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By

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Abstract

In a semigroup S with zero (0), an element a in S is called nilpotent if there exists a natural number n such that $a^n = 0$. In this note we obtain a formula for the total number of nilpotents in P_n, the partial symmetric semigroup and I_n, the symmetric inverse semigroup.

KEY WORDS: Semigroup, idempotent, nilpotent, symmetric semigroup, symmetric inverse semigroup, partial symmetric semigroup, Stirling numbers, Lah numbers, generalized Laguerre polynomials.

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Introduction

Let $X_n = \{1, 2, \ldots, n\}$ then a (partial) transformation $\alpha :$ Dom $\alpha \subseteq X_n \to$ Im α is said to be a *full* or *total* transformation if Dom $\alpha = X_n$; otherwise it is called *strictly partial.* Three fundamental semigroups of transformations under the usual composite that have been extensively studied are: T_n, the full transformation semigroup (or the symmetric semigroup); I_n, the semigroup of partial one-one mappings (or the symmetric inverse semigroup); and P_n the semigroup of partial transformations (or the partial symmetric semigroup). Partial one-one mappings are also known as subpermutations [2]. Enumerative problems of an essentially combinatorial nature arise naturally in the study of semigroups of transformations. If we denote by S_n, the symmetric group, then Howie [8] showed that $T_n \setminus S_n = \mathrm{Sing}_n$ is idempotent generated and Tainiter [11] showed that the total number of its idempotent is

$$\sum_{r=1}^{n-1} \binom{n}{r} r^{n-r}.$$

In a semigroup S with 0, an element a in S is called *nilpotent* if there exists a natural number n such that $a^n = 0$. Gomes and Howie [7] showed that $I_n \setminus S_n = SI_n$ is nilpotent generated if n is even, and also characterized the nilpotent generated subsemigroup if n is odd. Simultaneously and independently, Sullivan [10] obtained similar results for $P_n \setminus T_n = SP_n$. Moreover, for the nilpotent case, Garba [5 & 6] obtained analogous results to Howie and McFadden [9] in the idempotent case. Gomes and Howie [7] computed the number of nilpotents of I_n of height $n-1$ (*height* $\alpha = |\mathrm{Im}\ \alpha|$) to be $n!$. It is worth noting that the number of nilpotents of P_n of height $n-1$ is also $n!$, since they must be one-one.

The total number of nilpotents in P_n and I_n does not seem to have been investigated and in this note we set out to investigate these numbers.

The Main Result

First, let S be one of the semigroups P_n or I_n and for $r = 0, 1, 2, \ldots, n-1$; let

$$N(J_r) = \{\alpha \in S : |\text{Im } \alpha| = r \text{ and } \alpha \text{ is nilpotent}\} \tag{1}$$

be the set of all nilpotents in S of height r. Next, as in [9] we define the Stirling number of the second kind, $S(n, r)$ as the *total number of partitions of an n-element set into r nonempty subsets.* Alternatively, $S(n, r)$ can be defined by the initial conditions $S(n, 0) = S(n, 1) = S(n, n) = 1$ and the recurrence relation $S(n, r) = S(n-1, r-1) + rS(n-1, r)$, $(n \geq r,\ n \geq 1)$. Then the principal result of this paper is:

Theorem 1 *Let $N(J_r)$ be as defined in (1). Then for the semigroups P_n and I_n, respectively, we have*

(a) $\displaystyle\sum_{r=0}^{n-1} |N(J_r)| = \sum_{r=0}^{n-1} \binom{n}{r} S(n, r+1) r! = (n+1)^{n-1}$;

(b) $|N(J_r)| = \binom{n}{r}\binom{n-1}{r} r! = |L_{n,n-r}|$, *where* $L_{n,r} = (-1)^n \binom{n-1}{r-1} \frac{n!}{r!}$ *is the Lah number [3].*

As a first step towards the proof of the above theorem we show that [7, Lemma 2.1] which gives a characterization of nilpotents in I_n can be extended to P_n.

Lemma 2 *An element α in P_n is nilpotent if and only if there exists no nonempty subset A of Dom α such that $A\alpha = A$.*

Proof. The direct half of the proof is exactly as in [7] since the one-one property has not been used.

