King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR301

June 2003

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INITIAL INVERSE PROBLEM IN A TWO-LAYER HEAT CONDUCTION MODEL

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GLOSSARY

Symbol	Definition
i	$i=1$ correspods to the layer $0 \leq x \leq a$, and $i=2$ corresponds to the
	layer $a \leq x \leq b$
c_i	Thermal conductivity $i=1,2$
d_i	Thermal diffusivity $i=1,2$
$f_i(x)$	Final temperature distribution $i=1,2$
f_{in}	Fourier coefficients $i=1,2$
$g_i(x)$	Initial temperature distribution $i=1,2$
$L^2[.,.]$	The space of the square integrable functions
M_1	$\frac{c_1}{d_1}\int_0^a \phi_{1n}^2(\zeta)\,d\xi$
M_2	$\frac{c_2}{d_2}\int_a^b \phi_{2n}^2(\zeta)\,d\xi$
N_n	Normalizing constants
$\Vert.\Vert$	Norm
$\phi_{in}(x)$	Eigenfunctions $i=1,2$
λ_n	Eigenvalues

Abstract: We investigate the inverse problem associated with the heat equation involving recovery of initial temperature distribution in a two-layered model from the information of final temperature profile. An integral representation for the problem is found, from which a formula for initial temperature is derived using Picard's criterion and the singular system of the associated operators.

1. Introduction

The classical direct problem in heat conduction is to determine the temperature distribution of a body as the time progresses. The task of determining the initial temperature distribution from the final distribution is distinctly different from the direct problem and it is identified as the initial inverse heat conduction problem. This type of inverse problem is extremely ill-posed, see e.g. Engl [3]. There is another approach to this inverse problem that consists of a complete reformulation of the governing equation. The inverse problem based upon the parabolic heat equation is closely approximated by a hyperbolic heat equation; see e.g. Weber [11], and Elden [2]. This alternate formulation gives rise to an inverse problem, which is stable and well-posed and thus gives more reliable results. The need to consider the alternate formulation has some physical advantages. In many applications, one encounters the situation where the usual parabolic heat equation does not serve as a realistic model. For instance, if the speed of propagation of the thermal signal is finite, i.e. for short-pulse laser applications, then the hyperbolic differential equation correctly models the problem; see Vedavarz et al. [10] and Gratzke et al. [4] among others. Moreover, as we see later, the parabolic heat conduction model can be treated as a limiting case of the hyperbolic model.

The transient-temperature distribution in a composite medium consisting of several layers in contact has numerous applications in engineering, see e.g. Özişik [9]. In this

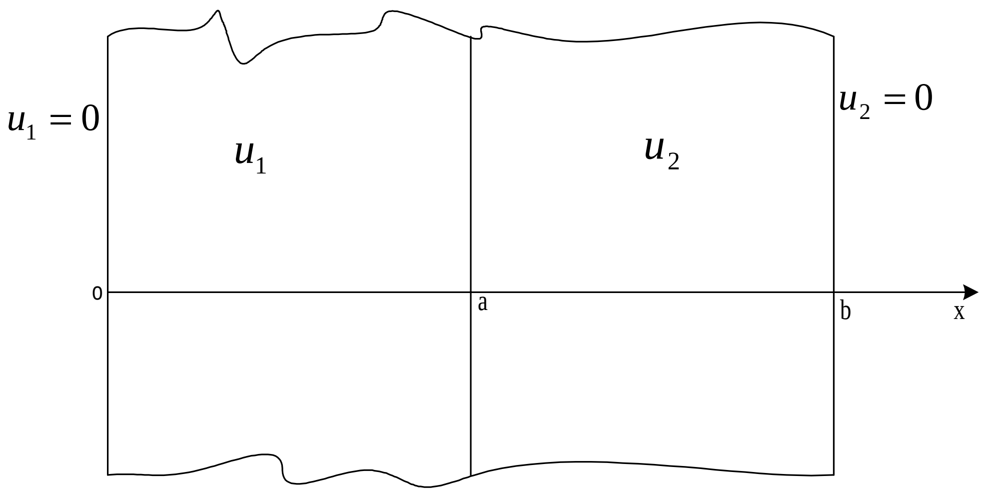

FIG. 1. Two-layer slab in perfect thermal contact at the interface.

paper, the mathematical formulation of the determination of the initial temperature distribution from the final temperature distribution in a composite medium consisting of two parallel layers of slabs shown in the Fig. 1 is presented. This problem can be transformed to an integral equation of the first kind, from that a formula for initial temperature distribution can be derived by using Picard's theorem and singular system of the associated operator, see e.g. Groetsch [5].

