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On the Continuity of the Free Boundary in a Heterogeneous Dam with a Leaky Boundary Condition

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Abstract

A heterogeneous dam is considered assuming the flux, at the bottoms of the reservoirs obeying to a nonlinear law : a leaky condition. The velocity and the pressure inside the porous medium are related by a nonlinear Darcy law. Under a general condition on the permeability we prove that the free boundary is locally represented by continuous curves.

Introduction

In reference [23] the second author studied a heterogeneous dam problem with the following nonlinear law

$$\mathbf{v} = -\mathcal{A}(\mathbf{X}, \nabla(\mathbf{p}+\mathbf{y}))$$

where $\mathbf{v}$ is the fluid velocity, p its pressure, $X=(x,y)$ and $\mathcal{A}$ is a mapping from $\Omega\times\mathbb{R}^2$ to $\mathbb{R}^2$. Moreover a leaky boundary condition

$$\mathbf{v}.\nu = -\beta(\mathbf{X}, \varphi - \mathbf{p})$$

was imposed at the reservoirs bottoms. Under the assumption

$$\mathcal{A}(X,e) = k(X)e, \text{ with } e=(0.1) \quad \text{and} \quad \frac{\partial k}{\partial y} \geq 0 \text{ in } \mathcal{D}'(\Omega), \tag{0.1}$$

where k is a real valued function, he proved that the free boundary is a continuous curve $y=\Phi(x)$.

Note that a similar result is proved in [14] in the case $\mathcal{A}(X,\xi) = a(X).\xi$, where $a(X) = (a_{ij}(X))$ the matrix permeability, is such that $a_{12}(X)=0$ and $\frac{\partial a_{12}}{\partial y} \geq 0$ in $\mathcal{D}'(\Omega)$. However the continuity of the free boundary under the reservoirs was established only when $a(X)$ is constant.

In this paper we shall be concerned with the same problem under the weaker assumption

$$div(\mathcal{A}(X,e)) \geq 0 \text{ in } \mathcal{D}'(\Omega) \tag{0.2}$$

and we follow a new idea introduced in [15] for the heterogeneous dam problem with Dirichlet conditions. It consists on the fact that under condition (0.2) the function χ describing the saturated zone is non-increasing along the orbits of the differential equation

$$X'(t) = \mathcal{A}(X(t), e)$$

which generalizes the fact that χ is non-increasing with respect to y under condition (0.1). As a consequence we show that if the pressure is positive at some point $X_0 = X(t_0)$ of the porous medium, where $X(.)$ is the orbit containing X_0, then it is also positive on the part of this orbit below X_0. This important property is then used to define the free boundary. This is done by introducing two C^1-diffeomorphisms related to the above ordinary differential equation. Then we show that the free boundary is represented locally by graphs of continuous functions. As a consequence we obtain an expression of $g = 1-\chi$ in terms of the characteristic function of the unsaturated part $[p = 0]$, β, $\mathcal{A}(., e)$, $X(.)$ and the outward unit normal vector to the reservoirs bottoms which extends expressions obtained in [14] and [23].

1 Statement of the problem and preliminary results

A porous medium that we denote by Ω is supplied by N_0 reservoirs filled with a fluid which infiltrates through Ω. We assume that Ω is a bounded C^1 domain of $\mathbb{R}^2$ with boundary $\partial\Omega = S_1 \cup S_2 \cup S_3$, where S_1 is the impervious part, S_2 is the part in contact with air and $S_3 = \bigcup_{i=1}^{i=N_0} S_{3,i}$ with $S_{3,i}$ $(i = 1, ..., N_0)$ the part in contact with the bottom of the i^{th} reservoir. We assume that the flow in Ω has reached a steady state and we look for the fluid pressure p and the saturated region S of the porous medium. The boundary ∂S of S is divided into four parts (see Figure 1)

- $\Gamma_1 \subset S_1$: the impervious part,
- $\Gamma_2 \subset \Omega$: the free boundary,
- $\Gamma_3 \subset S_3$: the part covered by fluid,
- $\Gamma_4 \subset S_2$: the part where the fluid flows outside Ω.

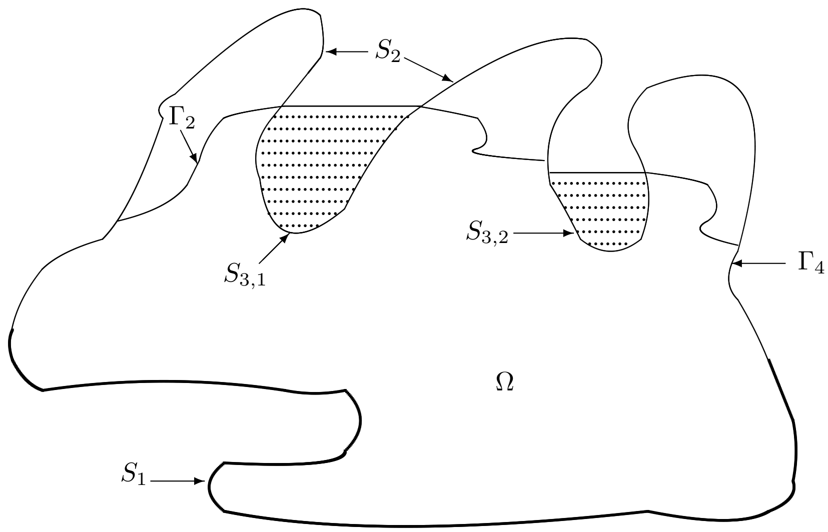

Figure 1

The flow is governed by the following nonlinear Darcy law

$$v = -\mathcal{A}(X, \nabla(p+y)) = -\mathcal{A}(X, \nabla u) \tag{1.1}$$

where $u = p+y$ and $\mathcal{A} : \Omega \times I\!R^2 \longrightarrow I\!R^2$ is a mapping that satisfies the following assumptions for some constants $q > 1$ and $0 < c_0 \le c_1 < \infty$

$$\left\{ \begin{array}{l} i)\ X \longmapsto \mathcal{A}(X,\xi) \text{ is measurable } \forall \xi \in I\!R^2, \\ ii)\ \xi \longmapsto \mathcal{A}(X,\xi) \text{ is continuous for a.e } X \in \Omega, \\ iii) \text{ for all } \xi \in I\!R^2 \text{ and for a.e. } X \in \Omega, \\ \qquad \mathcal{A}(X,\xi).\xi \ge c_0|\xi|^q \quad \text{and} \quad |\mathcal{A}(X,\xi)| \le c_1|\xi|^{q-1}, \\ iv) \text{ for all } \xi, \zeta \in I\!R^2 \text{ and for a.e. } X \in \Omega, \\ \qquad (\mathcal{A}(X,\xi) - \mathcal{A}(X,\zeta)).(\xi - \zeta) \ge 0. \end{array} \right. \tag{1.2}$$

Moreover we have the following boundary conditions

$$\left\{ \begin{array}{l} v.\nu = 0 \text{ on } \Gamma_1, \\ p = 0 \quad \text{and} \quad v.\nu = 0 \quad \text{on } \Gamma_2, \\ v.\nu = -\beta(X, \varphi - p) \text{ on } S_3, \\ v.\nu \ge 0 \text{ on } \Gamma_4 \end{array} \right. \tag{1.3}$$

where φ is a nonnegative Lipschitz continuous function in $\bar{\Omega}$ representing the exterior fluid pressure on S_3 and β a function on $S_3 \times \mathbb{R}$ with values in $\mathbb{R}$ such that

$$\begin{cases} X \mapsto \beta(X,u) \quad \text{is measurable } \forall u \in \mathbb{R}, \\ X \mapsto \beta(X,u) \quad \text{is nondecreasing a.e. } X \in S_3, \\ \beta(X,0) = 0 \quad \text{a.e. } X \in S_3. \end{cases} \tag{1.4}$$

The third condition in (1.3) is called a leaky boundary condition as it describes the phenomenon of partial filtration or leak through S_3 in opposition to the Dirichlet boundary condition that models total filtration.
Assuming the flow to be incompressible and taking into account (1.1) and (1.3), we are led (see [15]) to the following problem

$$(P) \begin{cases} \text{Find } (u,g) \in W^{1,q}(\Omega) \times L^\infty(\Omega) \text{ such that } : \\ (i) \quad u \geq y, \quad 0 \leq g \leq 1, \quad g(u-y) = 0 \ \text{ a.e. in } \Omega, \\ (ii) \quad u = \varphi + y = \psi \quad \text{on } S_2 \cup S_3, \\ (iii) \quad \displaystyle\int_\Omega \big(\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)\big).\nabla\xi dX \ \leq \int_{S_3} \beta(X,\psi - u)\xi d\sigma \\ \qquad\qquad\qquad\qquad\qquad \forall \xi \in W^{1,q}(\Omega), \quad \xi \geq 0 \text{ on } S_2. \end{cases}$$

Under assumptions (1.2) and (1.4), there exists a solution for problem (P) (see [22]). We recall also the following properties of the solution (see [22]. For the regularity result, the reader can consult the references [17] [24].

Proposition 1.1. *Let (u,g) be a solution of (P). Then we have*
i)

$$div\big(\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)\big) = 0 \qquad in\ \mathcal{D}'(\Omega). \tag{1.5}$$

ii) If $div\big(\mathcal{A}(X,e)\big) \geq 0$ in $\mathcal{D}'(\Omega)$, then

$$div\big(\mathcal{A}(X,\nabla u)\big) = div\big(g\mathcal{A}(X,e)\big) \geq 0 \qquad in\ \mathcal{D}'(\Omega). \tag{1.6}$$

iii) $u \in C^{0,\alpha}(\Omega \cup S_2)$ for some $\alpha \in (0,1)$ and $[u > y]$ is an open set.

iv) $div\big(\mathcal{A}(X,\nabla u)\big) = 0$ in $\mathcal{D}'([u > y])$ i.e. u is $\mathcal{A}$-harmonic in $[u > y]$.

