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Khalid Masood and F. D. Zaman

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

INVESTIGATION OF THE INITIAL INVERSE PROBLEM IN THE HEAT EQUATION

KHALID MASOOD: Hafr Al-Batin Community College, King Fahd University of Petroleum and Minerals, P.O. Box 5087, Dhahran 31261, Saudi Arabia, Email: masood@kfupm.edu.sa

F.D. ZAMAN: Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia, Email: fzaman@kfupm.edu.sa

NOMENCLATURE

Symbol	Definition
$\langle .,. \rangle$	Inner product
$\vert . \vert$	Absolute value
$\Vert . \Vert$	Norm
$A^* : Y \longrightarrow X$	Adjoint of A, i.e. for all $x \in X$ and $y \in Y$, $\langle Ax, y \rangle = \langle x, A^* y \rangle$
$\overline{A(X)}$	Closure of $A(X)$
$A(X)^{\perp}$	Orthogonal complement of $A(X)$
c_n	$\int_0^{\pi} g\,(x)\,\phi_n(\tau) d\tau$
f_n	$\int_0^{\pi} \phi_n(\zeta)\, f(\zeta)\, d\zeta$
$N\,(A)$	Null space of A
u_x	$\frac{\partial u}{\partial x}$
SNR	Signal to noise ratio
WKBJ	Wentzel, Kramers, Brillouin and Jeffreys
$\phi_n(x)$	Eigenfunctions

Abstract

The aim of this paper is to investigate the inverse problem in the heat equation involving the recovery of the initial temperature from measurements of the final temperature. This problem is extremely ill-posed and it is believed that only information in the first few modes can be recovered by classical methods. We will consider this problem with a regularizing parameter which approximates and regularizes the heat conduction model.

1. Introduction

The classical direct problem in heat conduction is to determine the temperature distribution of a body as the time progresses. The task of determining the initial temperature distribution from the final distribution is distinct from the direct problem and is identified as the initial inverse heat conduction problem. This type of inverse problem is extremely ill-posed, see e.g. Engle [1]. There is an alternative approach which consists of a reformulation of the classical heat equation by a hyperbolic heat equation, see Weber [2], Elden [3], and Masood et al. [4].

We will present an alternative approach which approximates and regularizes the initial inverse heat conduction solution. The need to consider the alternative formulation has some physical advantages. In many applications, one encounters a situation where the usual parabolic heat equation does not serve as a realistic model. Since the speed of propagation of the thermal signal is finite, e.g. for short-pulse laser applications, the hyperbolic differential equation correctly models the problem; see Vedavarz et al. [5] and Gratzke et al. [6] among others. This approach also makes it possible to recover information of higher modes whereas the classical heat conduction model only gives meaningful results up to third mode if the slightest noise is present in the data. Such a problem as this can be solved with the help of Picard's theorem which can be applied to an integral equation of the first kind with the associated singular system [1].

The initial inverse problem in the hyperbolic heat equation is stable and well posed. Moreover, numerical methods for hyperbolic problems are efficient and accurate. We will utilize the small value of the parameter and apply the WKBJ (Wentzel, Kramers, Brillouin and Jeffreys) method to solve the initial inverse problem, see Bender and Orszag [7]. We will also show that by controlling the size of the parameter the solution may give some information for higher modes when there is white Gaussian noise added to the data. In the second section we will solve the inverse problem by considering the parabolic heat equation. In the third section the inverse problem in the hyperbolic equation with a small parameter will be solved and compared with the inverse solution of the parabolic heat equation. An example is also provided to check the validity of the inverse solution. We will perform some numerical experiments and the results of these experiments will be analyzed in the fourth section. Conclusions of the results will be presented in the final section.

2. Initial inverse problem in the heat equation

Supposing we have a metal bar, which for the sake of convenience we take to extend over the interval $0 \leq x \leq \pi$, whose temperature at the point x and at time t is given by the function $u(x,t)$. Then, for an appropriate choice of units, $u(x,t)$ satisfies the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \pi, \quad t > 0, \tag{1}$$

with homogeneous Dirichlet boundary conditions

$$u(0,t) = u(\pi,t) = 0. \tag{2}$$

We assume the final temperature distribution of the bar at time $t = T$

$$f(x) = u(x,T)\,, \tag{3}$$

and we want to recover the initial temperature profile of the bar

$$g(x) = u(x,0)\,. \tag{4}$$

The condition (2) can be replaced by an insulated boundary, i.e. $u_x(0,t) = u_x(\pi,t) = 0$, since it is important in some applications, see for example Beck et. al. [8] and Al-Khalidy [9]. All details for this insulated boundary condition can be carried out in a similar manner to that described in this paper.

We assume by the separation of the variables, the solution of the direct problem of the form

$$u(x,t) = \sum_{n=1}^{\infty} v_n(t)\,\phi_n(x)\,. \tag{5}$$

The eigenfunctions of $\frac{d^2}{dx^2}$ given by $\phi_n(x) \doteq \sqrt{\frac{2}{\pi}}\sin(nx)$ form a complete orthonormal system in $L^2[0,\pi]$. Thus $g(x) \in L^2[0,\pi]$ can be expanded as

$$g\,(x) = \sum_{n=1}^{\infty} c_n\phi_n(x)\ ,\ \ x \in [0,\pi], \tag{6}$$

where

$$c_n = \int_0^{\pi} g\,(x)\,\phi_n(\tau)d\tau. \tag{7}$$

The solution of the form (5) when substituted into (1) leads to the following ordinary differential equation

$$\frac{dv_n(t)}{dt} = -n^2 v_n(t)\,, \qquad t > 0, \tag{8}$$

subject to

$$v_n(0) = c_n. \tag{9}$$

So, we can write the solution of the direct problem (1) in the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp\Big[-n^2 t\Big]\, \phi_n(x)\,. \tag{10}$$

Now by applying (3), we can write

$$f(x) = \int_0^{\pi} K(x,\zeta)\, g(\zeta)\, d\zeta, \tag{11}$$

which is an integral equation of the first kind. This integral equation relates the final profile $f(x)$ with the initial profile $g(x)$ via a kernel function which is given by

$$K(x,\zeta) = \sum_{n=1}^{\infty} \exp\Big[-n^2 T\Big]\, \phi_n(\zeta)\, \phi_n(x)\,. \tag{12}$$

By solving the heat equation backward in time we mean to solve (11) for the initial temperature distribution $g(x)$, given the temperature distribution $f(x)$ at a later time. This problem is ill-posed due to the fact that the mapping $f \rightarrow g$ is discontinuous, while in any practical situation f is not known precisely and only a measured approximation can be found. Therefore small errors in the measured f may lead to large errors in the solution g. The singular system of the integral equation (11) is given by

$$\Big\{\exp\Big[-n^2 T\Big]\,,\ \phi_n(x)\,,\ \phi_n(x)\Big\} \tag{13}$$

Theorem (Picard) Let $(\sigma_n; v_n, u_n)$ be a singular system for compact linear operator $A : X \longrightarrow Y$. Then the equation of first kind $Ax = y$ is solvable if and only if

$$y \in N(A^*)^{\perp} = \overline{A(X)} \ \mathit{and} \qquad \sum_{n=1}^{\infty} \frac{|\langle y, u_n\rangle|^2}{\sigma_n^2} < \infty.$$