In the second section the heat conduction problem in a two-layer medium is formulated. In the third section the direct problem is solved. The initial inverse problem is considered in the fourth section. Regularization of the initial inverse problem in the parabolic heat equation by an alternate approach is presented in the fifth section. Finally, in the last section a summary of results is presented.

2. Formulation of the Problem

We consider a two-layer slab consists of the first layer in $0 \le x \le a$ and the second layer in $a \le x \le b$, which are in perfect thermal contact at $x = a$ as illustrated in Fig.1. Let c_1 and c_2 be the thermal conductivities, and d_1 and d_2 the thermal diffusivities for the first and second layer, respectively. The temperature distribution at the point x and t in the first layer is given by $u_1(x,t)$ and in the second layer by $u_2(x,t)$. These temperature distributions satisfy the following governing equation in the two regions

$$d_1 \frac{\partial^2 u_1(x,t)}{\partial x^2} = \frac{\partial u_1(x,t)}{\partial t} \qquad \text{in} \qquad 0 \le x \le a, \quad t > 0, \tag{1}$$

$$d_2 \frac{\partial^2 u_2(x,t)}{\partial x^2} = \frac{\partial u_2(x,t)}{\partial t} \qquad \text{in} \qquad a \le x \le b, \quad t > 0, \tag{2}$$

subject to the boundary conditions

$$u_1(0,t) = u_2(b,t) = 0, \;\; t > 0, \tag{3}$$

$$u_1(a,t) = u_2(a,t)\,, \;\; t > 0, \tag{4}$$

$$c_1 \frac{\partial u_1(a,t)}{\partial x} = c_2 \frac{\partial u_2(a,t)}{\partial x}, \;\; t > 0. \tag{5}$$

The boundary condition (3) can be replaced by an insulated boundary, i.e. $\partial_x u_1(0,t) = \partial_x u_2(b,t) = 0$, or by a radiating type boundary condition depending upon how the boundary of the bar is kept in a given situation. If the energy dissipates at the boundary then the condition (3) can be replaced by convective type boundary conditions. The analysis for these boundary conditions can be carried out in a manner similar to that described in this paper. We assume the final temperature distribution of two regions at time $t = T$ is given by

$$f_i(x) = u_i(x,T)\,, \quad i = 1, 2. \tag{6}$$

Our aim is to recover initial temperature profiles of the first and the second layer given by

$$g_i(x) = u_i(x,0)\,, \quad i = 1, 2. \tag{7}$$

3. The Direct Problem

We assume the solution of the direct problem in the form

$$u_i(x,t) = \sum_{n=1}^{\infty} v_n(t)\,\phi_{in}(x)\,, \quad i = 1, 2. \tag{8}$$

The corresponding eigenvalue problem is

$$\frac{d^2\phi_{in}(x)}{dx^2} + \frac{\lambda_n^2}{d_i}\phi_{in}(x) = 0, \qquad 0 \le x \le a, \quad i = 1, 2, \tag{9}$$

subject to the boundary conditions

$$\phi_{1n}(0) = \phi_{2n}(b) = 0, \tag{10}$$

$$c_1 \frac{d\phi_{1n}(a)}{dx} = c_2 \frac{d\phi_{2n}(a)}{dx}. \tag{11}$$

The general solution of the eigenvalue problem (9) can be found to be

$$\phi_{in}(x) = A_{in} \sin\left(\frac{\lambda_n x}{\sqrt{d_i}}\right) + B_{in} \cos\left(\frac{\lambda_n x}{\sqrt{d_i}}\right), \quad i = 1, 2. \tag{12}$$