Conversely, suppose that α is not nilpotent. Then by finiteness

$$\text{Dom } \alpha \supseteq \text{Dom } \alpha^2 \supseteq \ldots \supseteq \text{Dom } \alpha^n = \text{Dom } \alpha^{n+1}$$

for some natural number n and Dom α^n = Dom $\alpha^{n+1} \neq \emptyset$, by nonnilpotency. Now let $\beta = \alpha^n$ and fix $x \in$ Dom β. Next for $i = 1, 2, \ldots$ define

$$A_i = \{x\beta^i, x\beta^{i+1}, \ldots\}$$

then it follows that

$$A_1 \supseteq A_2 \supseteq \ldots \supseteq A_k = A_{k+1}$$

for some natural number k. Hence $x\beta^k \in A_k = A_{k+1}$, and so $x\beta^k = x\beta^{k+r}$ for some natural number r. Put $x\beta^k = y$ then $y\beta^r = y$ and so $y\alpha^{nr} = y$. Finally, let

$$A = \{y, y\alpha, \ldots, y\alpha^{t-1}\} \neq \emptyset \ \ (t = nr)$$

then

$$A\alpha = \{y\alpha, y\alpha^2 ..., y\alpha^{t-1}, y\} = A,$$

as required.

An immediate consequence of the above lemma is that nilpotent partial transformations are cycle-free (in particular, have no fixed points) and they are strictly partial (i. e., $X_n \setminus \text{Dom } \alpha$ is nonempty).

Proof of Theorem 1.
(a) We begin by determining the cardinality of $N(J_r)$. For each partition of X_n into $r+1$ nonempty subsets, we can define an element α in $N(J_r)$ in the following way. We let one of the $r+1$ subsets be $X_n \setminus \text{Dom } \alpha$ and let the remaining r subsets be pre-images of α. Note that we can partition X_n into $r+1$ nonempty subsets in $S(n, r+1)$ ways. Next we can choose the r images of α from X_n, say $\{x_1, x_2, \ldots, x_r\}$ in $\binom{n}{r}$ ways. Finally, we have to tie these r images to their pre-images in a one-one fashion and simultaneously avoiding cycles. We achieve this by identifying the forbidden pre-images of x_i $(i = 1, 2, \ldots, r)$.

First note that x_1 cannot have as pre-image the partition class containing itself, but can have any of the other r classes as possible pre-images and so x_1 has r degrees of freedom. Next, note that x_2 cannot have $x_1\alpha^{-1}$ as a pre-image and also the partition class containing itself. However, note that the partition class containing x_2 could be $x_1\alpha^{-1}$ in which case $x_2 \in x_1\alpha^{-1}$ (equivalently, $x_2\alpha = x_1$). Then x_2 cannot also have the partition class containing x_1 as a pre-image, for that will mean $x_1\alpha = x_2$ and so we have a cycle (x_2x_1) in α. Moreover, the partition class containing x_1 and $x_1\alpha^{-1}$ are necessarily distinct by the non-choice of the former as a possible pre-image of x_1 and so in either eventuality x_2 has $r-1$ degrees of freedom. Now, in a recursive way we define the set of forbidden pre-images of x_k (for $k = 2, 3, \ldots, r$) as

$$\{x_1\alpha^{-1}, x_2\alpha^{-1}, \ldots, x_{k-1}\alpha^{-1}, A_k\},$$

where A_i is the partition class containing x_i $(i \in \{1, 2, \ldots, k\})$. Then as in the case of x_2 above, A_k could be one of the already chosen pre-images say, $x_i\alpha^{-1}$ (for some $i < k$). If this happens then x_k cannot also have A_i as a pre-image, and so A_i replaces the last A_k in the set of forbidden pre-images. Should this A_i be one of the already chosen pre-images then we repeat the same argument until we find a required forbidden pre-image that is distinct from all the previously chosen pre-images. Of course, this process cannot go on indefinitely, because of finiteness, and the fact that there are no cycles up to the choice of $x_{k-1}\alpha^{-1}$. Moreover, the $r+1$ partition classes that we have as compared to the r images guarantees that we can find a required forbidden partition class that is distinct from all the previously chosen pre-images. Thus x_k has $(r+1) - k$ degrees of freedom. Therefore the total number of ways of tying the x_i's to their pre-images in a one-one fashion and simultaneously avoiding cycles is $r(r-1)(r-2)\cdots 2.1 = r!$. Hence, we have

$$|(N(J_r)| = \binom{n}{r} S(n, r+1)r!. \quad (2)$$

In particular, we have from (2) that

$$|N(J_{n-1})| = \binom{n}{n-1} S(n,n)(n-1)! = n!,$$

as found by Gomes and Howie [7, page 388].

Before stating the next proposition let us define the *descending (or falling) factorial of a natural number* x as

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1).$$

The following well known identity is sometimes used to define the Stirling numbers of the second kind

$$\sum_{r=0}^{n} S(n,r)(x)_r = x^n.$$

Now we have

Proposition 3 $\sum_{r=0}^{n} S(n,r+1)(x)_r = (x+1)^{n-1}$.