In this case the solution is given by

$$x = \sum_{n=1}^{\infty} \frac{\langle y, u_n \rangle}{\sigma_n} v_n.$$

Now by application of Picard's theorem the inverse problem can be solved if

$$\sum_{n=1}^{\infty} \exp\left[2n^2 T\right] |f_n|^2 < \infty, \tag{14}$$

where

$$f_n = \int_0^{\pi} \phi_n(\zeta)\, f(\zeta)\, d\zeta, \tag{15}$$

are the classical Fourier coefficients of f. Now again using Picard's theorem, we can recover the initial profile by the following expression

$$g\,(x) = \sum_{n=1}^{\infty} \exp\left[n^2 T\right] f_n \phi_n\,(x)\,. \tag{16}$$

Picard's theorem demonstrates the ill-posed nature of the problem considered. If we perturb the data by setting $f^{\delta} = f + \delta\phi_n$ we obtain a perturbed solution $g^{\delta} = g + \delta\phi_n \exp[n^2 T]$. Hence the ratio $\left\|g^{\delta} - g\right\| / \left\|f^{\delta} - f\right\| = \exp[n^2 T]$ can be made arbitrarily large due to the fact that the singular values $\exp[-n^2 T]$ decay exponentially. The influence of errors in the data f is obviously affected by the rate of this decay. These errors can also be controlled further by choosing a small value of T, for example for $T = 1$, a small error in the n-th Fourier coefficient is amplified by the factor $\exp[n^2]$. So in regularizing, we will confine ourselves to lower modes by only retaining the first few terms in the series (16). This technique of truncating the series is known as truncated singular value decomposition (TSVD), see Hansen [10]. There is a necessary compromise between the exact solution and the approximate solution obtained by truncating the series (16) in case of the noisy data.

3. Regularization of the solution of the Classical Heat equation with the Hyperbolic model

The method we apply is similar to the quasi-inversion method of Lions [11]. The quasi-inversion method is based on replacing the problem $(1)-(4)$ by a problem for equation of higher order with a small parameter. It consists of the following boundary value problem

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}-\alpha\frac{\partial^4 u}{\partial x^4},\quad \alpha>0,\quad 0<x<\pi, 0<t<T, \tag{17}$$

$$u(0,t)=u(\pi,t)=0,\quad 0\le t\le T, \tag{18}$$

$$\frac{\partial^2 u(0,t)}{\partial x^2}=\frac{\partial^2 u(\pi,t)}{\partial x^2}=0,\quad 0\le t\le T, \tag{19}$$

$$u(x,0)=g(x),\quad 0\le x\le \pi. \tag{20}$$

This problem is well-posed and Lions showed that it approximates solution of (1).

There are several methods for solving ill-posed problems. The quasi-solution method to solve the equation of the first kind was introduced by Ivanov [12]. The essence of this method is to change the notion of solution of an ill-posed problem so that, for certain conditions, the problem of its determination will be well-posed. Tikhonov's regularization method id widely used for solving linear and nonlinear operator equations of the first kind, see Tikhonov and Arsenin [13]. Iterative methods are applied to solve different problems and particularly these methods can also be applied to solve operator equations of the first kind. Moultanovsky [14] applied such an iterative method to solve an initial inverse heat transfer problem. The projective methods for solving various ill-posed problems are based on the representation of the approximate solution as a finite linear combination of a certain functional system, see e.g. Vasin and Ageev [15].

The methods mentioned in the previous paragraph may be applied for solving the extensive class of inverse problems. These methods do not take into account the specific character of concrete inverse problems. The Lion's method and the method we present in this paper take into account peculiarities of the inverse problem. There is an alternative approach to the inverse heat conduction problem [2, 3], which consists of introducing a small damping parameter with the term $\frac{\partial^2 u}{\partial t^2}$. By controlling the size of the parameter we intend to regularize the solution of the inverse heat conduction problem. So, let us consider the following hyperbolic heat equation

$$\epsilon \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad \epsilon > 0, \quad 0 < x < \pi, \tag{21}$$

together with conditions $(2-4)$ and one additional condition

$$\frac{\partial u}{\partial t}(x, 0) = 0. \tag{22}$$

Following the same procedure as that for the parabolic heat equation and assuming the solution of the form (5), for $\epsilon \to 0^+$we get the following ordinary differential equation

$$\epsilon \frac{d^2 a_n(t)}{dt^2} + \frac{d a_n(t)}{dt} + n^2 a_n(t) = 0, \qquad \epsilon > 0, \quad t > 0, \tag{23}$$

subject to

$$a_n(0) = c_n, \tag{24}$$

and

$$\frac{d a_n(0)}{dt} = 0, \tag{25}$$

This is a singular perturbation problem, so we seek the WKBJ solution to this problem [7]. The WKBJ solution to (23) is

$$a_n(t) = \left(\frac{\epsilon n^2 - 1}{2\epsilon n^2 - 1}\right) c_n \exp\left[-n^2 t\right] + \left(\frac{\epsilon n^2 c_n}{2\epsilon n^2 - 1}\right) \exp\left[n^2 t - \frac{t}{\epsilon}\right]. \tag{26}$$

The singular system for this problem with $\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$ is

$$\left\{\left(\frac{\epsilon n^2 - 1}{2\epsilon n^2 - 1}\right) \exp\left[-n^2 T\right] + \left(\frac{\epsilon n^2}{2\epsilon n^2 - 1}\right) \exp\left[n^2 T - \frac{T}{\epsilon}\right];\ \phi_n(x), \phi_n(x)\right\}. \tag{27}$$

Using Picard's theorem the solution exists if

$$\sum_{n=1}^{\infty} \frac{|f_n|^2}{\left\{\left(\frac{\epsilon n^2-1}{2\epsilon n^2-1}\right)\exp\left[-n^2 T\right] + \left(\frac{\epsilon n^2}{2\epsilon n^2-1}\right)\exp\left[n^2 T - \frac{T}{\epsilon}\right]\right\}^2} < \infty, \tag{28}$$

and the solution is given by

$$g(x) = \sum_{n=1}^{\infty} \frac{f_n \phi_n(x)}{\left\{\left(\frac{\epsilon n^2-1}{2\epsilon n^2-1}\right)\exp\left[-n^2 T\right] + \left(\frac{\epsilon n^2}{2\epsilon n^2-1}\right)\exp\left[n^2 T - \frac{T}{\epsilon}\right]\right\}}. \tag{29}$$

Supposing $\epsilon \longrightarrow 0^+$ in the expression (29), we get the solution of the heat conduction problem (16). So, the solution of the heat equation can also be treated as the limiting case of the hyperbolic heat equation.