The next step is to apply conditions (10) and (11) for the determination of four coefficients A_{in}, B_{in}. Without loss of generality, we set of the non-vanishing coefficients, say A_{1n} equal to unity. This leads to the following form of eigenfunctions

$$\phi_{1n}(x) = \sin\left(\frac{\lambda_n x}{\sqrt{d_1}}\right), \tag{13}$$

$$\phi_{2n}(x) = A_{2n} \sin\left(\frac{\lambda_n x}{\sqrt{d_2}}\right) + B_{2n} \cos\left(\frac{\lambda_n x}{\sqrt{d_2}}\right), \tag{14}$$

where A_{2n} and B_{2n} are given by

$$A_{2n} = \sin\left(\frac{\lambda_n a}{\sqrt{d_1}}\right)\sin\left(\frac{\lambda_n a}{\sqrt{d_2}}\right) + \frac{c_1}{c_2}\sqrt{\frac{d_2}{d_1}}\cos\left(\frac{\lambda_n a}{\sqrt{d_1}}\right)\cos\left(\frac{\lambda_n a}{\sqrt{d_2}}\right), \tag{15}$$

$$B_{2n} = \sin\left(\frac{\lambda_n a}{\sqrt{d_1}}\right)\cos\left(\frac{\lambda_n a}{\sqrt{d_2}}\right) - \frac{c_1}{c_2}\sqrt{\frac{d_2}{d_1}}\cos\left(\frac{\lambda_n a}{\sqrt{d_1}}\right)\sin\left(\frac{\lambda_n a}{\sqrt{d_2}}\right), \tag{16}$$

The eigenvalues λ_n are solution of the following transcendental equation

$$\begin{vmatrix} \sin\left(\frac{\lambda_n a}{\sqrt{d_1}}\right) & -\sin\left(\frac{\lambda_n a}{\sqrt{d_2}}\right) & -\cos\left(\frac{\lambda_n a}{\sqrt{d_2}}\right) \\ \frac{c_1}{c_2}\sqrt{\frac{d_2}{d_1}}\cos\left(\frac{\lambda_n a}{\sqrt{d_1}}\right) & -\cos\left(\frac{\lambda_n a}{\sqrt{d_2}}\right) & \sin\left(\frac{\lambda_n a}{\sqrt{d_2}}\right) \\ 0 & \sin\left(\frac{\lambda_n b}{\sqrt{d_2}}\right) & \cos\left(\frac{\lambda_n b}{\sqrt{d_2}}\right) \end{vmatrix} = 0. \tag{17}$$

This transcendental equation can be solved for λ_n by assigning numerical values to the constants, see [9], pages 47-50.

The eigenfunctions given by (13) and (14) are complete in $L^2[0,b]$. Therefore $g_i(x) \in L^2[0,a]$, $i = 1,2$, can be expanded as

$$g_i(x) = \sum_{n=1}^{\infty} k_n \phi_{in}(x), \qquad i = 1,2, \tag{18}$$

where

$$k_n = \frac{1}{N_n}\left[\frac{c_1}{d_1}\int_0^a \phi_{1n}(x)\, g_1(x)\, dx + \frac{c_2}{d_2}\int_a^b \phi_{2n}(x)\, g_2(x)\, dx\right], \tag{19}$$

and

$$N_n = \frac{c_1}{d_1}\int_0^a \phi_{1n}^2(\zeta)\, d\xi + \frac{c_2}{d_2}\int_a^b \phi_{2n}^2(\zeta)\, d\xi. \tag{20}$$

We use the solution assumed by separation of variables (8) in (1) and (2) which leads to the following ordinary differential equation

$$\frac{dv_n(t)}{dt} + \lambda_n^2 v_n(t) = 0, \tag{21}$$

together with

$$v_n(0) = k_n. \tag{22}$$

The direct solution of the problem is now complete and the temperature distribution $u_i(x,t)$, $i = 1, 2$ in any one of the two regions is given by

$$u_i(x,t) = \sum_{n=1}^{\infty} \exp\left(-\lambda_n^2 t\right) \frac{1}{N_n} \phi_{in}(x) \begin{bmatrix} \frac{c_1}{d_1} \int_0^a \phi_{1n}(\zeta)\, g_1(\zeta)\, d\zeta \\ +\frac{c_2}{d_2} \int_a^b \phi_{2n}(\zeta)\, g_2(\zeta)\, d\zeta \end{bmatrix}, \quad i = 1, 2. \tag{23}$$

4. The Inverse Solution

The method we use to solve the inverse problem is based on the reduction of the direct problem to an integral equation of the first kind. The expression (23) together with condition (6) leads to an integral equation of the first kind. That integral equation can be inverted by the application of Picard's theorem using the singular system of the integral operator involved.