In what follows, we make the following assumption on $\mathcal{A}$ and Ω

$$\partial\Omega \quad \text{is of class} \quad \mathcal{C}^1 \tag{1.7}$$

$$\mathcal{A}(.,e) = (a^1(.), a^2(.)) \in \mathcal{C}^1(\bar{\Omega}) \tag{1.8}$$

$$div\big(\mathcal{A}(X,e)\big) \geq 0 \quad \text{in } \mathcal{C}^0(\Omega) \tag{1.9}$$

$$\mathcal{A}(.,e).\nu \neq 0 \quad \text{on } \partial\Omega \tag{1.10}$$

$$\sum_{1\le i,j\le 2}\frac{\partial \mathcal{A}_{i,j}}{\partial \zeta_j}(X,\zeta)\xi_i\xi_j \ge \lambda_0(\kappa+|\zeta|^{q-2})|\xi|^2 \tag{1.11}$$

$$|\frac{\partial \mathcal{A}_{i,j}}{\partial \zeta_j}(X,\zeta)| \le \lambda_1(\kappa+|\zeta|^{q-2}) \quad \forall i,j\in\{1,2\} \tag{1.12}$$

with $\lambda_0, \lambda_1, \kappa \in \mathbb{R}^+, \lambda_1 \ge \lambda_0$ and

$$\|\mathcal{A}(X,\zeta)-\mathcal{A}(Y,\zeta)\| \le \lambda_1(1+|\zeta|^{q-1})\|X-Y\|^\sigma, \qquad 0<\sigma<1. \tag{1.13}$$

Remark 1.1. *Under assumptions (1.11)-(1.13), one has (see [16]) $u \in C^{1,\gamma}_{loc}([u > y])$ for some $\gamma \in (0,1)$.*

Following [15] we consider the differential equation

$$\begin{cases} X'(t,\omega,h) = \mathcal{A}(X(t,\omega,h),e) \\ X(0,\omega,h) = (\omega,h) \end{cases} \tag{$E(\omega,h)$}$$

where $h \in \pi_y(\Omega)$, $\omega \in \pi_x(\Omega \cap [y=h])$, π_x and π_y are respectively the orthogonal projection on the x and y axes. By the classical theory of ordinary differential equations there exists a unique maximal solution $X(.,\omega,h)$ of $E(\omega,h)$ which is defined on $[\alpha_-(\omega,h),\alpha_+(\omega,h)]$ with $X(\alpha_-(\omega,h),\omega,h) \in \partial\Omega \cap ([y < h]), X(\alpha_+(\omega,h),\omega,h) \in \partial\Omega \cap ([y > h])$. In the sequel we will denote, for simplicity, $X(t,\omega,h)$, $\alpha_-(\omega,h)$ and $\alpha_+(\omega,h)$ respectively by $X(t,\omega)$, $\alpha_-(\omega)$ and $\alpha_+(\omega)$. We note that (1.10) means that the orbits of $E(\omega,h)$ do not cross $\partial\Omega$ tangentially. Moreover under the assumptions (1.7)-(1.8) and (1.10), one can prove (see [15])

Proposition 1.2. $\alpha_-, \alpha_+ \in C^1(\pi_x(\Omega\cap[y=h]))$.

For each $h \in \pi_y(\Omega)$ we define the set

$$D_h = \{(t,\omega) \,/\, \omega \in \pi_x(\Omega\cap[y=h]),\, t\in(\alpha_-(\omega),\alpha_+(\omega))\}$$

and consider the mappings T_h and S_h defined by

$$\begin{array}{rcl} T_h : D_h & \longrightarrow & T_h(D_h) \\ (t,\omega) & \longmapsto & T_h(t,\omega) = X(t,\omega) = (T_h^1, T_h^2)(t,\omega) \end{array}$$

$$S_h : D_h \quad \longrightarrow \quad S_h(D_h)$$

$$(t,\omega) \quad \longmapsto \quad S_h(t,\omega) = (\omega, L_h(t,\omega)) = (\omega,\tau)$$

where $L_h(t,\omega) = \displaystyle\int_{\alpha_-(\omega)}^{t} |\mathcal{A}(X(s,\omega),e)|ds = \int_{\alpha_-(\omega)}^{t} |X'(s,\omega)|ds$ represents the arc length of the curve $X(.,\omega)$ from the point $X(\alpha_-(\omega),\omega)$ to the point $X(t,\omega)$. Then we obtain as in [15]

Proposition 1.3.

$$\Omega = \bigsqcup_{h\in\pi_y(\Omega)} T_h(D_h), \quad T_h \quad \textit{and} \quad S_h \textit{ are } C^1 \textit{ diffeomorphisms,}$$
$$Y_h(t,\omega) = det(\mathcal{J}T_h) = -a^2(X(0,\omega))\exp\Big(div\big(\mathcal{A}(.,e)\big)\Big),$$
$$det\mathcal{J}S_h = -|\mathcal{A}(X(t,\omega),e)| < 0,$$

where we denote by $\mathcal{J}F$ *the jacobian matrix of the transformation* F *and by* $det(\mathcal{J}F)$ *the determinant of* $\mathcal{J}F$.

Remark 1.2. *We remark that far from* S_3*, our problem behaves as the one with Dirichlet condition. Therefore the two problems share many properties which are often given here without proof and the reader is refereed to [15]. The most important one is the monotonicity property of* g.

Theorem 1.1. *Let* (u,g) *be a solution of* (P)*. We have for each* $h\in\pi_y(\Omega)$

$$\frac{\partial}{\partial\tau}\big(\widetilde{g}_h.(-Y_h o S_h^{-1})\big) \geq 0 \qquad \textit{in } \mathcal{D}'(S_h(D_h))$$

where $\widetilde{g}_h = goT_hoS_h^{-1}$.

Remark 1.3. *In order to avoid complicated notations we will use in the sequel* $\widetilde{f}_h$ *to denote the function* $foT_hoS_h^{-1}$ *for any function* f *defined on* $T_h(D_h)$*. The functions* $T_hoS_h^{-1}$ *and* $-Y_hoS_h^{-1}$ *will be denoted by* $\mathcal{T}_h$ *and* $\mathcal{Y}_h$ *respectively.*

Theorem 1.2. *Let* (u,g) *be a solution of* (P) *and* $X_0 = \mathcal{T}_h(\omega_0,\tau_0) = (x_0,y_0) \in \Omega$.
i) If $p(X_0) = \widetilde{p}_h(\omega_0,\tau_0) > 0$*, then there exists* $\epsilon > 0$ *such that*

$$\widetilde{p}_h(\omega,\tau) > 0 \qquad \forall(\omega,\tau) \in C_\epsilon = \{(\omega,\tau)\in S_h(D_h)\,/\,|\omega-\omega_0|<\epsilon,\ \tau<\tau_0+\epsilon\}$$

ii) If $p(X_0) = \widetilde{p}_h(\omega_0,\tau_0) = 0$*, then* $\quad \widetilde{p}_h(\omega_0,\tau) = 0 \qquad \forall\tau\geq\tau_0$.

Remark 1.4. *The result of Theorem 1.2 means that if a point X_0 belongs to the saturated region S, then the part of the curve $X(.,\omega)$ passing through X_0 at t_0 is also in S for all $t \leq t_0$. This allows us to define the free boundary $\partial([u > y]) \cap \Omega$ locally. Indeed for each $h \in \pi_y(\Omega)$ we define the function Φ_h on $\pi_\omega(S_h(D_h))$ by*

$$\Phi_h(\omega) = \begin{cases} \sup\{\tau / (\omega,\tau) \in S_h(D_h), \quad \widetilde{p}_h(\omega,\tau) > 0\} \\ \qquad\qquad\qquad \textit{if this set is not empty} \\ 0 \qquad\qquad\quad \textit{elsewhere.} \end{cases} \tag{1.14}$$

Φ_h is well defined and we have

Proposition 1.4. *Φ_h is lower semi-continuous on $\pi_\omega(S_h(D_h))$. Moreover*

$$[\widetilde{p}_h(\omega,\tau) > 0] = [\tau < \Phi_h(\omega)].$$

The following lemma plays an important role in the continuity proof of the free boundary. The proof can be obtained by combining the proofs of Lemma 4.1 in [15] and Theorem 2.2 in [23]. We assume that the function $X \mapsto \beta(X,\varphi(X))$ extends to S_2 so that $\beta(X,\varphi(X)) = 0$ a.e. $X \in S_2$.

Lemma 1.1. *Let (u,g) be a solution of (P). Let $(\omega_1,\tau_0),(\omega_2,\tau_0) \in S_h(D_h)$ with $\omega_1 < \omega_2$ and*

$$\widetilde{p}_h(\omega_i,\tau) = 0 \qquad \forall(\omega_i,\tau) \in S_h(D_h), \quad \tau > \tau_0 \geq 0.$$

Set $Z_{\tau_0} = \mathcal{T}_h\Big(\big((\omega_1,\omega_2)\times(\tau_0,+\infty)\big) \cap S_h(D_h)\Big)$.
Let $y_0 \in I\!R$ such that $D_{y_0,\tau_0} = [y > y_0] \cap Z_{\tau_0} \neq \emptyset$. Then we have

$$\int_{D_{y_0,\tau_0}} \big(\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)\big).ef(y)dX \leq \int_{D_{y_0,\tau_0}} \gamma_h \mathcal{A}(X,e).ef(y)dX$$

for any nonnegative function $f \in W^{1,q}(\Omega)$ depending only on y and where γ_h is defined by

$$\gamma_h(X(t,\omega)) = \frac{Y_h(\alpha_+(\omega),\omega)}{Y_h(t,\omega)}.\Big(\frac{\beta(.,\varphi(.))}{|\mathcal{A}(.,e).\nu(.)|}\Big)(X(\alpha_+(\omega),\omega)) \qquad a.e.\ (t,\omega) \in D_h$$

where ν is the outward unit normal to S_3.

2 Characterization of g in the unsaturated region

Theorem 2.1. *Assume that $(X,u) \mapsto \beta(X,u)$ is bounded. Let (u,g) be a solution of (P) and let $X_0 = (x_0,y_0) = \mathcal{T}_h(\omega_0,\tau_0)$ be a point in Ω, $(\omega_0,\tau_0) \in S_h(D_h)$. We denote by $B_r(\omega_0,\tau_0)$ a ball with center (ω_0,τ_0) and radius r contained in $S_h(D_h)$.*
If $\widetilde{p}_h = 0$ in $B_r(\omega_0,\tau_0)$ then we have

$$\begin{aligned} &\widetilde{p}_h = 0 \quad in \quad C_r = \{(\omega,\tau) \in S_h(D_h), |\omega-\omega_0| < r, \tau > \tau_0\} \cup B_r(\omega_0,\tau_0), \\ &\widetilde{g}_h = 1 - \widetilde{\gamma}_h \quad a.e.\ in \quad C_r. \end{aligned}$$

Proof. By Theorem 1.2 ii), we have $\widetilde{p}_h = 0$ in C_r. Moreover without loss of generality, we can assume that $Z_{\tau_0} = \mathcal{T}_h\Big(\big((\omega_0 - r, \omega_0 + r) \times (\tau_0, +\infty)\big) \cap S_h(D_h)\Big)$ is such that $\bar{Z}_{\tau_0} \cap S_3 = \emptyset$ or $\bar{Z}_{\tau_0} \cap S_2 = \emptyset$ and we will therefore consider these two cases only.

i) $\bar{Z}_{\tau_0} \cap S_3 = \emptyset$:

Applying Lemma 1.2 with domains $D_{\tau_1} = \mathcal{T}_h\Big(\big((\omega_1,\omega_2) \times (\tau_1, +\infty)\big) \cap S_h(D_h)\Big)$, we obtain

$$\int_{[y>y_1]\cap D_{\tau_1}} (1-g)\mathcal{A}(X,e).e dX \le 0 \qquad \forall y_1 \in I\!R \text{ such that } [y>y_1] \cap D_{\tau_1} \neq \emptyset.$$

So by (1.2)$iii)$ $g = 1$ a.e. in Z_{τ_0}. Since this holds for all domains Z_{τ_1} in C_r, we get $g = 1$ a.e. in C_r.

ii)$\bar{Z}_{\tau_0} \cap S_2 = \emptyset$:

By (1.5), we have for $\zeta \in \mathcal{D}(\mathcal{T}_h(C_r))$

$$\int_{\mathcal{T}_h(C_r)} (1-g)\mathcal{A}(X,e)\nabla\zeta dX = 0.$$

Using the change of variables T_h, we get

$$\int_{S_h^{-1}(C_r)} (1 - goT_h)(-Y_h(t,\omega))\frac{\partial(\zeta o T_h)}{\partial t} dt d\omega = 0$$

which means that

$$\theta(\omega) = (1 - goT_h)(-Y_h(t,\omega)) \in L^\infty(S_h^{-1}(C_r)) \tag{2.1}$$

does not depend on t.
Now, let $\xi \in W^{1,q}(S_h^{-1}(C_r))$ such that $\xi = 0$ on $\partial(S_h^{-1}(C_r)) \cap D_h$. Then $\xi o T_h^{-1} \in W^{1,q}(\mathcal{T}_h(C_r))$ and $\xi o T_h^{-1} = 0$ on $\partial(\mathcal{T}_h(C_r)) \cap \Omega$. So $\chi(\mathcal{T}_h(C_r))\xi o T_h^{-1}$ is a test function for (P) and we have

$$\int_{\mathcal{T}_h(C_r)} (1-g)\mathcal{A}(X,e).\nabla(\xi o T_h^{-1}) = \int_{S_3 \cap \partial(\mathcal{T}_h(C_r))} \beta(X,\varphi)\xi o T_h^{-1} d\sigma. \tag{2.2}$$

Using the change of variables T_h and taking into account (2.1), the left hand side of (2.2) becomes

$$\int_{\mathcal{T}_h(C_r)} (1-g)\mathcal{A}(X,e).\nabla(\xi o T_h^{-1})dX = \int_{S_h^{-1}(C_r)} (1-goT_h)(-Y_h(t,\omega))\frac{\partial \xi}{\partial t}dtd\omega$$
$$= \int_{S_h^{-1}(C_r)} \theta(\omega)\frac{\partial \xi}{\partial t}dtd\omega = \int_{\partial(S_h^{-1}(C_r))} \theta(\omega)\xi\nu_t d\sigma^*$$
$$= \int_{T_h^{-1}(S_3)\cap\partial(S_h^{-1}(C_r))} \theta(\omega)\xi\nu_t d\sigma^*. \tag{2.3}$$

Note that we have

$$T_h^{-1}(S_3) \cap \partial(S_h^{-1}(C_r)) = \{(\alpha_+(\omega),\omega),\ \omega \in (\omega_0 - r, \omega_0 + r)\}.$$

Then the outward unit normal to $T_h^{-1}(S_3)\cap\partial(S_h^{-1}(C_r))$ is given by $\nu = \dfrac{(1,-\alpha'_+(\omega))}{\sqrt{1+\alpha'^2_+(\omega)}}$,

$d\sigma^* = \sqrt{1+\alpha'^2_+(\omega)}d\omega$ and we have

$$\int_{T_h^{-1}(S_3)\cap\partial(S_h^{-1}(C_r))} \theta(\omega)\xi\nu_t d\sigma^* = \int_{\omega_0-r}^{\omega_0+r} \theta(\omega)\xi(\alpha_+(\omega),\omega)d\omega. \tag{2.4}$$

Shrinking if necessary, one can assume by (1.7) that there exists a C^1 function σ such that one of the following situations holds

$$\begin{array}{ll} i) & \sigma(X_1(\alpha_+(\omega),\omega)) = X_2(\alpha_+(\omega),\omega) \quad \forall \omega \in (\omega_0 - r, \omega_0 + r), \\ ii) & \sigma(X_2(\alpha_+(\omega),\omega)) = X_1(\alpha_+(\omega),\omega) \quad \forall \omega \in (\omega_0 - r, \omega_0 + r). \end{array}$$

Assume for example that $i)$ holds. The case $ii)$ can be treated similarly.
Since $x \mapsto (x,\sigma(x))$ is a parameterization of $S_3 \cap \partial(\mathcal{T}_h(C_r))$, the right hand side of (2.2) denoted by J, is given by

$$J = \int_{\pi_x(S_3\cap\partial(\mathcal{T}_h(C_r)))} \big(\beta(.,\varphi(.))\xi o T_h^{-1}(.)\big)(x,\sigma(x))\sqrt{1+\sigma'^2(x)}dx \tag{2.5}$$

Now we have $(x,\sigma(x)) = T_h(\alpha_+(\omega),\omega)$ and for $\omega \in (\omega_0 - r, \omega_0 + r)$, set $f(\omega) = x = T_h^1(\alpha_+(\omega),\omega)$. Then f is a C^1 function and $f'(\omega) = \alpha'_+(\omega)a^1(X(\alpha_+(\omega),\omega)) + \dfrac{\partial X_1}{\partial \omega}$. Arguing as in [15], we can deduce by the implicit function theorem that

$$\alpha'_+(\omega) = \Big(\frac{\sigma' o T_h^1.\partial X_1/\partial\omega - \partial X_2/\partial\omega}{a^2 o T_h - \sigma' o T_h^1.a^1 o T_h}\Big)(\alpha_+(\omega),\omega).$$

Hence

$$f'(\omega) = \Big(\frac{Y_h}{a^2 o T_h - \sigma' o T_h^1.a^1 o T_h}\Big)(\alpha_+(\omega),\omega) = \frac{Y_h(\alpha_+(\omega),\omega).(1+\sigma'^2(x))^{-1/2}}{\mathcal{A}(X(\alpha_+(\omega),\omega),e).\nu(X(\alpha_+(\omega),\omega))} \neq 0$$

where $\nu(X(\alpha_+(\omega),\omega))$ denotes the outward unit normal to S_3 given by $\nu(X) = \dfrac{(-\sigma'(x),1)}{\sqrt{1+\sigma'^2(x)}}$.
Therefore one can use the change of variable f in (2.5) to show that

$$J = \int_{\omega_0-r}^{\omega_0+r} \Big(\frac{\beta(.,\varphi(.))}{|\mathcal{A}(.,e).\nu(.)|}\Big)(X(\alpha_+(\omega),\omega)).|Y_h(\alpha_+(\omega),\omega)|\xi(\alpha_+(\omega),\omega))d\omega \quad (2.6)$$

Using (2.2)-(2.6), we deduce that

$$\theta(\omega) = -\Big(\frac{\beta(.,\varphi(.))}{|\mathcal{A}(.,e).\nu(.)|}\Big)(X(\alpha_+(\omega),\omega)).Y_h(\alpha_+(\omega),\omega) \quad \text{ a.e. } \omega\in I$$

and the expression of g holds. □

Remark 2.1. *i) When $a^1 = 0$, we have $\mathcal{A}(X(\alpha_+(\omega),\omega),e).\nu = a^2(X(\alpha_+(\omega),\omega))\nu_2$, $Y(t,\omega) = -a^2(X(t,\omega))$. Therefore $goT_h(t,\omega) = 1 - \dfrac{\beta(X(\alpha_+(\omega),\omega),\varphi(X(\alpha_+(\omega),\omega)))}{a^2(X(t,\omega)).\nu_2}$ a.e. in C_r, which is the result established by the second author in [23].*

ii) Since $\partial Y_h o S_h^{-1}/\partial\tau \leq 0$, we deduce that if $\widetilde{p}_h = 0$ in D, then $\widetilde{g}_h \leq 1 - \Big(\beta(.,\varphi(.))/|\mathcal{A}(.,e).\nu(.)|\Big)$ and then we have necessarily $\beta(.,\varphi(.))/|\mathcal{A}(.,e).\nu(.)| \leq 1$ in D.

The following result is a sort of a maximum principle.

Theorem 2.2. *Let (u,g) be a solution of (P), $X_0 = (x_0,y_0) = \mathcal{T}_h(\omega_0,\tau_0)$ be a point of Ω and B_r be the open ball in $S_h(D_h)$ with center (ω_0,τ_0) and radius r. Then we cannot have the following situations*

$$(i) \quad \begin{cases} \widetilde{p}_h(\omega_0,\tau) = 0 & \forall\tau\in(\tau_0-r,\tau_0+r) \\ \widetilde{p}_h(\omega,\tau) > 0 & \forall(\omega,\tau)\in B_r\setminus S_{eg}, \qquad S_{eg} = \{\omega_0\}\times(\tau_0-r,\tau_0+r), \end{cases}$$

$$(ii) \quad \begin{cases} \widetilde{p}_h(\omega,\tau) = 0 & \forall(\omega,\tau)\in B_r\cap[\omega\leq\omega_0] \\ \widetilde{p}_h(\omega,\tau) > 0 & \forall(\omega,\tau)\in B_r\cap[\omega>\omega_0], \end{cases}$$

$$(iii) \quad \begin{cases} \widetilde{p}_h(\omega,\tau) = 0 & \forall(\omega,\tau)\in B_r\cap[\omega\geq\omega_0] \\ \widetilde{p}_h(\omega,\tau) > 0 & \forall(\omega,\tau)\in B_r\cap[\omega<\omega_0]. \end{cases}$$

Proof. *i*) Since $u > y$ a.e. in $\mathcal{T}_h(B_r)$, we have $g = 0$ a.e. in $\mathcal{T}_h(B_r)$. Then u is $\mathcal{A}$-harmonic in $\mathcal{T}_h(B_r)$ and we get a contradiction with the strong maximum principle (see [23]).

ii) Let $\xi \in \mathcal{D}(\mathcal{T}_h(B_r))$, $\xi \geq 0$. Since $\pm\xi$ are test functions for (P) and g is given explicitly in $B_r \cap [\omega \leq \omega_0]$ (see Theorem 2.1), we get by setting $\tau^*(\omega) = \int_{\alpha_-(\omega)}^{\alpha_+(\omega)} |\mathcal{A}(X(s,\omega),e)|ds$,

$$\begin{aligned}
&\int_{\mathcal{T}_h(B_r)} \mathcal{A}(X,\nabla u).\nabla\xi dX = \int_{\mathcal{T}_h(B_r)} g\mathcal{A}(X,e).\nabla\xi dX \\
&= \int_{B_r\cap[\omega\leq\omega_0]} (1-\widetilde{\gamma}_h)(\omega,\tau).(-\mathcal{Y}_h)(\omega,\tau).\frac{\partial\widetilde{\xi}_h}{\partial\tau} d\omega d\tau \\
&= \int_{B_r\cap[\omega\leq\omega_0]} -\mathcal{Y}_h(\omega,\tau)\frac{\partial\widetilde{\xi}_h}{\partial\tau} d\omega d\tau \\
&+ \int_{B_r\cap[\omega\leq\omega_0]} \mathcal{Y}_h(\omega,\tau^*(\omega))\Big(\frac{\beta(.,\varphi(.))}{\mathcal{A}(.,e).\nu(.)}\Big)(\mathcal{T}_h(\omega,\tau^*(\omega)))\frac{\partial\widetilde{\xi}_h}{\partial\tau} d\omega d\tau \\
&= \int_{B_r} -\mathcal{Y}_h(\omega,\tau)\frac{\partial\widetilde{\xi}_h}{\partial\tau} d\omega d\tau + \int_{B_r\cap[\omega>\omega_0]} \mathcal{Y}_h(\omega,\tau)\frac{\partial\widetilde{\xi}_h}{\partial\tau} d\omega d\tau \\
&+ \int_{B_r\cap[\omega\leq\omega_0]} \mathcal{Y}_h(\omega,\tau^*(\omega))\Big(\frac{\beta(.,\varphi(.))}{\mathcal{A}(.,e).\nu(.)}\Big)(\mathcal{T}_h(\omega,\tau^*(\omega)))\frac{\partial\widetilde{\xi}_h}{\partial\tau} d\omega d\tau \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where we have