Example: Let us consider the initial temperature distribution of the form $g(x) = \sqrt{\frac{2}{\pi}}\sin(mx)$, where m is some fixed integer, then the final data for (29) and (16) can be given by

$$f_m = \left(\frac{\epsilon m^2 - 1}{2\epsilon m^2 - 1}\right) \exp\left[-m^2 T\right] + \left(\frac{\epsilon m^2}{2\epsilon m^2 - 1}\right) \exp\left[m^2 T - \frac{T}{\epsilon}\right], \tag{30}$$

and

$$f_m = \exp\left[-m^2 T\right]. \tag{31}$$

The expression (30) represents the final data for the hyperbolic model corresponding to the assumed initial profile $g(x)$ in the absence of the noise. Using (30) in the expression (29), the initial profile $g(x)$ can be recovered exactly. The expression (31) represents the final data for the parabolic model corresponding to the assumed initial profile $g(x)$ in the absence of the noise and the initial profile can be recovered exactly by using it in the expression (16).

4. Numerical Experiments

We consider the case $m = 2$ and $T = 1$ in Figs. $1-2$. In Figs.$1-2$, using the final data for the hyperbolic heat equation given by (30) in the parabolic heat conduction solution (16) which is represented by a dashed line and compare it with the exact initial profile which is represented by a thick solid line for different values of the parameter ϵ. Also we use the final data for the parabolic heat conduction model given by (31) in the hyperbolic heat solution (29) which is represented by a thin solid line and compare it with the exact initial profile. It is clear from Figs. $1-2$ that both dashed and thin lines get closer to the exact initial profile as the size of ϵ decreases and the hyperbolic heat equation approximates the heat conduction model for $\epsilon \leq 0.01$. It is demonstrated with some numerical examples in [7] that the WKBJ solution stays valid for large values of the parameter such as $\epsilon = 0.5$. Our aim in this work is not only the existence of the solution but also require that the solution to stay closer to the heat equation solution. There is a necessary compromise between the size of the parameter ($\epsilon > 0.01$) and the exact solution as the magnitude of noise increases. The higher value of ϵ helps to filter out the noise but it also damps the solution. As it can be seen from the hyperbolic model Figs. $1-2$ that the recovered profile stay below the peak and stay above the trough. The larger values of ϵ as we will use later due to increase in the magnitude of the noise, it is possible to recover the behavior of the initial profile.

The final distribution of the temperature predicted by the parabolic heat conduction model by solving the direct problem correspond to the final data (31). Now we analyze the models by adding white Gaussian noise to the data (31). The reason to use (31) as final data is that the exact measured data would be of this form. In Figs.

3−14, we use the noisy data (white Gaussian noise+(31)) in both parabolic heat conduction and hyperbolic heat models and see the mean behavior of 100 independent realizations. The noisy data used in the parabolic heat conduction solution (16) is represented by a dotted line and in the hyperbolic heat solution (29) by a thin solid line and the exact initial profile by a thick solid line.

We have considered the second mode, that is, $m = 2$ in Figs. 3−5 as well as retaining the first three terms ($N = 3$) in series (16) and (29). The number of terms N in the series (16) depends on the mode for which the information to be recovered. The recovered initial profile will deviate more from the noiseless initial profile as the number of terms N increase. In Fig. 3, the signal to noise ratio (SNR) is equal to 50 dB (we have chosen SNR=50 dB to ensure that both the models appear clearly in the Figure where SNR is defined as SNR=10 log(variance of the signal/variance of the noise)dB) and $\epsilon = 0.04$. The hyperbolic heat model behaves better than the parabolic heat conduction model for this low level of noise. We have increased the level of noise in Figs. 4−5 to SNR=20 dB. In Fig. 4, the approximation of exact initial profile by the hyperbolic model is demonstrated with an appropriate choice of ϵ. How to choose ϵ is discussed in the last paragraph of this section. The inherent instability of the parabolic heat conduction model is clear from Fig. 5 by observing the range of the vertical axis.

In Figs. 6−8, we have considered $m = 4$ and $N = 4$. In Fig. 6, we set SNR=100 dB, $\epsilon = 0.02$, and see the effects of this very low noise on both models as compared to the exact profile. We decrease SNR to 20 dB and observe the behavior of the parabolic heat conduction model in Fig. 8, noting that the vertical axis is given in units of 10^4. However for the hyperbolic model in Fig. 7 with SNR=20 dB and $\epsilon = 0.065$, some information of the initial profile may be recovered. So, from the above analysis of figures, we conclude that the hyperbolic model behaves much better than the parabolic heat conduction model in the case of noisy data. Even for lower modes, if the magnitude of noise increases, the parabolic heat conduction model becomes highly unstable.

The same analysis applies to higher modes, see Figs. 9−10. To see the effects of the size of parameter T in both models, we set $T = 2$ in Figs. 11−14. Comparing Figs. 11−12 with Figs. 4−5 and Figs. 13−14 with Figs. 7−8. For the parabolic heat conduction model, the error is more than double. However for the hyperbolic model, there is very little deviation.

To choose ϵ, we start from a higher value of ϵ for which there is no signal appearing on the graph. We gradually reduce the size and note the values of ϵ for which the signal starts to appear. We reduce the size further and note the values of ϵ for which the signal amplifies significantly, for example spikes range from -8 to 8. Then we take the mean of the two values of ϵ, which will give an appropriate choice of ϵ. For example, in Fig. 7, the signal starts to appear for $\epsilon = 0.08$ and amplifies to a significant level for $\epsilon = 0.04$. So the appropriate choice of ϵ is approximately 0.06 and it may be refined further by checking neighboring values of 0.06 for which the spikes are less pronounced. We have observed that the same procedure of finding ϵ works for higher modes as well as for lower modes.

Conclusions

The inverse solution of the heat conduction model is characterized by discontinuous dependence on the data. A small error in the nth Fourier coefficient is amplified by the factor $\exp[n^2T]$. Thus it depends on the rate of decay of singular values and this rate of decay also depends on the size of the parameter T. In order to get some meaningful information, one has to consider at most first three degrees of freedom in the data and has to filter out everything else if there is slightest noise present in the data. It is possible to recover some information about the higher modes by the proposed hyperbolic model.

It is shown that a complete reformulation of the heat conduction problem as a hyperbolic equation produces meaningful results as compared to the parabolic model. The hyperbolic model with a small parameter approximates and regularizes the heat conduction equation. It is also shown that in case of noisy data, the hyperbolic model approximates the exact initial profile better than the parabolic heat conduction model. Further, in the case of noisy data, the information about the initial profile cannot be recovered for higher modes by the parabolic heat conduction model but by the hyperbolic model some useful information may be recovered about the initial profile if the value of parameter ϵ is chosen appropriately.