Consider the case $g_1(x) \neq 0$ and $g_2(x) = 0$. The expression (23) in this case is

$$u_i(x,t) = \sum_{n=1}^{\infty} \exp\left(-\lambda_n^2 t\right) \frac{1}{N_n} \phi_{in}(x) \left[\frac{c_1}{d_1} \int_0^a \phi_{1n}(\zeta)\, g_1(\zeta)\, d\zeta\right], \quad i = 1, 2. \tag{24}$$

Using condition (6) in the expression (24) leads to

$$f_i(x) = \int_0^a K_i(x,\zeta)\, g_1(\zeta)\, d\zeta, \quad i = 1, 2, \tag{25}$$

where

$$K_i(x,\zeta) = \frac{c_i}{d_i} \sum_{n=1}^{\infty} \exp\left(-\lambda_n^2 T\right) \frac{1}{N_n} \phi_{in}(x)\, \phi_{1n}(\zeta), \quad i = 1, 2. \tag{26}$$

Our aim is to solve the integral equation (25) for the unknown initial temperature distribution $g_1(x)$. to accomplish this goal, we record the final profile in the first layer, that is $i = 1$. Therefore expressions (25) and (26) reduce to

$$f_1(x) = \int_0^a K_1(x,\zeta)\, g_1(\zeta)\, d\zeta, \tag{27}$$

where

$$K_1(x,\zeta) = \frac{c_1}{d_1}\sum_{n=1}^{\infty} \exp\left(-\lambda_n^2 T\right) \frac{1}{N_n} \phi_{1n}(x)\, \phi_{1n}(\zeta)\,. \tag{28}$$

Thus the inverse problem is reduced to solving integral equation of the first kind. The singular system in the first layer of the integral operator in (27) is

$$\left\{\frac{M_1}{N_n}\exp\left(-\lambda_n^2 T\right);\ \ \sqrt{\frac{c_1}{M_1 d_1}}\phi_{1n}(x)\,,\ \sqrt{\frac{c_1}{M_1 d_1}}\phi_{1n}(x)\right\}, \tag{29}$$

where

$$M_1 = \frac{c_1}{d_1}\int_0^a \phi_{1n}^2(\zeta)\, d\xi. \tag{30}$$

In the expression (29), the first term in the braces is the singular values and the next two terms correspond to the singular functions. Such a decomposition of an operator is called the singular value decomposition, see Engl [3].

Now by application of Picard's theorem (see [3]) the inverse problem is solvable iff

$$\sum_{n=1}^{\infty}\left(\frac{N_n}{M_1}\right)^2 \exp\left[2\lambda_n^2 T\right] |f_{1n}|^2 < \infty, \tag{31}$$

where

$$f_{1n} = \sqrt{\frac{c_1}{M_1 d_1}}\int_0^a \phi_{1n}(\zeta)\, f_1(\zeta)\, d\zeta, \tag{32}$$

are the classical Fourier coefficients of f_1. In this case by Picard's theorem, we can recover the initial profile by the following expression

$$g_1(x) = \sqrt{\frac{c_1}{M_1 d_1}} \sum_{n=1}^{\infty} \frac{N_n}{M_1} \exp\left[\lambda_n^2 T\right] f_{1n} \phi_{1n}(x). \tag{33}$$

Picard's theorem demonstrates the ill-posed nature of the problem considered. If we perturb the data by setting $f^\delta = f + \delta\phi_n$ we obtain a perturbed solution $g^\delta = g + \delta\phi_n \exp\left[\lambda_n^2 T\right]$. Hence the ratio $\left\|g^\delta - g\right\| / \left\|f^\delta - f\right\| = \exp\left[\lambda_n^2 T\right]$ can be made arbitrarily large due to the fact that the singular values $\exp\left[-\lambda_n^2 T\right]$ decay exponentially. This rate of decay depends on the size of the eigenvalues and on the size of the time displacement. It is also intuitively clear that for large values of T, the influence of the initial condition on the solution reduces and thus initial condition may not be recoverable, see [8] for the effect of T on recovery of the initial profile.