$$\begin{aligned}
I_1 &= \int_{\mathcal{T}_h(B_r)} \mathcal{A}(X,e)\nabla\xi dX \\
I_2 &= \int_{B_r\cap[\omega>\omega_0]} -\frac{\partial\mathcal{Y}_h}{\partial\tau}\widetilde{\xi}_h + \int_{S_{eg}} \mathcal{Y}_h\widetilde{\xi}_h\nu_\tau \geq 0 \\
&\qquad \text{since } \nu_\tau = 0 \text{ and } \frac{\partial\mathcal{Y}_h}{\partial\tau} < 0 \\
I_3 &= 0 \text{ since } \frac{\beta(\mathcal{T}_h(\omega,\tau^*(\omega)),\varphi o\mathcal{T}_h(\omega,\tau^*(\omega)))}{\mathcal{A}(\mathcal{T}_h(\omega,\tau^*(\omega)),e).\nu(\mathcal{T}_h(\omega,\tau^*(\omega)))} \text{ does not depend on } \tau
\end{aligned}$$

and $\nu_\tau = 0$ on S_{eg}.
It follows that

$$\int_{\mathcal{T}_h(B_r)} \big(\mathcal{A}(X,\nabla u) - \mathcal{A}(X,\nabla y)\big)\nabla\xi dX \geq 0.$$

Since $u \geq y$ in $\mathcal{T}_h(B_r)$, $u = y$ in $\mathcal{T}_h(B_r \cap [\omega \leq \omega_0])$, $\nabla y = e \neq 0$, we deduce by the strong maximum principle that $u = y$ in $\mathcal{T}_h(B_r)$ which contradicts the fact that $u > y$ in $\mathcal{T}_h(B_r \cap [\omega > \omega_0])$.

iii) We argue as in ii). □

3 An Upper Bound of g

In what follows, we assume that

$$\mathcal{A}(X,e).\nu > 0 \quad \text{on } \partial\Omega. \tag{3.1}$$

The main result of this section is the following theorem that gives a sharp upper bound for g which is reached in the unsaturated region. A similar result was proved in [23] in the case $a_1 = 0$ and $a_2(X) = a_2(y)$.

Theorem 3.1. *Let (u, g) be a solution of (P). Then we have*

$$0 \le \widetilde{g}_h \le \widetilde{\mu}_h = (1-\widetilde{\gamma}_h)^+ \qquad a.e.\ in\ S_h(D_h), \quad h \in \pi_y(\Omega). \tag{3.2}$$

To prove this Theorem, we need the following lemma

Lemma 3.1. *Let I be an interval contained in $\pi_\omega(T_h^{-1}(S_3))$. Then we have*

$$\begin{aligned}&\int_{(\mathbb{R}\times I)\cap D_h} (goT_h - \mu_h oT_h)^+(-Y_h(t,\omega))\frac{\partial \xi}{\partial t}dtd\omega\\ &= \int_{T_h((\mathbb{R}\times I)\cap D_h)} (g-\mu_h)^+\mathcal{A}(X,e).\nabla(\xi oT_h)dX \le 0\\ &\qquad \forall \xi \in \mathcal{D}((\mathbb{R}\times I)\setminus T_h^{-1}(S_1)), \xi \ge 0.\end{aligned}$$

Proof. Set $B = (\mathbb{R}\times I)\setminus T_h^{-1}(S_1)$ and let $\xi \in \mathcal{D}(B)$ with $\xi \ge 0$. For $\epsilon > 0$, $\pm(H_\epsilon(u-y)-1)\xi oT_h^{-1}.\chi(T_h(B)\cap\Omega)$ are test functions for (P) and we have with $S_3^B = S_3 \cap T_h(B)$

$$\begin{aligned}&\int_{T_h(B)\cap\Omega} (\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)).\nabla\big((H_\epsilon(u-y)-1)\xi oT_h^{-1}\big)dX\\ &= \int_{S_3^B} \beta(X,\psi-u)(H_\epsilon(u-y)-1)\xi oT_h^{-1}d\sigma\end{aligned}$$

from which we deduce

$$\begin{aligned}&\int_{T_h(B)\cap\Omega} g\mathcal{A}(X,e).\nabla(\xi oT_h^{-1})dX\\ &= \int_{T_h(B)\cap\Omega} (1-H_\epsilon(u-y))\big(\mathcal{A}(X,\nabla u)-\mathcal{A}(X,e)\big).\nabla(\xi oT_h^{-1})dX\\ &+ \int_{T_h(B)\cap\Omega} (1-H_\epsilon(u-y))\mathcal{A}(X,e).\nabla(\xi oT_h^{-1})dX\end{aligned}$$

$$
\begin{aligned}
&- \int_{T_h(B)\cap\Omega} \xi o T_h^{-1} H'_\epsilon(u-y)\mathcal{A}(X,\nabla u).\nabla(u-y)dX \\
&+ \int_{T_h(B)\cap\Omega} g\mathcal{A}(X,e).\nabla(H_\epsilon(u-y)\xi o T_h^{-1})dX \\
&+ \int_{S_3^B} \beta(X,\psi-u)(H_\epsilon(u-y)-1)\xi o T_h^{-1} d\sigma
\end{aligned} \tag{3.3}
$$

The second integral in the right hand side of (3.3) can be written as

$$
\begin{aligned}
&\int_{T_h(B)\cap\Omega} (1-H_\epsilon(u-y))\mathcal{A}(X,e).\nabla(\xi o T_h^{-1})dX \\
&= \int_{T_h(B)\cap\Omega} (H_\epsilon(u-y)-1)div(\mathcal{A}(X,e))\xi o T_h^{-1} dX \\
&+ \int_{T_h(B)\cap\Omega} \xi o T_h^{-1} H'_\epsilon(u-y)\mathcal{A}(X,e).\nabla(u-y)dX \\
&+ \int_{S_3^B} (1-H_\epsilon(u-y))\mathcal{A}(X,e).\nu\xi o T_h^{-1} d\sigma.
\end{aligned} \tag{3.4}
$$

Taking into account (3.4), the monotonicity of $\mathcal{A}$ and H_ϵ, we get

$$
\begin{aligned}
&\int_{T_h(B)\cap\Omega} g\mathcal{A}(X,e).\nabla(\xi o T_h^{-1})dX \\
&\leq \int_{T_h(B)\cap\Omega} (1-H_\epsilon(u-y))\big(\mathcal{A}(X,\nabla u)-\mathcal{A}(X,e)\big).\nabla(\xi o T_h^{-1})dX \\
&\quad + \int_{T_h(B)\cap\Omega} (H_\epsilon(u-y)-1)div(\mathcal{A}(X,e))\xi o T_h^{-1} dX \\
&\quad + \int_{S_3^B} (1-H_\epsilon(u-y))\big(\mathcal{A}(X,e).\nu-\beta(X,\psi-u)\big)\xi o T_h^{-1} d\sigma.
\end{aligned} \tag{3.5}
$$

Now set $\lambda_h(X) = \min(1,\gamma_h(X))$ and $\mu_h(X) = 1-\lambda_h(X)$. Then we have

$$
\begin{aligned}
\int_{T_h(B)\cap\Omega} -\mu_h\mathcal{A}(X,e).\nabla(\xi o T_h^{-1})dX &= \int_{T_h(B)\cap\Omega} div(\mathcal{A}(X,e)).\xi o T_h^{-1} dX \\
&- \int_{S_3^B} \mathcal{A}(X,e).\nu\xi o T_h^{-1} d\sigma + \int_{S_3^B} \lambda_h\mathcal{A}(X,e).\nu\xi o T_h^{-1} d\sigma \\
&+ \int_{B\cap D_h} \xi\frac{\partial}{\partial t}\big(\lambda_h o T_h.Y_h(t,\omega)\big)dtd\omega.
\end{aligned}
$$

Since $\dfrac{\partial}{\partial t}\big(\lambda_h o T_h.Y_h(t,\omega)\big) = \dfrac{\partial Y_h}{\partial t}\chi[\gamma_h o T_h > 1] \leq 0$, we get

$$\int_{T_h(B)\cap\Omega} -\mu_h \mathcal{A}(X,e).\nabla(\xi o T_h^{-1})dX \leq \int_{T_h(B)\cap\Omega} div(\mathcal{A}(X,e)).\xi o T_h^{-1}dX$$
$$+\int_{S_3^B}(\lambda_h(X)-1)\mathcal{A}(X,e).\nu\xi o T_h^{-1}d\sigma. \tag{3.6}$$

Adding (3.5) and (3.6), we get

$$\begin{aligned}
&\int_{T_h(B)\cap\Omega}(g-\mu_h)\mathcal{A}(X,e).\nabla(\xi o T_h^{-1})dX \\
&\leq \int_{T_h(B)\cap\Omega}(1-H_\epsilon(u-y))\big(\mathcal{A}(X,\nabla u)-\mathcal{A}(X,e)\big).\nabla(\xi o T_h^{-1})dX \\
&\quad+\int_{T_h(B)\cap\Omega} H_\epsilon(u-y)div(\mathcal{A}(X,e))\xi o T_h^{-1}dX \\
&\quad+\int_{S_3^B}\big[(1-H_\epsilon(u-y))\big(\mathcal{A}(X,e).\nu-\beta(X,\psi-u)\big)+(\lambda_h-1)\mathcal{A}(X,e).\nu\big]\xi o T_h^{-1}d\sigma.
\end{aligned} \tag{3.7}$$

The last term of (3.7) can be written as

$$\begin{aligned}
&\int_{S_3^B}(1-H_\epsilon(u-y))(\beta(X,\varphi)-\beta(X,\psi-u))\xi o T_h^{-1}d\sigma \\
&+\int_{S_3^B}[(1-H_\epsilon(u-y))\big(1-\frac{\beta(X,\varphi)}{\mathcal{A}(X,e).\nu}\big)+(\lambda_h-1)]\mathcal{A}(X,e).\nu\xi o T_h^{-1}d\sigma \\
&= J_1^\epsilon + J_2^\epsilon.
\end{aligned} \tag{3.8}$$

First, we have

$$\lim_{\epsilon\to 0} J_1^\epsilon = \int_{S_3^B} \chi[u=y](\beta(X,\varphi)-\beta(X,\psi-y))\xi d\sigma = 0. \tag{3.9}$$