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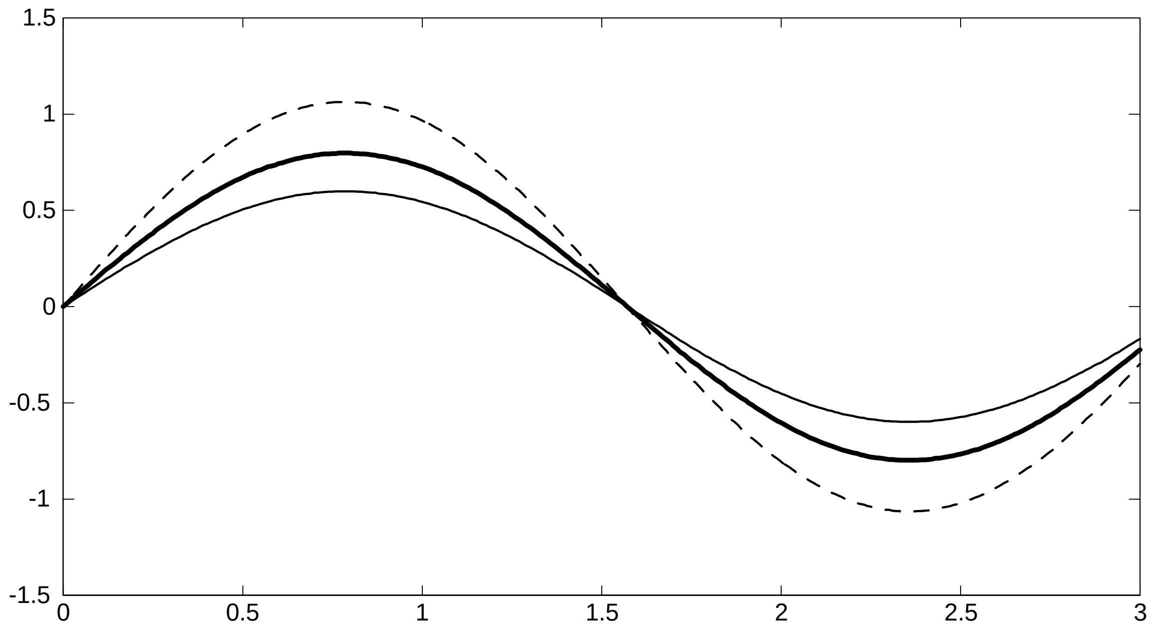

FIG. 1. The case $m = 2, T = 1$, $\epsilon = 0.05$. The thick solid line represents the exact initial profile, the thin solid line represents the response of the hyperbolic (29) model to the classical heat data (31) and the dashed line represents the response of the classical heat model (16) to the hyperbolic data (30).

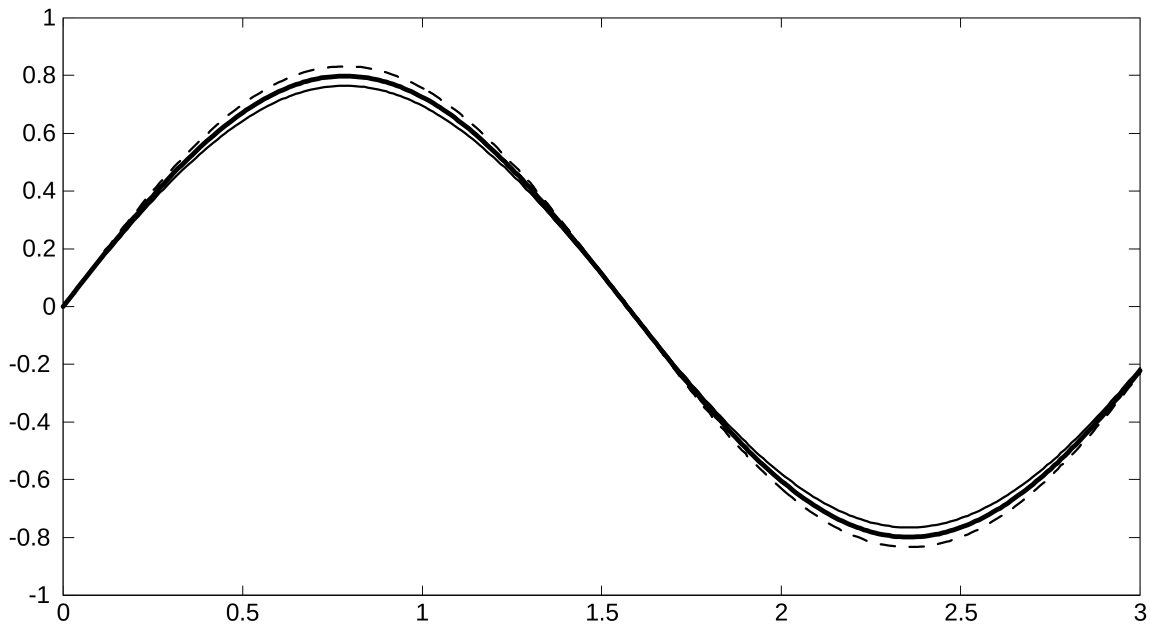

FIG. 2. The case $m = 2, T = 1$, $\epsilon = 0.01$. The thick solid line represents the exact initial profile, the thin solid line represents the response of the hyperbolic (29) model to the classical heat data (31) and the dashed line represents the response of the classical heat model (16) to the hyperbolic data (30).

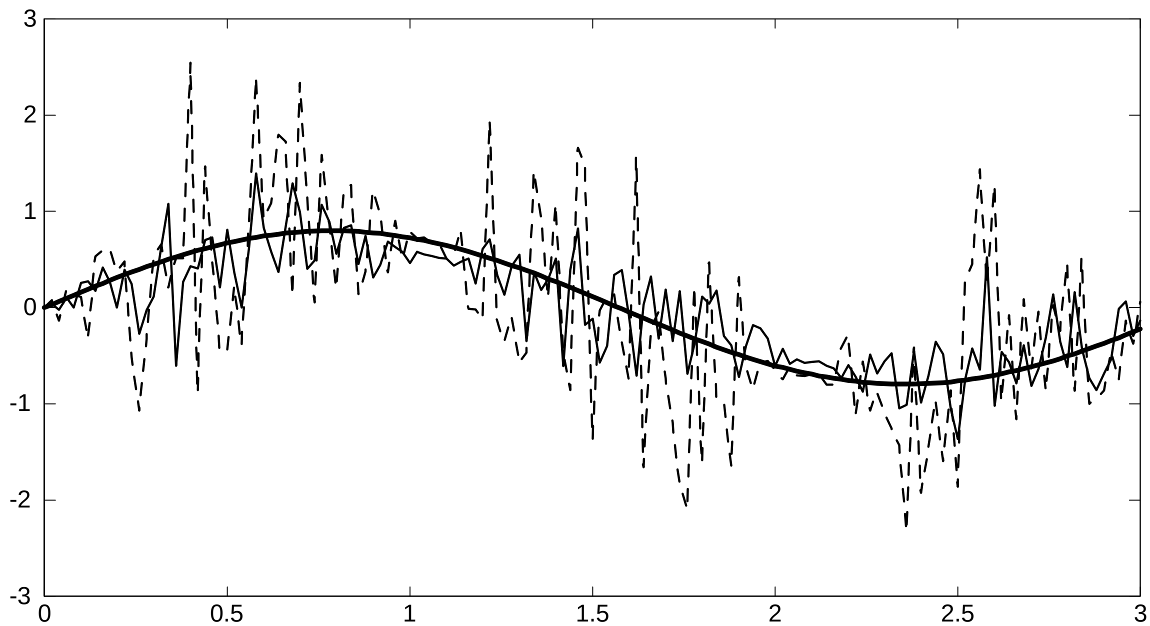

FIG. 3. The case of noisy data with SNR=50 dB, $N = 3$, $m = 2, T = 1$, $\epsilon = 0.04$. The noisy data used in the heat conduction solution (16) is represented by the dotted line and in the damped wave solution (29) by the thin solid line and the noiseless temperature by the thick solid line.