In case we choose $g_1(x) = 0$ and $g_2(x) \neq 0$, then a similar procedure leads to the following singular system in the second layer

$$\left\{\frac{M_2}{N_n} \exp\left(-\lambda_n^2 T\right);\ \sqrt{\frac{c_2}{M_2 d_2}} \phi_{2n}(x),\ \sqrt{\frac{c_2}{M_2 d_2}} \phi_{2n}(x)\right\}, \tag{34}$$

where

$$M_2 = \frac{c_2}{d_2} \int_a^b \phi_{2n}^2(\zeta)\, d\xi. \tag{35}$$

The condition for existence of solution is

$$\sum_{n=1}^{\infty} \left(\frac{N_n}{M_2}\right)^2 \exp\left[2\lambda_n^2 T\right] |f_{2n}|^2 < \infty, \tag{36}$$

where

$$f_{2n} = \sqrt{\frac{c_2}{M_2 d_2}} \int_a^b \phi_{2n}(\zeta)\, f_2(\zeta)\, d\zeta, \tag{37}$$

are the classical Fourier coefficients of f_2. In this case the initial profile is given by the following expression

$$g_2(x) = \sqrt{\frac{c_2}{M_2 d_2}} \sum_{n=1}^{\infty} \frac{N_n}{M_2} \exp\left[\lambda_n^2 T\right] f_{2n} \phi_{2n}(x). \tag{38}$$

5. Regularizing the Inverse solution

In order to overcome the ill-posedness of the inverse problem, we may model the problem by introducing a hyperbolic term with a small parameter in the classical heat equation. It is well established that this new model regularizes the problem in classical heat model, see e.g. Masood and Zaman [7], Masood et. al. [8], Zaman and Masood [12]. The hyperbolic heat conduction model in two layers has the following form

$$d_1 \frac{\partial^2 u_1(x,t)}{\partial x^2} = \frac{\partial u_1(x,t)}{\partial t} + \epsilon \frac{\partial^2 u_1(x,t)}{\partial t^2} \qquad \text{in} \qquad 0 \le x \le a, \quad t > 0, \tag{39}$$

$$d_2 \frac{\partial^2 u_2(x,t)}{\partial x^2} = \frac{\partial u_2(x,t)}{\partial t} + \epsilon \frac{\partial^2 u_1(x,t)}{\partial t^2} \qquad \text{in} \qquad a \le x \le b, \quad t > 0, \tag{40}$$

where the parameter ϵ is assumed to be small and $\epsilon \to 0^+$. Together with conditions $(3) - (7)$ and one additional condition given below

$$\frac{\partial u_i}{\partial t}(x,0) = 0, \quad i = 1, 2. \tag{41}$$

In this case instead of (21), we get the following ordinary differential equation

$$\epsilon \frac{d^2 v_n(t)}{dt^2} + \frac{d v_n(t)}{dt} + \lambda_n^2 v_n(t) = 0, \tag{42}$$

together with

$$v_n(0) = k_n, \tag{43}$$

and

$$\frac{\partial v_n(0)}{\partial t} = 0. \tag{44}$$

Since $\epsilon \to 0^+$, this is a singular perturbation problem. We apply the WKBJ method [1] to obtain an asymptotic representation for the solution of (42) containing parameter ϵ; the representation is to be valid for small values of the parameter. It is demonstrated in [1] that the solution stays closer to the exact solution for large values such as $\epsilon = 0.5$. The solution of (42) is given by

$$v_n(t) = \left(\frac{\epsilon\lambda_n^2 - 1}{2\epsilon\lambda_n^2 - 1}\right) k_n \exp\left[-\lambda_n^2 t\right] + \left(\frac{\epsilon\lambda_n^2 k_n}{2\epsilon\lambda_n^2 - 1}\right) \exp\left[\lambda_n^2 t - \frac{t}{\epsilon}\right]. \tag{45}$$