Next, since on S_3^B we have $\lambda_h = \min\big(1, \beta(.,\varphi)/\mathcal{A}(X,e).\nu\big)$ and $\mathcal{A}(X,e).\nu > 0$, then

$$J_2^\epsilon \leq \int_{S_3^B} -H_\epsilon(u-y)(1-\lambda_h)\mathcal{A}(X,e).\nu\xi o T_h^{-1}d\sigma \leq 0.$$

Therefore

$$\begin{aligned}
&\int_{T_h(B)\cap\Omega}(g-\mu_h)\mathcal{A}(X,e).\nabla(\xi o T_h^{-1})dX \\
&\leq \int_{T_h(B)\cap\Omega}(1-H_\epsilon(u-y))\big(\mathcal{A}(X,\nabla u)-\mathcal{A}(X,e)\big).\nabla(\xi o T_h^{-1})dX
\end{aligned}$$

$$+\int_{T_h(B)\cap\Omega} H_\epsilon(u-y)div(\mathcal{A}(X,e))\xi o T_h^{-1} dX + J_1^\epsilon.$$

Letting $\epsilon \to 0$, we get $\forall \xi \in \mathcal{D}(B)$ with $\xi \geq 0$

$$\begin{aligned}\int_{B\cap D_h} (goT_h - \mu_h oT_h)(-Y_h(t,\omega))\frac{\partial\xi}{\partial t}dtd\omega &= \int_{T_h(B)\cap\Omega}(g-\mu_h)\mathcal{A}(X,e).\nabla(\xi o T_h^{-1})dX \\ \leq \int_{T_h(B)\cap\Omega}\chi([u>y])div(\mathcal{A}(X,e)).(\xi o T_h^{-1})dX \\ = \int_{B\cap D_h}\chi([poT_h>0])(-\frac{\partial Y_h}{\partial t})\xi dtd\omega. \end{aligned} \tag{3.10}$$

In what follows we denote by Y_h, $(goT_h - \mu_h oT_h)$ and Θ the extensions outside $B \cap D_h$ by 0 of Y_h, $(goT_h - \mu_h oT_h)$ and $\chi([u > y])(-\partial Y_h/\partial t)$ respectively. Set $\epsilon_0 = d(supp\xi, T_h^{-1}(S_1) \cup (\bar{B} \cap D_h))$ and let $\epsilon \in (0, \epsilon_0/2)$. For all $(t', \omega') \in B_\epsilon(0)$, the function $(t', \omega') \mapsto \xi(t+t', \omega+\omega')$ remains in $\mathcal{D}(B)$ and therefore we obtain by plugging it in (3.10)

$$\int_{\mathbb{R}^2}(goT_h - \mu_h oT_h)(-Y_h(t,\omega))\frac{\partial\xi}{\partial t}(t+t',\omega+\omega')dtd\omega \leq \int_{\mathbb{R}^2}\Theta(t,\omega)\xi(t+t',\omega+\omega')dtd\omega$$

from which we deduce

$$\begin{aligned}\int_{\mathbb{R}^2}\rho_\epsilon(t',\omega')\Big(\int_{\mathbb{R}^2}(\bar{G}-\bar{\mu})\frac{\partial\xi}{\partial t}(t+t',\omega+\omega')dtd\omega\Big)dt'd\omega' \\ \leq \int_{\mathbb{R}^2}\rho_\epsilon(t',\omega')\Big(\int_{\mathbb{R}^2}\Theta(t,\omega)\xi(t+t',\omega+\omega')dtd\omega\Big)dt'd\omega'\end{aligned}$$

where $\bar{G} = goT_h.(-Y_h(t,\omega))$, $\bar{\mu} = \mu_h oT_h.(-Y_h(t,\omega))$ and ρ_ϵ a smooth function satisfying $\rho_\epsilon \geq 0$, $supp\rho_\epsilon \subset B_\epsilon(0)$ and $\int_{\mathbb{R}^2}\rho_\epsilon = 1$.
Setting $\bar{G}_\epsilon = \rho_\epsilon * \bar{G}$, $\bar{\mu}_\epsilon = \rho_\epsilon * \bar{\mu}$ and $\Theta_\epsilon = \rho_\epsilon * \Theta$, we get

$$\int_{\mathbb{R}^2}(\bar{G}_\epsilon - \bar{\mu}_\epsilon)\frac{\partial\xi}{\partial t}dtd\omega \leq \int_{\mathbb{R}^2}\Theta_\epsilon\xi dtd\omega \qquad \forall\xi \in \mathcal{D}(B),\ \xi \geq 0.$$

By density, the last inequality remains valid for $\xi \in H_0^1(B), \xi \geq 0$ with $supp\xi \subset B$. Therefore it is in particular satisfied by the function $\min\big(1, \frac{(\bar{G}_\epsilon - \bar{\mu}_\epsilon)^+}{\delta}\big)\xi$, with $\delta > 0$, $\xi \in \mathcal{D}(B)$, $\xi \geq 0$ and we have

$$\int_{\mathbb{R}^2}(\bar{G}_\epsilon - \bar{\mu}_\epsilon)\frac{\partial}{\partial t}\Big(\min\big(1, \frac{(\bar{G}_\epsilon - \bar{\mu}_\epsilon)^+}{\delta}\big)\xi\Big)dtd\omega \leq \int_{\mathbb{R}^2}\Theta_\epsilon\min\big(1, \frac{(\bar{G}_\epsilon - \bar{\mu}_\epsilon)^+}{\delta}\big)\xi dtd\omega.$$

Letting $\delta \to 0$, we get

$$* \quad \int_{\mathbb{R}^2} \min\big(1, \frac{(\bar{G}_\epsilon - \bar{\mu}_\epsilon)^+}{\delta}\big)(\bar{G}_\epsilon - \bar{\mu}_\epsilon)\frac{\partial \xi}{\partial t} \longrightarrow \int_{\mathbb{R}^2} (\bar{G}_\epsilon - \bar{\mu}_\epsilon)^+ \frac{\partial \xi}{\partial t}$$

$$* \quad \int_{\mathbb{R}^2} (\bar{G}_\epsilon - \bar{\mu}_\epsilon)\xi \frac{\partial}{\partial t}\Big(\min\big(1, \frac{(\bar{G}_\epsilon - \bar{\mu}_\epsilon)^+}{\delta}\big)\Big) = \int_{\mathbb{R}^2 \cap [0 < \bar{G}_\epsilon - \bar{\mu}_\epsilon < \delta]} \frac{\xi}{\delta}(\bar{G}_\epsilon - \bar{\mu}_\epsilon)\frac{\partial}{\partial t}(\bar{G}_\epsilon - \bar{\mu}_\epsilon)$$

$$= \int_{\mathbb{R}^2} \frac{\xi}{2\delta}\frac{\partial}{\partial t}\big(\min(\delta, (\bar{G}_\epsilon - \bar{\mu}_\epsilon)^+)\big)^2 = -\frac{1}{2\delta}\int_{\mathbb{R}^2} \big(\min(\delta, (\bar{G}_\epsilon - \bar{\mu}_\epsilon)^+\big)^2 \frac{\partial \xi}{\partial t} \to 0,$$

$$* \quad \int_{\mathbb{R}^2} \Theta_\epsilon \min\big(1, \frac{(\bar{G}_\epsilon - \bar{\mu}_\epsilon)^+}{\delta}\big)\xi \longrightarrow \int_{\mathbb{R}^2} \Theta_\epsilon \chi[\bar{G}_\epsilon - \bar{\mu}_\epsilon > 0]\xi.$$

It follows that

$$\int_{\mathbb{R}^2} (\bar{G}_\epsilon - \bar{\mu}_\epsilon)^+ \frac{\partial \xi}{\partial t} \leq \int_{\mathbb{R}^2} \Theta_\epsilon \chi[\bar{G}_\epsilon - \bar{\mu}_\epsilon > 0]\xi$$

which leads by letting $\epsilon \to 0$, to

$$\int_{B \cap D_h} (goT_h - \mu_h oT_h)^+ (-Y_h(t,\omega))\frac{\partial \xi}{\partial t} \leq \int_{B \cap D_h} \chi[poT_h > 0].\chi[goT_h \geq \mu_h oT_h]\big(-\frac{\partial Y_h}{\partial t}\big)\xi.$$

Now if we write the above inequality for $\xi = (1 - H_\epsilon(uoT_h - T_h^2))\zeta$, with $\zeta \in \mathcal{D}(B)$, $\zeta \geq 0$, we obtain since $g(u-y) = 0$

$$\int_{B \cap D_h} (goT_h - \mu_h oT_h)^+ (-Y_h(t,\omega))(1 - H_\epsilon(uoT_h - T_h^2))\frac{\partial \zeta}{\partial t}$$
$$\leq \int_{B \cap D_h} \chi[poT_h > 0].\chi[goT_h \geq \mu_h oT_h]\big(-\frac{\partial Y_h}{\partial t}\big)(1 - H_\epsilon(uoT_h - T_h^2))\zeta.$$

Letting $\epsilon \to 0$, the lemma follows. □

Proof of Theorem 3.1. First below S_2, the inequality reduces to $0 \leq g \leq \mu(X) = 1$ since $\beta(X,\varphi) = 0$ on S_2, which is satisfied of course. So let Z be a domain below $T_h^{-1}(S_{3,i})$ for $i \in \{1, ..., N\}$ such that $Z = ((\omega_1, \omega_2) \times (t_0, +\infty)) \cap D_h$ with $(\omega_1, \omega_2) \times \{t_0\} \subset D_h$. By Lemma 3.1, we have for $\xi = l(\omega)(t - t_0)\chi(Z)$ with $l(\omega) = (\omega - \omega_1)(\omega_2 - \omega)$

$$\int_Z (-Y_h(t,\omega))(goT_h - \mu_h oT_h)^+ l(\omega)dtd\omega \;=\; 0.$$

This leads to $(goT_h - \mu_h oT_h)^+ = 0$ a.e. in Z and $goT_h \leq \mu_h oT_h$ a.e. in Z. This is being true for all such domains Z, we have proved that $goT_h \leq \mu_h oT_h$ a.e. in D_h. □

Corollary 3.1. *Let $\triangle$ be a subset of $Int(S_3)$. If $\beta(.,\varphi(.)) \geq \mathcal{A}(.,e).\nu(.) > 0$ a.e. on $\triangle$, then the region $Z_\triangle$ of Ω located below $\triangle$, in the following sense,*

$$Z_\triangle = \cup_{h\in\pi_y(\Omega),T_h(\bar{D}_h)\cap S_1\neq\emptyset}\ T_h\Big(\big(\mathbb{R}\times\pi_\omega(T_h^{-1}(\triangle))\big)\cap D_h\Big)$$

is completely saturated.