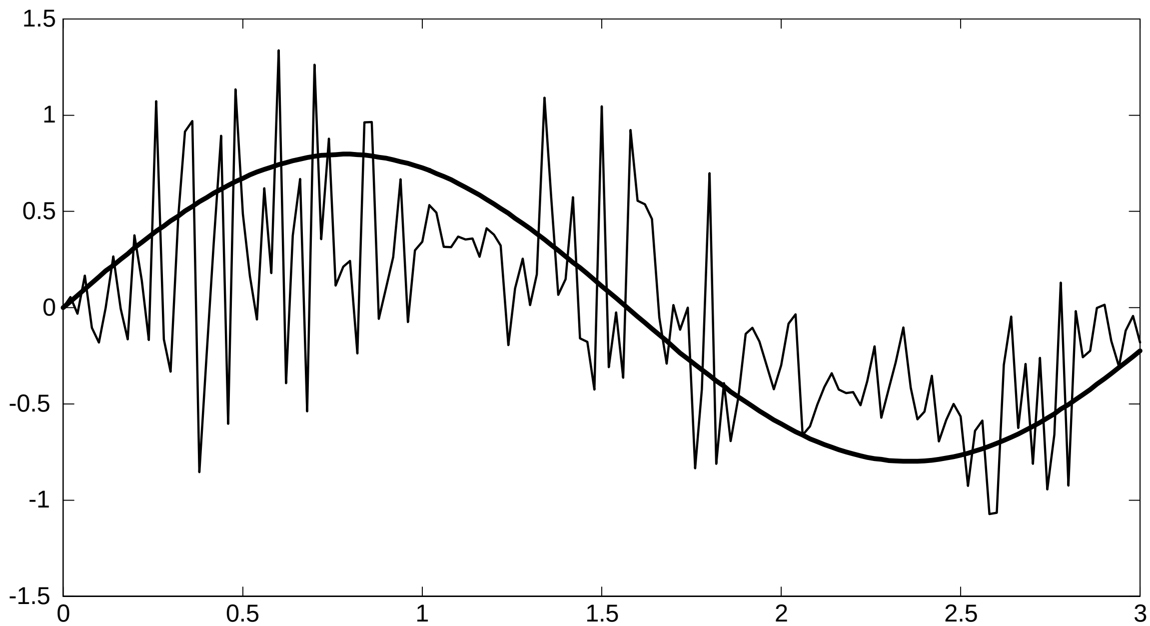

FIG. 4. Response of the damped model (29) in the case of noisy data with SNR=20 dB, $N = 3$, $m = 2, T = 1$, $\epsilon = 0.07$.

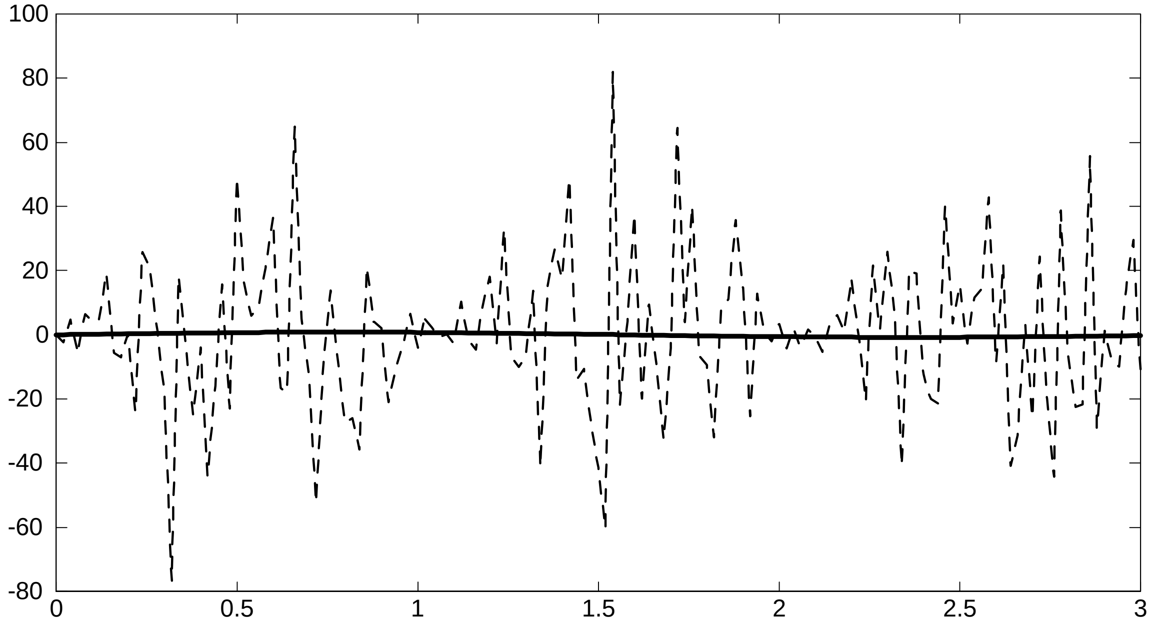

FIG. 5. Response of the classical heat model (16) in the case of noisy data with SNR=20 dB, $N = 3$, $m = 2, T = 1$.

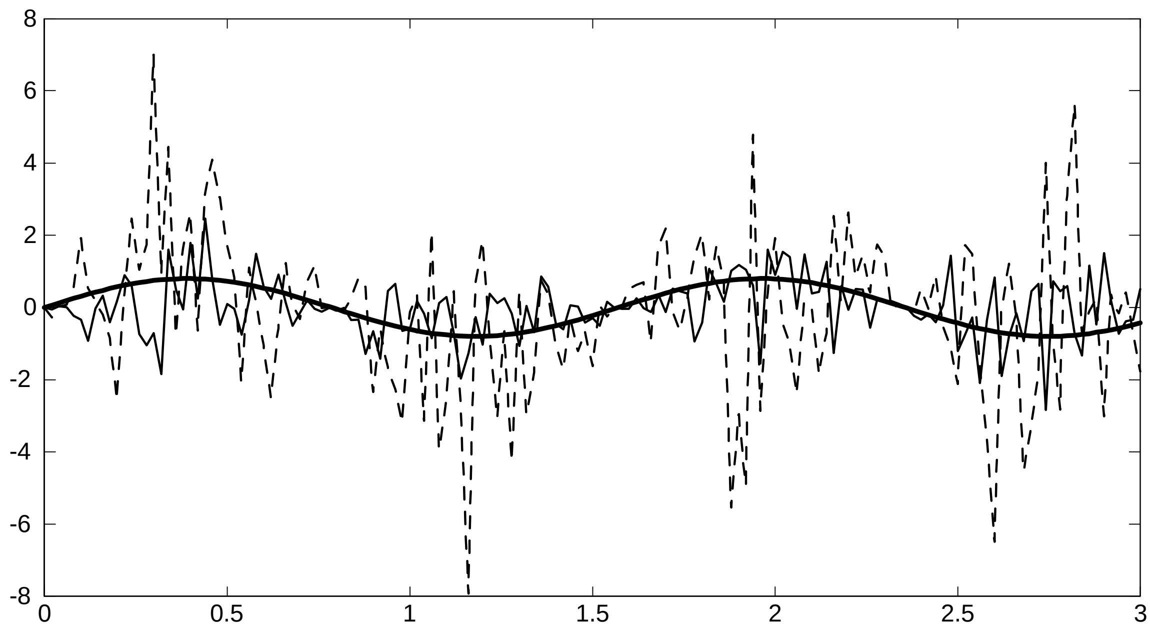

FIG. 6. The case of noisy data with SNR=100 dB, $N = 4, T = 1$, $m = 4$, $\epsilon = 0.02$. The noisy data used in the heat conduction solution (16) is represented by the dotted line and in the damped wave solution (29) by the thin solid line and the noiseless temperature by the thick solid line.

