n+iq_0\right]}. \tag{3.10}$$

The Fourier transform of the reflection coefficient can be calculated from the integral (Kay and Moses [8])

$$R\left(\zeta\right)=\frac{-iq_0}{4\pi}\int_{-\infty}^{\infty}\frac{\exp\left(ik_n\zeta\right)}{k_n+\frac{iq_0}{2}}dk_n. \tag{3.11}$$

The pole of the integrand is at $k_n=-\frac{iq_0}{2}$. If $\zeta>0$, the contour can be closed in the upper half plane and since there is no singularity lying inside the contour, by Cauchy's theorem the integral will be zero. If, on the other hand, $\zeta<0$, then the contour will be closed in the lower half plane and by Cauchy's theorem we have

$$R\left(\zeta\right)=-\frac{q_0}{2}\exp\left(\frac{q_0\zeta}{2}\right)H\left(-\zeta\right). \tag{3.12}$$

Where $H\left(\zeta\right)$ is the Heaviside function. The bound state solutions occur if the poles of the reflection coefficient lie in the upper half plane. So the potential $q\left(x\right)$ appearing in equation(3.2) has no bound state solutions as should be expected. To recover the potential, all we need is the impulse response function $R\left(\zeta\right)$ given by equation (3.12). The Gelfand-Levitan and Marchenko integral equation is given by (Marchenko [10])

$$K\left(x,\xi\right)+R\left(x+\xi\right)+\int_x^{\infty}K\left(x,\theta\right)R\left(\xi+\theta\right)d\theta=0. \tag{3.13}$$

Now we use equation (3.12) in (3.13) to get

$$K\left(x,\xi\right)-\frac{q_0}{2}\exp\left[\frac{q_0\left(x+\xi\right)}{2}\right]H\left[-\left(x+\xi\right)\right]-\frac{q_0}{2}\int_x^{-\xi}K\left(x,\theta\right)\exp\left[\frac{q_0\left(\xi+\theta\right)}{2}\right]d\theta=0. \tag{3.14}$$

If $x > -\xi$, then $K(x,\xi) = 0$ identically. In the opposite case $x + \xi < 0$, the integral equation (3.14) can be satisfied by taking $K(x,\xi)$ to be a constant which equals $\frac{q_0}{2}$. This then gives

$$K(x,\xi) = \frac{q_0}{2} H\left[-(x+\xi)\right], \tag{3.15}$$

$$\begin{aligned} q(x) &= -2\frac{d}{dx}\left[K(x,x)\right], \\ &= q_0 \delta(x). \end{aligned} \tag{3.16}$$

It may be noted that we can transform equation (2.7) to the Schrödinger equation in a way similar to that used to transform equation (2.6). We can then apply a similar procedure to recover the inhomogeneity in the half space.

4. Conclusions

It has been established that the governing equation for the Love wave can be transformed to one-dimensional Schrödinger equation in a more direct and straightforward manner provided that there are small variations in the propagation speed across the inhomogeneity. The method outlined here can be applied to the problems where material parameter is known up to a small perturbation, and varies in one direction only.

Acknowledgments

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