Proof.

$$\begin{aligned}(1+x)^{n-1} &= \sum_{k=0}^{n-1} \binom{n-1}{k} x^k = \sum_{k=0}^{n-1} \binom{n-1}{k} \sum_{j=0}^{k} S(k,j)(x)_j \\ &= \sum_{j=0}^{n-1}\sum_{k=j}^{n-1} \binom{n-1}{k} S(k,j)(x)_j\end{aligned}$$

By Theorem B in [4, page 209], we have

$$S(n,j+1) = \sum_{k=j}^{n-1} \binom{n-1}{k} S(k,j).$$

Therefore

$$(x+1)^{n-1} = \sum_{r=0}^{n-1} S(n,r+1)(x)_r,$$

as required. Thus the proof of Theorem 1(a) is complete.

(b) Here we begin with the image set, say $\{x_1, x_2, \ldots, x_r\}$ whose elements can be chosen from X_n in $\binom{n}{r}$ ways. Now to tie each of these images to their preimages in a one-one fashion and simultaneously avoiding cycles we argue exactly as in (a) above to get the first equality of Theorem 1(b), that is

$$|N(J_r)| = \binom{n}{r}(n-1)_r = \binom{n}{r}\binom{n-1}{r} r!. \tag{3}$$

In particular, we have from (3) that

$$|J_{n-1}| = \binom{n}{n-1}\binom{n-1}{n-1}(n-1)! = n!.$$

To complete the proof of Theorem 1(b), we let

$$U_{n,k} = \binom{n}{k}\binom{n-1}{k} k!.$$

Now replacing k by $n-k$ we get from [4, page 156]

$$U_{n,n-k} = \binom{n-1}{k-1}\frac{n!}{k!} = |L_{n,k}|.$$

A closed formula for $\sum U_{n,k}$ seems out of reach, however, a recurrence relation is possible. The generalized Laguerre polynomial of degree n is given in [3, page 145] by

$$L_n^{(a)}(x) = \sum_{k=0}^{n} \binom{n+a}{n-k} \frac{(-x)^k}{k!} \tag{4}$$

so that

$$U_n = \sum_{k=0}^{n} U_{n,k} = n! L_n^{(-1)}(-1).$$

From [3, page 153, equation (2.2)], we deduce

$$n\frac{U_n}{n!} = (2n-1)\frac{U_{n-1}}{(n-1)!} - (n-2)\frac{U_{n-2}}{(n-2)!}$$

which gives

Proposition 4 $U_n = (2n-1)U_{n-1} - (n-1)(n-2)U_{n-2}$.

Finally, if we denote by C_n the order of the symmetric inverse semigroup I_n, that is

$$C_n = \sum_{k=0}^{n} \binom{n}{k}^2 k!,$$

then we can easily deduce that

$$U_n = C_n - nC_{n-1},$$

which gives an alternative way of computing U_n since we know from [1] that C_n satisfies the following recurrence relation:

Theorem 5 *[1, Proposition 2.1(b)]* $C_n = 2nC_{n-1} - (n-1)^2 C_{n-2}$.

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References

[1] D. Borwein, S. Rankin and L. Renner, Enumeration of injective partial transformations, *Discrete Math.* **73**(1989), 291–296.

[2] P.J. Cameron and M. Deza, On permutation geometries, *J. London Math. Soc.* (2) **20**(1979), 373–386.

[3] T.S. Chihara, "An introduction to orthogonal polynomials," Gordon and Breach Publishers, Inc., Paris: 1978.

[4] L. Comtet, "Advanced Combinatorics: the art of finite and infinite expansions," D. Reidel Publishing Company, Dordrecht, Holland: 1974.

[5] G.U. Garba, On the nilpotent rank of partial transformation semigroups, *Portugaliae Mathematica*, **51**(1994), 163–172.

[6] G.U. Garba, On the nilpotent ranks of certain semigroups of transformations, *Glasgow Math. J.*, **36**(1994), 1–9.

[7] G.M.S. Gomes and J.M. Howie, Nilpotents in finite symmetric inverse semigroups, *Proc. Edinburgh Math. Soc.* **30**(1978), 383–395.

[8] J.M. Howie, The subsemigroup generated by the idempotents of a full transformation semigroup, *J. London Math. Soc.* **41**(1966), 707–716.

[9] J.M. Howie and R.B. McFadden, Idempotent rank in finite full transformation semigroups, *Proc. Roy. Soc. Edinburgh*, **114**(1990), 161–167.

[10] R.P. Sullivan, Semigroups generated by nilpotent transformations, *J. Algebra* **110**(1987), 324–345.

[11] M. Tainiter, A characterization of idempotents in semigroups, *J. Combinatorial Theory* **5**(1968), 370–373.

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