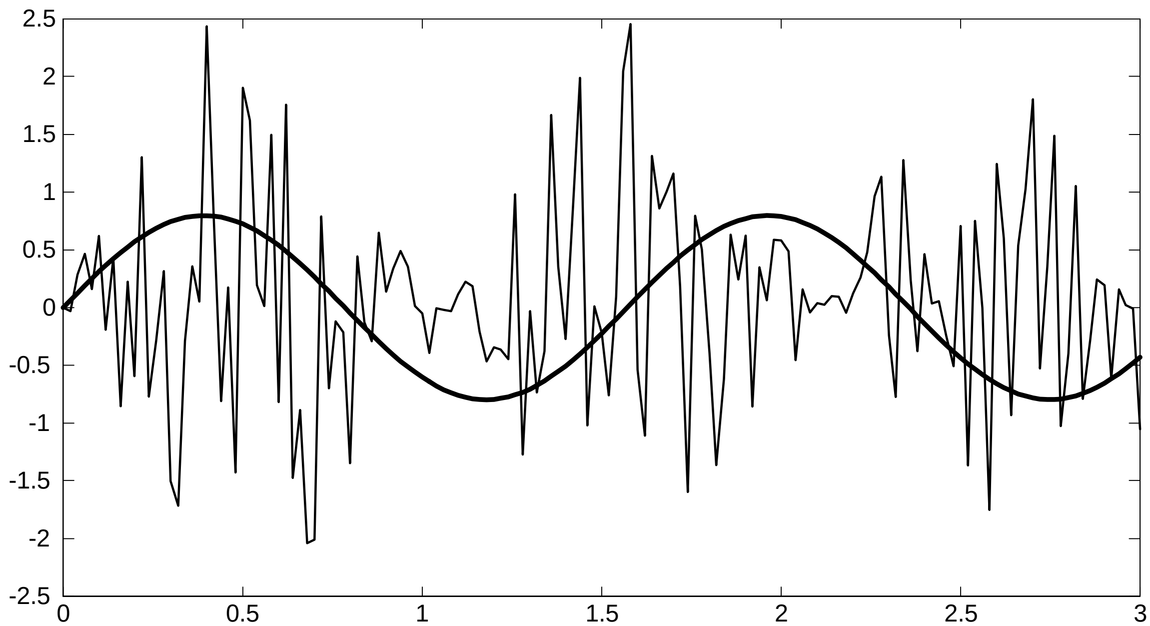

FIG. 7. Response of the damped model (29) in the case of noisy data with SNR=20 dB, $N = 4, T = 1$, $m = 4$, $\epsilon = 0.065$.

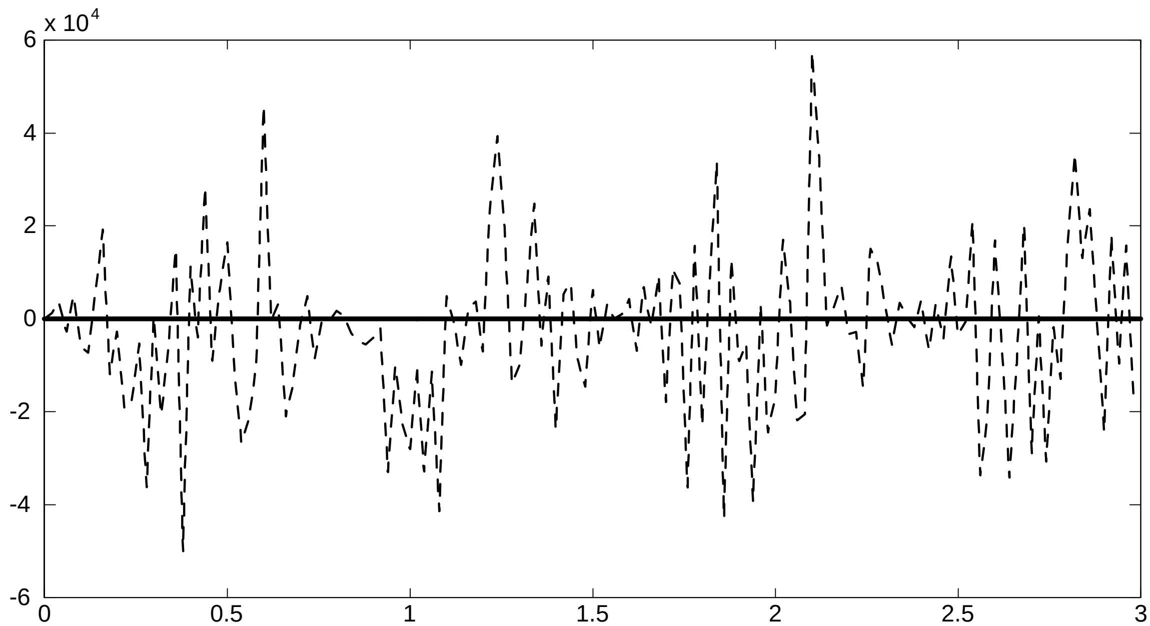

FIG. 8. Response of the classical heat model (16) in the case of noisy data with SNR=20 dB, $N = 4, T = 1$, $m = 4$.