The remaining procedure of finding the inverse solution is same as in the previous section. The inverse solutions (33) and (38) for the hyperbolic heat conduction model can be written as

$$g_1(x) = \sqrt{\frac{c_1}{M_1 d_1}} \sum_{n=1}^{\infty} \frac{N_n}{M_1} \frac{f_{1n}\phi_{1n}(x)}{\left\{ \begin{array}{l} \left(\frac{\epsilon\lambda_n^2 - 1}{2\epsilon\lambda_n^2 - 1}\right) k_n \exp\left[-\lambda_n^2 T\right] \\ + \left(\frac{\epsilon\lambda_n^2 k_n}{2\epsilon\lambda_n^2 - 1}\right) \exp\left[\lambda_n^2 T - \frac{T}{\epsilon}\right] \end{array} \right\}}. \tag{46}$$

$$g_2(x) = \sqrt{\frac{c_2}{M_2 d_2}} \sum_{n=1}^{\infty} \frac{N_n}{M_2} \frac{f_{2n}\phi_{2n}(x)}{\left\{ \begin{array}{l} \left(\frac{\epsilon\lambda_n^2 - 1}{2\epsilon\lambda_n^2 - 1}\right) k_n \exp\left[-\lambda_n^2 T\right] \\ + \left(\frac{\epsilon\lambda_n^2 k_n}{2\epsilon\lambda_n^2 - 1}\right) \exp\left[\lambda_n^2 T - \frac{T}{\epsilon}\right] \end{array} \right\}}. \tag{47}$$

The solutions given by (33) and (38) can be recovered by letting $\epsilon \to 0^+$ in equation (46) and (47) respectively. This shows that the parabolic heat conduction model can be treated as a limiting case of the hyperbolic heat conduction model. It is shown in [8] that by choosing an appropriate value of the parameter ϵ the hyperbolic heat conduction model behaves much better than the parabolic heat conduction model.

Example 1 *Consider the initial temperature distributions of the form*

$$g_1(x) = \phi_{12}(x) = \sin\left(\frac{\lambda_2 x}{\sqrt{d_1}}\right), \quad \textit{and} \quad g_2(x) = 0. \tag{48}$$

The initial profiles should be of the form so that the integrals appearing in the expression (23) *exist and converge to the initial profiles as the time approaches to zero. One such criterion for the integrals to exist is that* $g_1(x)$ *is continuous in* $0 < x < a$ *and* $g_2(x)$ *is continuous in* $a < x < b$. *More generally, we can choose the initial profiles which are measurable and satisfy a criteria of boundedness, see John [6].*

To see that the initial profile (48) *is recovered by the processing formula* (33), *first we calculate the final data* f_{12} *given by* (32). *The expression* (27) *yields*

$$f_1(x) = \frac{M_1}{N_2}\exp\left(-\lambda_2^2 T\right)\phi_{12}(x), \tag{49}$$

and the expression (32) *yields*

$$f_{12} = \sqrt{\frac{d_1}{c_1 M_1}}\frac{M_1^2}{N_2}\exp\left(-\lambda_2^2 T\right). \tag{50}$$

Now the initial profile given by (48) *can be recovered if we use the final data given by* (50) *in the processing formula* (33). *The initial profile given by* (48) *can also be recovered exactly by the processing formula* (46).

Example 2 *Consider the initial temperature distribution of the form*

$$g_2(x) = \phi_{22}(x), \quad \textit{and} \quad g_1(x) = 0. \tag{51}$$

By following a similar procedure as in the previous example, this initial profile can be recovered by the processing formula (38). *This initial profile can also be recovered by the processing formula* (47).

At present, the method presented in this paper may not seem to work if both $g_1(x)$ and $g_2(x)$ are non-zero. Some other method or the method presented in this paper with some modifications may extend it to the case where both the initial profiles are non-zero.

6. Conclusions

The recovery of initial temperature distribution from the observation of final temperature distribution in a two-layer model is presented. It is shown that if the initial temperature distribution in either one of the layer is zero then the problem can be solved using singular value decomposition.

The inverse solution of the heat conduction model is characterized by discontinuous dependence on the data. A small error in the nth Fourier coefficient is amplified by the factor $\exp\left[\lambda_n^2 T\right]$. Thus it depends on the rate of decay of singular values and this rate of decay also depends on the size of the parameter T. In order to get some meaningful information, one has to consider first few degrees of freedom in the data and has to filter out everything else depending on the rate of decay of singular values and the size of parameter T. It is shown that a complete reformulation of the heat conduction problem as a hyperbolic equation may produce meaningful results. The hyperbolic model with a small parameter is stable and regularizes the heat conduction equation.

Acknowledgment:

The authors wish to acknowledge support provided by the King Fahd University of Petroleum and Minerals.

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