Proof. By assumption, we have $\widetilde{\gamma}_h \geq 1$ on $\mathcal{T}_h^{-1}(\triangle)$ $\forall h \in \pi_y(\Omega)$. So $\widetilde{\gamma}_h \geq 1$ in $\big(\mathbb{R}\times\pi_\omega(T_h^{-1}(\triangle))\big)\cap D_h$ since $\partial\widetilde{\gamma}_h/\partial\tau \leq 0$. We deduce from Theorem 3.1 that $\widetilde{g}_h = 0$ a.e. in $S_h\Big(\big(\mathbb{R}\times\pi_\omega(T_h^{-1}(\triangle))\big)\cap D_h\Big)$. So $g = 0$ a.e. in $Z_\triangle$.
By Proposition 1.1 i), we get : $div(\mathcal{A}(X,\nabla u)) = 0$ in $\mathcal{D}'(Z_\triangle)$. By the strong maximum principle we deduce that $u > y$ or $u = y$ in $Z_\triangle$. Assume that $u = y$ in $Z_\triangle$ and let $\xi \in \mathcal{D}(\overline{Z}_\triangle \setminus \partial Z_\triangle \cap \Omega)$. Then $\pm\xi\chi(Z_\triangle)$ are test functions for (P) and we have

$$\int_{Z_\triangle} \mathcal{A}(X,e).\nabla\xi dX = \int_\triangle \beta(X,\varphi)\xi d\sigma$$

from which we deduce that $\mathcal{A}(X,e).\nu = 0$ on $S_1 \cap \partial Z_\triangle$. This contradicts the assumption (1.10). □

Remark 3.1. *Assume that $\beta \in C^0(S_3\times\mathbb{R})$. Let $h \in \pi_y(\Omega)$ such that $T_h(\bar{D}_h)\cap S_1 \neq \emptyset$. Then if $X_0 \in T_h(D_h)$ is such that $\gamma_h(X_0) > 1$, we have $u > y$ and thus $g = 0$ a.e. in a neighborhood of X_0.*
Indeed let $X_0 \in \Omega$ such that $\gamma_h(X_0) > 1$. Then by continuity, $\gamma_h > 1$ on a ball $B_r(X_0) \subset T_h(D_h)$. So by Theorem 3.1, $g = 0$ a.e. in $B_r(X_0)$. By the monotonicity of g, we deduce that $g = 0$ a.e. in the region

$$Z_B = \cup_{h\in\pi_y(\Omega),T_h(\bar{D}_h)\cap S_1\neq\emptyset}\ T_h\Big(\big(\mathbb{R}\times\pi_\omega(T_h^{-1}(B_r))\big)\cap D_h\Big).$$

Therefore u is $\mathcal{A}$-harmonic in Z_B and by the strong maximum principle, it follows that $u = y$ or $u > y$ in Z_B. Arguing as in the end of the Corollary 3.1 proof, we obtain that $u > y$ in $B_r(X_0)$.

4 Continuity of the Free Boundary

In this section we assume that $\mathcal{A}$ is strictly monotone in the following sense

$$\big(\mathcal{A}(X,\xi)-\mathcal{A}(X,\zeta)\big).(\xi-\zeta) > 0 \qquad \forall\xi\neq\zeta, \quad \forall X\in\Omega. \tag{4.1}$$

Then the main result of this section is the continuity of the free boundary. First, under S_2, the problem behaves as if we have Dirichlet boundary conditions. Therefore, we obtain by arguing as in [15]

Theorem 4.1. *Continuity under* S_2
For each $h \in \pi_y(\Omega)$ *the function* Φ_h *is continuous on* $\pi_\omega o S_h o T_h^{-1}(S_2)$.

Regarding the continuity of the free boundary under S_3, we need the following assumptions on the permeability and on β

$$\mathcal{A}(X, re) \;=\; r^{q-1}\mathcal{A}(X, e) \qquad \text{for all } X \in \Omega \text{ and } r \in \mathbb{R}^+. \tag{4.2}$$

$$\beta \in C^0(S_3 \times \mathbb{R}). \tag{4.3}$$

Under these assumptions we have

Theorem 4.2. *Continuity under* S_3
Let $h \in \pi_y(\Omega)$ *and* $\omega_0 \in Int\big(\pi_\omega o S_h o T_h^{-1}(S_3)\big)$ *such that*

$$(\omega_0, \tau_0) = (\omega_0, \Phi_h(\omega_0)) \in S_h(D_h), \qquad \widetilde{\gamma}_h(\omega_0, \tau_0) \in [0, 1).$$

Then Φ_h *is continuous at* ω_0.

It suffices to prove that Φ_h is upper semi-continuous. We first give a simple proof in the case $div(\mathcal{A}(X, e)) = 0$.

4.1 Proof of Theorem 4.2 in the Case where $div(\mathcal{A}(X, e)) = 0$

Let $\epsilon > 0$ small enough. Since $\widetilde{\gamma}_h$ is continuous at (ω_0, τ_0), there exists a ball $B_{\epsilon_0}(\omega_0, \tau_0) \subset S_h(D_h)$ with $0 < \epsilon_0 < \epsilon$ such that

$$\widetilde{\gamma}_h(\omega, \tau) < (1-\epsilon)^{q-1} \qquad \forall (\omega, \tau) \in B_{\epsilon_0}(\omega_0, \tau_0).$$

Since $\partial\widetilde{\gamma}_h/\partial\tau \leq 0$, we deduce that

$$\widetilde{\gamma}_h(\omega, \tau) < (1-\epsilon)^{q-1} \qquad \forall (\omega, \tau) \in \big((\omega_0-\epsilon_0, \omega_0+\epsilon_0)\times(\tau_0, +\infty)\big)\cap S_h(D_h). \tag{4.4}$$

By continuity of $\widetilde{u}_h$ at (ω_0, τ_0), there exists $0 < \epsilon' < \epsilon$ such that

$$\begin{cases} \widetilde{u}_h(\omega, \tau) \leq T_h^2 o S_h^{-1}(\omega, \tau) + \epsilon^2 & \forall (\omega, \tau) \in B_{\epsilon'}(\omega_0, \tau_0) \\ X(\alpha_+(\omega), \omega) \in Int(S_3) & \forall \omega \in (\omega_0 - \epsilon', \omega_0 + \epsilon'). \end{cases}$$

Using Theorem 1.5, one of the following situations holds

$$\begin{array}{lllll} a) & \exists(\omega_1,\tau_1)\in B_{\epsilon'}(\omega_0,\tau_0) & \text{such that } \omega_1<\omega_0 & \text{and} & \widetilde{p}_h(\omega_1,\tau_1)=0 \\ b) & \exists(\omega_2,\tau_2)\in B_{\epsilon'}(\omega_0,\tau_0) & \text{such that } \omega_2>\omega_0 & \text{and} & \widetilde{p}_h(\omega_2,\tau_2)=0. \end{array}$$

Assume that for example $a)$ is satisfied and set $X_1 = \mathcal{T}_h^{-1}(\omega_1,\tau_1)$ and $\tau_M = \max(\tau_0,\tau_1)$. Then we have by Theorem 1.2

$$\widetilde{p}_h(\omega_i,\tau)=0 \qquad \forall(\omega_i,\tau)\in S_h(D_h) \quad \text{such that } \tau>\tau_M \ (i=0,1).$$

Set $Z_{\tau_M} = \mathcal{T}_h\Big(\big((\omega_1,\omega_0)\times(\tau_M,+\infty)\big)\cap S_h(D_h)\Big)$ and let $y_0\in I\!R$ such that $\mathcal{T}_h^{-1}([y=y_0])\cap B_{\epsilon'}(\omega_0,\tau_0)\cap[\tau>\tau_M]\neq\emptyset$. Note that

$$\begin{aligned} \mathcal{T}_h^{-1}([y=y_0]) &= \{(\omega,\tau_{y_0}(\omega))\in S_h(D_h)\,/\tau_{y_0}(\omega) \\ &= \int_{\alpha_-(\omega)}^{t_{y_0}(\omega)} |\mathcal{A}(X(s,\omega),e)|ds,\ X_2(t_{y_0}(\omega),\omega)=y_0\}. \end{aligned}$$

Set

$$\begin{cases} D_{y_0,\tau_M}=[y>y_0]\cap Z_{\tau_M}\neq\emptyset, & S_3^D=S_3\cap\bar{D}_{y_0,\tau_M} \\ v(y)=\epsilon(\epsilon+y_0-y)^+ + y, & \xi(x,y)=\chi(D_{y_0,\tau_M})(u-v)^+. \end{cases}$$

Since $v\ge y=u$ on $\partial D_{y_0,\tau_M}\setminus\big([y=y_0]\cup S_3^D\big)$ and $v(y_0)=\epsilon^2+y_0\ge u(x,y_0)$, it follows that $\pm\xi$ are test functions for (P) and we have

$$\int_{D_{y_0,\tau_M}} \big(\mathcal{A}(X,\nabla u)-g\mathcal{A}(X,e)\big).\nabla(u-v)^+dX = \int_{S_3^D}\beta(X,\psi-u)(u-v)^+d\sigma. \tag{4.5}$$

We also have

$$\begin{aligned} &\int_{D_{y_0,\tau_M}} \big(\mathcal{A}(X,\nabla v)-\chi([v=y])\mathcal{A}(X,e)\big).\nabla(u-v)^+dX \\ &\qquad = \int_{D_{y_0,\tau_M}\cap[v>y]} (1-\epsilon)^{q-1}\mathcal{A}(X,e).\nabla(u-v)^+dX. \end{aligned} \tag{4.6}$$

Since $div(\mathcal{A}(X,e))=0$, we have

$$\int_{D_{y_0,\tau_M}\cap[v>y]} (1-\epsilon)^{q-1}\mathcal{A}(X,e).\nabla(u-v)^+dX =$$
$$-\int_{D_{y_0,\tau_M}\cap[v=y]} (1-\epsilon)^{q-1}\mathcal{A}(X,e).\nabla(u-v)^+dX$$

$$+\int_{S_3^D}(1-\epsilon)^{q-1}\mathcal{A}(X,e).\nu(u-v)^+d\sigma.$$

So, we obtain by subtracting (4.6) from (4.5),

$$\begin{aligned}&\int_{D_{y_0,\tau_M}}\big(\mathcal{A}(X,\nabla u)-\mathcal{A}(X,\nabla v)+(\chi([v=y])-g)\mathcal{A}(X,e)\big).\nabla(u-v)^+dX\\&=\int_{S_3^D}\beta(X,\psi-u)(u-v)^+d\sigma-\int_{S_3^D}(1-\epsilon)^{q-1}\mathcal{A}(X,e).\nu(u-v)^+d\sigma\\&\qquad+\int_{D_{y_0,\tau_M}\cap[v=y]}(1-\epsilon)^{q-1}\mathcal{A}(X,e).\nabla(u-v)^+dX\end{aligned}$$

which can be written as

$$\begin{aligned}&\int_{D_{y_0,\tau_M}\cap[v>y]}(\mathcal{A}(X,\nabla u)-\mathcal{A}(X,\nabla v)).\nabla(u-v)^+dX\\&\qquad+\int_{D_{y_0,\tau_M}\cap[v=y]}(\mathcal{A}(X,\nabla u)-g\mathcal{A}(X,e)).\nabla(u-v)^+dX\\&=\int_{D_{y_0,\tau_M}\cap[v=y]}(1-\epsilon)^{q-1}\mathcal{A}(X,e).\nabla(u-v)^+dX\\&\qquad+\int_{S_3^D}[\beta(X,\psi-u)-(1-\epsilon)^{q-1}\mathcal{A}(X,e).\nu](u-v)^+d\sigma.\end{aligned}\tag{4.7}$$