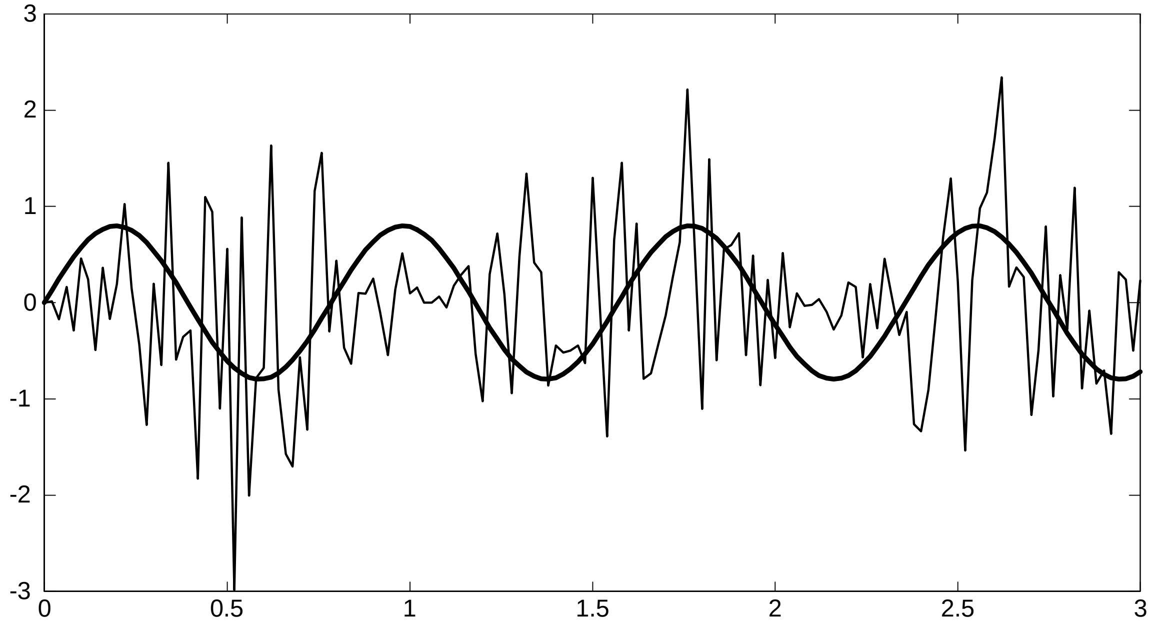

FIG. 9. Response of the damped model (29) in the case of noisy data with SNR=20 dB, $N = 8, T = 1$, $m = 8$, $\epsilon = 0.062$.

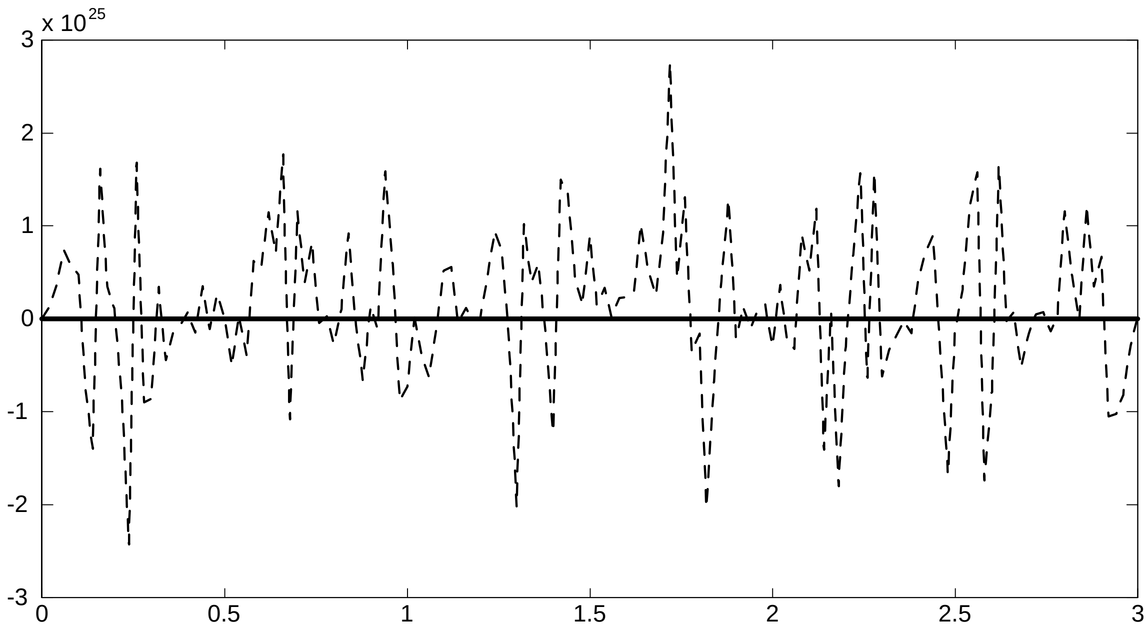

FIG. 10. Response of the classical heat model (16) in the case of noisy data with SNR=20 dB, $N = 8, T = 1$, $m = 8$.

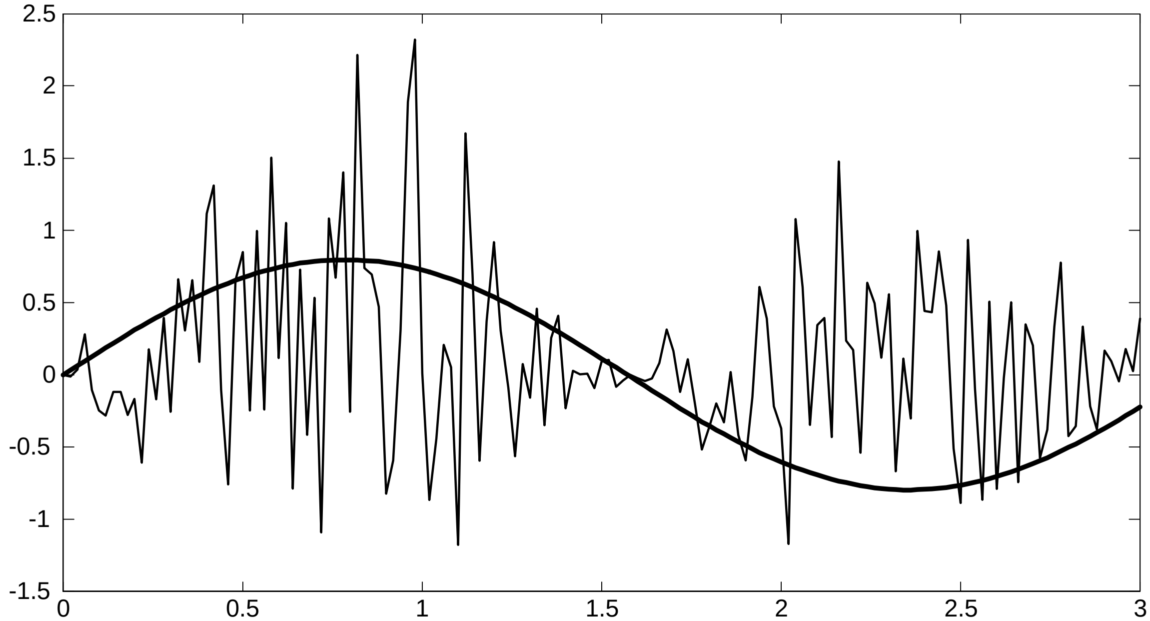

FIG. 11. Response of the damped model (29) in the case of noisy data with SNR=20 dB, $N = 3$, $m = 2, T = 2$, $\epsilon = 0.12$.

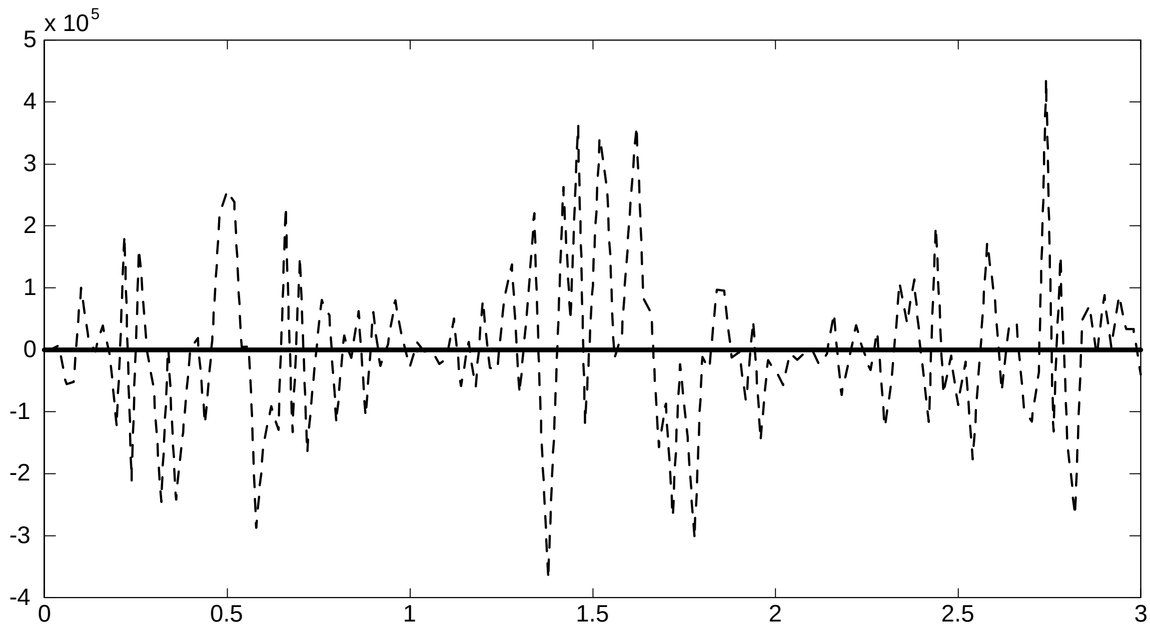

FIG. 12. Response of the classical heat model (16) in the case of noisy data with SNR=20 dB, $N = 3$, $m = 2, T = 2$.

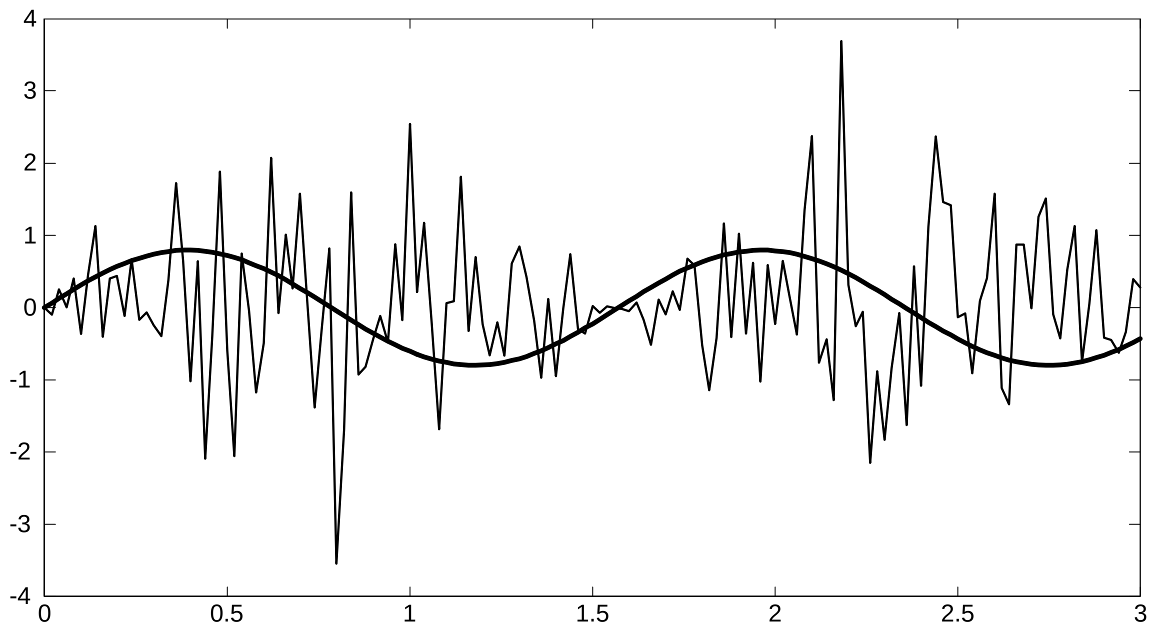

FIG. 13. Response of the damped model (29) in the case of noisy data with SNR=20 dB, $N = 4$, $m = 4, T = 2$, $\epsilon = 0.095$.

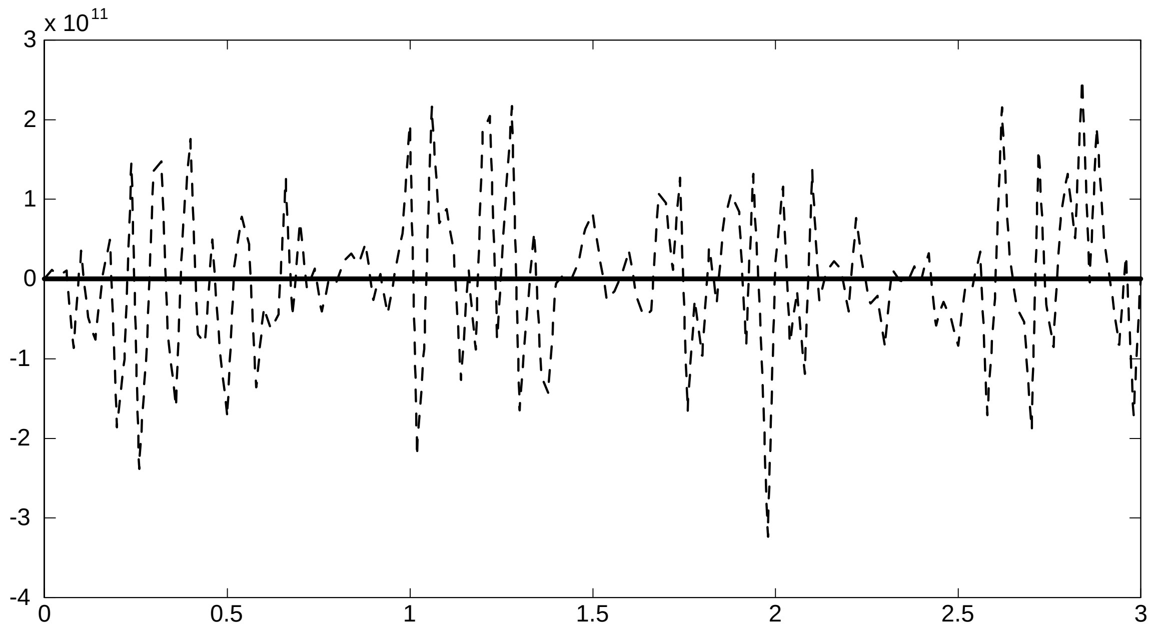

FIG. 14. Response of the classical heat model (16) in the case of noisy data with SNR=20 dB, $N = 4$, $m = 4, T = 2$.