Note that

$$\begin{aligned}&\int_{D_{y_0,\tau_M}\cap[v=y]}(\mathcal{A}(X,\nabla u)-g\mathcal{A}(X,e)).\nabla(u-v)^+dX\\&=\int_{D_{y_0,\tau_M}\cap[v=y]}(\mathcal{A}(X,\nabla u)-\mathcal{A}(X,(1-\epsilon)e)).(\nabla u-(1-\epsilon)e)dX\\&+\int_{D_{y_0,\tau_M}\cap[v=y]}(1-\epsilon)^{q-1}\mathcal{A}(X,e).\nabla(u-y)dX\\&+\int_{D_{y_0,\tau_M}\cap[v=y]}\epsilon((1-\epsilon)^{q-1}-g)\mathcal{A}(X,e).edX\\&-\int_{D_{y_0,\tau_M}\cap[v=y]}\epsilon(\mathcal{A}(X,\nabla u)-g\mathcal{A}(X,e)).edX.\end{aligned}$$

Therefore (4.7) becomes

$$\int_{D_{y_0,\tau_M}\cap[v>y]}(\mathcal{A}(X,\nabla u)-\mathcal{A}(X,\nabla v)).\nabla(u-v)^+dX+$$

$$\int_{D_{y_0,\tau_M}\cap[v=y]\cap[u>y]}(\mathcal{A}(X,\nabla u)-\mathcal{A}(X,(1-\epsilon)e)).(\nabla u-(1-\epsilon)e)dX$$
$$+\int_{D_{y_0,\tau_M}\cap[v=y]\cap[u>y]}\epsilon(1-\epsilon)^{q-1}\mathcal{A}(X,e).edX$$
$$+\int_{D_{y_0,\tau_M}\cap[v=y]\cap[u=y]}\epsilon(1-g)\mathcal{A}(X,e).edX$$
$$=\int_{D_{y_0,\tau_M}\cap[v=y]}\epsilon(\mathcal{A}(X,\nabla u)-g\mathcal{A}(X,e)).edX$$
$$+\int_{S_3^D}(\beta(X,\psi-u)-\beta(X,\varphi))(u-y)^+d\sigma$$
$$+\int_{S_3^D}(\gamma_h(X)-(1-\epsilon)^{q-1})\mathcal{A}(X,e).\nu(u-y)d\sigma. \quad (4.8)$$

We remark that the second, third and fourth integrals of the left hand side of (4.8) are non-negative. The second integral of the right hand side of (4.8) is non-positive since β is nondecreasing with respect to the second variable. The last integral in (4.8) is non-positive by (4.4). Then by using Lemma 1.1, we obtain

$$\int_{D_{y_0,\tau_M}\cap[v>y]}(\mathcal{A}(X,\nabla u)-\mathcal{A}(X,\nabla v)).\nabla(u-v)^+dX$$
$$\int_{D_{y_0,\tau_M}\cap[v=y]\cap[u>y]}\epsilon\big((1-\epsilon)^{q-1}-\gamma_h\big)\mathcal{A}(X,e).edX$$
$$+\int_{D_{y_0,\tau_M}\cap[v=y]\cap[u=y]}\epsilon(1-g-\gamma_h)\mathcal{A}(X,e).edX\leq 0. \quad (4.9)$$

By (4.4) and Theorem 3.1, the second and last integrals of (4.9) are nonnegative. Finally, we get

$$\int_{D_{y_0,\tau_M}\cap[v>y]}(\mathcal{A}(X,\nabla u)-\mathcal{A}(X,\nabla v)).\nabla(u-v)^+dX\ \leq\ 0$$

and we conclude as for the continuity proof of the free boundary under S_2 (see [15]).

4.2 Proof of Theorem 4.2 in the Case where $div(\mathcal{A}(X,e))\geqslant 0$

We need four Lemmas

Lemma 4.1. *There exists $\delta>0$ and $\eta>0$ such that* *i)* $\quad \gamma_h(x,y) < \exp(-M\delta) \qquad \forall (x,y) \in Z$

where

$$Z = \mathcal{T}_h\Big(\big((\omega_0-\eta,\omega_0+\eta)\times(\tau_0,+\infty)\big)\cap S_h(D_h)\Big), \qquad M = \sup_{X\in\bar\Omega} \frac{div(\mathcal{A}(X,e))}{\mathcal{A}(X,e).e}.$$

ii) $\quad X_2(S_h^{-1}(\omega_0,\tau_0)) < h_0 < \sigma(X_1(S_h^{-1}(\omega,\tau^*(\omega)))) < h_0+\delta \quad \forall \omega \in (\omega_0-\eta,\omega_0+\eta)$

where $\quad h_0 = \sigma(X_1(S_h^{-1}(\omega_0,\tau^*(\omega_0)))) - \delta/2.$

Proof. Since $\widetilde{\gamma}_h$ is continuous at (ω_0,τ_0), there exists a ball $B_{\epsilon_0}(\omega_0,\tau_0)\subset S_h(D_h)$ such that : $\widetilde{\gamma}_h(\omega,\tau)\le\gamma_+<1$ $\forall(\omega,\tau)\in B_{\epsilon_0}(\omega_0,\tau_0)$ with $\gamma_+\in(0,1)$. Moreover since $\widetilde{\gamma}_h$ is non-increasing in τ, we obtain

$$\widetilde{\gamma}_h(\omega,\tau)\le\gamma_+<1 \qquad \forall(\omega,\tau)\in\big((\omega_0-\epsilon_0,\omega_0+\epsilon_0)\times(\tau_0,+\infty)\big)\cap S_h(D_h).$$

By (1.7), we assume for example that

$$\{T_h(\alpha_+(\omega),\omega),\,\omega\in(\omega_0-\epsilon_0,\omega_0+\epsilon_0)\} = \{(x,\sigma(x)),\,x\in I_{\epsilon_0}\}, \qquad I_{\epsilon_0} \text{ is an interval.}$$

There exists $\delta>0$ small enough such that

$$X_2(S_h^{-1}(\omega_0,\tau_0)) < \sigma(X_1(S_h^{-1}(\omega_0,\tau^*(\omega_0)))) - \delta/2 = h_0$$
and
$$\gamma_h(x,y)\le\gamma_+<\exp(-M\delta) \quad \forall(x,y)\in\mathcal{T}_h\Big(\big((\omega_0-\epsilon_0,\omega_0+\epsilon_0)\times(\tau_0,+\infty)\big)\cap S_h(D_h)\Big).$$

By continuity of $\sigma o T_h^1 o S_h^{-1}$ at $(\omega_0,\tau^*(\omega_0))$, there exists $\eta\in(0,\epsilon_0)$ small enough such that

$$\begin{aligned}\sigma o T_h^1 o S_h^{-1}(\omega_0,\tau^*(\omega_0)) - \delta/2 &< \sigma o T_h^1 o S_h^{-1}(\omega,\tau^*(\omega))\\ &< \sigma o T_h^1 o S_h^{-1}(\omega_0,\tau^*(\omega_0)) + \delta/2 \qquad \forall\omega\in(\omega_0-\eta,\omega_0+\eta)\end{aligned}$$

and therefore

$$X_2(S_h^{-1}(\omega_0,\tau_0)) < h_0 < \sigma(X_1(S_h^{-1}(\omega,\tau^*(\omega)))) < h_0+\delta \qquad \forall\omega\in(\omega_0-\eta,\omega_0+\eta)$$

which is ii). □

Lemma 4.2. *Let* $\epsilon\in\Big(0,\frac{1}{3}\Big(\min_{w\in(\omega_0-\eta,\omega_0+\eta)}\sigma(X_1(\alpha_+(\omega),\omega)) - h_0\Big)\Big)$, $y_0\in\mathbb{R}$ *and*

$$\kappa(s) = 1 - \lambda(s)^{\frac{1}{q-1}}, \quad \lambda(s) = \exp(-M(s - y_0))$$
$$v(y) = y + \int_y^{\max(y, y_0+\epsilon)} \kappa(s) ds,$$
$$Z_\tau(\omega, \omega') = \mathcal{T}_h\Big(\big((\omega, \omega') \times (\tau, +\infty)\big) \cap S_h(D_h)\Big),$$
$$D_{y_0,\tau}(\omega, \omega') = Z_\tau(\omega, \omega') \cap [y > y_0], \quad \omega' \in (\omega_0 - \eta, \omega_0 + \eta).$$

Then $\exists \eta_1 \in \big(0, \min(\eta, c_0\epsilon/c_1)\big)$ *such that one of the following situations holds :*

i) $\exists \omega_1 \in (\omega_0 - \eta_1, \omega_0)$*,* $\tau_M \in (\tau_0 - \eta_1, \tau_0 + \eta_1)$*,* $\forall y_0 \in \pi_y(\Omega)$ *such that* $[y = y_0] \cap \big(\mathcal{T}_h(B_{\eta_1}(\omega_0, \tau_{h_0}(\omega_0))) \cap [\tau > \tau_M]\big) \neq \emptyset$*, we have*

$$u \leqslant v \qquad on \quad \partial D_{y_0,\tau_M}(\omega_1, \omega_0) \setminus S_3.$$

ii) $\exists \omega_2 \in (\omega_0, \omega_0 + \eta_1)$*,* $\tau_M \in (\tau_0 - \eta_1, \tau_0 + \eta_1)$*,* $\forall y_0 \in \pi_y(\Omega)$ *such that* $[y = y_0] \cap \big(\mathcal{T}_h(B_{\eta_1}(\omega_0, \tau_{h_0}(\omega_0))) \cap [\tau > \tau_M]\big) \neq \emptyset$*, we have*

$$u \leqslant v \qquad on \quad \partial D_{y_0,\tau_M}(\omega_0, \omega_2) \setminus S_3.$$

Proof. Consider $\epsilon > 0$ small enough such that

$$h_0 + 3\epsilon < \sigma(X_1(\alpha_+(\omega), \omega)) \qquad \forall \omega \in (\omega_0 - \eta, \omega_0 + \eta).$$

Let $t_{h_0}(\omega_0)$ be the unique element in $(\alpha_-(\omega_0), \alpha_+(\omega_0))$ such that $X_2(t_{h_0}(\omega_0), \omega_0) = h_0$ and let $\tau_{h_0}(\omega_0) = \int_{\alpha_-(\omega_0)}^{t_{h_0}(\omega_0)} |\mathcal{A}(X(s, \omega_0), e)| ds$.
Since $\widetilde{u}_h(\omega_0, \tau_{h_0}(\omega_0)) = T_h^2 o S_h^{-1}(\omega_0, \tau_{h_0}(\omega_0))$ and $\widetilde{u}_h$ is continuous at $(\omega_0, \tau_{h_0}(\omega_0))$, there exists $\eta_1, 0 < \eta_1 < \min\big(\eta, \dfrac{c_0}{c_1}\epsilon\big)$ such that

$$\widetilde{u}_h(\omega, \tau) \leq T_h^2 o S_h^{-1}(\omega, \tau) + \int_0^\epsilon \Big(1 - \exp\Big(\frac{-Ms}{q-1}\Big)\Big) ds \quad \forall (\omega, \tau) \in B_{\eta_1}(\omega_0, \tau_{h_0}(\omega_0))$$
$$X(\alpha_+(\omega), \omega) \in Int(S_3) \qquad \forall \omega \in (\omega_0 - \eta, \omega_0 + \eta).$$

Using Theorem 2.2, one of the following situations occurs

a) $\exists (\omega_1, \tau_1) \in B_{\eta_1}(\omega_0, \tau_{h_0}(\omega_0))$ such that $\omega_1 < \omega_0$ and $\widetilde{p}_h(\omega_1, \tau_1) = 0$

b) $\exists (\omega_2, \tau_2) \in B_{\eta_1}(\omega_0, \tau_{h_0}(\omega_0))$ such that $\omega_2 > \omega_0$ and $\widetilde{p}_h(\omega_2, \tau_2) = 0$.

Assume that for example *a)* is satisfied.
Set $\tau_M = \max(\tau_1, \tau_{h_0}(\omega_0)),\ Z_{\tau_M} = \mathcal{T}_h\Big(\big((\omega_1, \omega_0) \times (\tau_M, +\infty)\big) \cap S_h(D_h)\Big)$.
Let $y_0 \in I\!R$ such that $[y = y_0] \cap \big[\mathcal{T}_h(B_{\eta_1}(\omega_0, \tau_{h_0}(\omega_0))) \cap [\tau > \tau_M]\big] \neq \emptyset$.
Note that we have $h_0 < y_0 \leq h_0 + \epsilon$. Indeed if

$$z=\int_0^{t_z(\omega_0)} a^2(X(s,\omega_0))ds+h \quad \text{and} \quad \tau_z(\omega_0)=\int_{\alpha_-(\omega_0)}^{t_z(\omega_0)}|\mathcal{A}(X(s,\omega_0),e)|ds,$$

then $t_{y_0}(\omega_0)>t_{h_0}(\omega_0)$ since

$$\int_{t_{h_0}(\omega_0)}^{t_{y_0}(\omega_0)}|\mathcal{A}(X(s,\omega_0),e)|ds=\tau_{y_0}(\omega_0)-\tau_{h_0}(\omega_0)>\tau_M-\tau_{h_0}(\omega_0)>0.$$

We deduce that $y_0>h_0$. Moreover we have

$$y_0-h_0\leq c_1(t_{y_0}(\omega_0)-t_{h_0}(\omega_0))\leq \frac{c_1}{c_0}(\tau_{y_0}(\omega_0)-\tau_{h_0}(\omega_0))<\eta_1\frac{c_1}{c_0}<\epsilon.$$

Finally we have $\forall x\in\pi_x(D_{y_0,\tau_M}(\omega_1,\omega_0))$

$$\begin{aligned} v(y_0)&=y_0+\int_{y_0}^{y_0+\epsilon}\Big(1-\exp\Big(\frac{-M(s-y_0)}{q-1}\Big)\Big)ds\\ &=y_0+\int_0^{\epsilon}\Big(1-\exp\Big(\frac{-Mt}{q-1}\Big)\Big)dt\geq u(x,y_0). \end{aligned}$$

□

Lemma 4.3. *Under assumptions of Lemma 4.2, we have in case i) holds*

$$u\leqslant v \qquad \text{in } D_{y_0,\tau_M}(\omega_1,\omega_0)$$

and consequently

$$p=0 \qquad \text{in } D_{y_0,\tau_M}(\omega_1,\omega_0)\cap[y\geqslant y_0+\epsilon].$$

Proof. From the previous lemma, we deduce that $\pm\xi=\pm(u-v)^+\chi(D_{y_0,\tau_M})$ ($D_{y_0,\tau_M}(\omega_1,\omega_0)$ is simply denoted by D_{y_0,τ_M} when there is no confusion) are test functions for (P). So we have

$$\int_{D_{y_0,\tau_M}}\big(\mathcal{A}(X,\nabla u)-g\mathcal{A}(X,e)\big).\nabla(u-v)^+dX=\int_{S_3^D}\beta(X,\psi-u)(u-v)^+d\sigma. \tag{4.10}$$

We also have

$$\int_{D_{y_0,\tau_M}}\big(\mathcal{A}(X,\nabla v)-\chi([v=y])\mathcal{A}(X,e)\big).\nabla(u-v)^+dX$$

$$= \int_{D_{y_0,\tau_M}\cap[v>y]} (1-\kappa(y))^{q-1}\mathcal{A}(X,e).\nabla(u-v)^+ dX. \quad (4.11)$$

The right hand side of (4.11) reads

$$\begin{aligned}
&\int_{D_{y_0,\tau_M}\cap[v>y]} (1-\kappa(y))^{q-1}\mathcal{A}(X,e).\nabla(u-v)^+ dX = \\
&-\int_{D_{y_0,\tau_M}\cap[v=y]} (1-\kappa(y))^{q-1}\mathcal{A}(X,e).\nabla(u-v)^+ dX \\
&-\int_{D_{y_0,\tau_M}} div\big((1-\kappa(y))^{q-1}\mathcal{A}(X,e)\big)(u-v)^+ dX \\
&+\int_{S_3^D} (1-\kappa(y))^{q-1}\mathcal{A}(X,e).\nu(u-v)^+ d\sigma.
\end{aligned}$$

So we obtain by subtracting (4.11) from (4.10)

$$\begin{aligned}
&\int_{D_{y_0,\tau_M}} \big(\mathcal{A}(X,\nabla u)-\mathcal{A}(X,\nabla v)+(\chi([v=y])-g)\mathcal{A}(X,e)\big).\nabla(u-v)^+ dX \\
&= \int_{S_3^D} \beta(X,\psi-u)(u-v)^+ d\sigma - \int_{S_3^D} (1-\kappa(y))^{q-1}\mathcal{A}(X,e).\nu(u-v)^+ d\sigma \\
&+\int_{D_{y_0,\tau_M}\cap[v=y]} (1-\kappa(y))^{q-1}\mathcal{A}(X,e).\nabla(u-v)^+ dX \\
&+\int_{D_{y_0,\tau_M}} div\big((1-\kappa(y))^{q-1}\mathcal{A}(X,e)\big)(u-v)^+ dX.
\end{aligned}$$

which can be written

$$\begin{aligned}
&\int_{D_{y_0,\tau_M}\cap[v>y]} (\mathcal{A}(X,\nabla u)-\mathcal{A}(X,\nabla v)).\nabla(u-v)^+ dX \\
&\qquad +\int_{D_{y_0,\tau_M}\cap[v=y]} (\mathcal{A}(X,\nabla u)-g\mathcal{A}(X,e)).\nabla(u-v)^+ dX \\
&= \int_{D_{y_0,\tau_M}\cap[v=y]} (1-\kappa(y))^{q-1}\mathcal{A}(X,e).\nabla(u-v)^+ dX \\
&\qquad +\int_{S_3^D} [\beta(X,\psi-u)-(1-\kappa(y))^{q-1}\mathcal{A}(X,e).\nu](u-v)^+ d\sigma \\
&+\int_{D_{y_0,\tau_M}} div\big((1-\kappa(y))^{q-1}\mathcal{A}(X,e)\big)(u-v)^+ dX. \qquad (4.12)
\end{aligned}$$

Note that the second term of the left hand side of (4.12) is equal to

$$\int_{D_{y_0,\tau_M}\cap[v=y]} (\mathcal{A}(X,\nabla u)-\mathcal{A}(X,(1-\kappa(y))e)).(\nabla u-(1-\kappa(y))e) dX$$

$$+\int_{D_{y_0,\tau_M}\cap[v=y]}(1-\kappa(y))^{q-1}\mathcal{A}(X,e).\nabla(u-y)dX$$
$$+\int_{D_{y_0,\tau_M}\cap[v=y]}\kappa(y)((1-\kappa(y))^{q-1}-g)\mathcal{A}(X,e).edX$$
$$-\int_{D_{y_0,\tau_M}\cap[v=y]}\kappa(y)(\mathcal{A}(X,\nabla u)-g\mathcal{A}(X,e)).edX.$$

We then get from (4.12)

$$\int_{D_{y_0,\tau_M}\cap[v>y]}(\mathcal{A}(X,\nabla u)-\mathcal{A}(X,\nabla v)).\nabla(u-v)^+dX$$
$$+\int_{D_{y_0,\tau_M}\cap[v=y]\cap[u>y]}(\mathcal{A}(X,\nabla u)-\mathcal{A}(X,(1-\kappa(y))e)).(\nabla u-(1-\kappa(y))e)dX$$
$$+\int_{D_{y_0,\tau_M}\cap[v=y]\cap[u>y]}\kappa(y)(1-\kappa(y))^{q-1}\mathcal{A}(X,e).edX$$
$$+\int_{D_{y_0,\tau_M}\cap[v=y]\cap[u=y]}\kappa(y)(1-g)\mathcal{A}(X,e).edX$$

$$=\int_{D_{y_0,\tau_M}\cap[v=y]}\kappa(y)(\mathcal{A}(X,\nabla u)-g\mathcal{A}(X,e)).edX$$
$$+\int_{S_3^D}(\beta(X,\psi-u)-\beta(X,\varphi))(u-y)^+d\sigma$$
$$+\int_{S_3^D}(\gamma_h(X)-(1-\kappa(y))^{q-1})\mathcal{A}(X,e).\nu(u-y)d\sigma$$
$$+\int_{D_{y_0,\tau_M}}div\big((1-\kappa(y))^{q-1}\mathcal{A}(X,e)\big)(u-v)^+dX. \qquad (4.13)$$

We remark that the second integral of the left hand side of (4.13) is nonnegative. Moreover since β is nondecreasing with respect to the second variable and because of the upper bound of $\gamma_h(X)$ (see Lemma 4.1 i)), we obtain by using Theorem 3.1,

$$\int_{D_{y_0,\tau_M}\cap[v>y]}(\mathcal{A}(X,\nabla u)-\mathcal{A}(X,\nabla v)).\nabla(u-v)^+dX$$
$$+\int_{D_{y_0,\tau_M}\cap[v=y]\cap[u>y]}\kappa(y)\big((1-\kappa(y))^{q-1}-\gamma_h\big)\mathcal{A}(X,e).edX$$
$$+\int_{D_{y_0,\tau_M}\cap[v=y]\cap[u=y]}\kappa(y)(1-g-\gamma_h)\mathcal{A}(X,e).edX$$
$$\leq\int_{D_{y_0,\tau_M}}div\big((1-\kappa(y))^{q-1}\mathcal{A}(X,e)\big)(u-v)^+dX.$$

Again since $\gamma_h(X) \leq \lambda(y)$ and $div\big((1-\kappa(y))^{q-1}\mathcal{A}(X,e)\big) \leq 0$ a.e. in D_{y_0,τ_M}, we get by Theorem 3.1,

$$\int_{D_{y_0,\tau_M}\cap[v>y]} (\mathcal{A}(X,\nabla u) - \mathcal{A}(X,\nabla v)).\nabla(u-v)^+ dX \ \leq 0.$$

This leads to $u \leq v$ in D_{y_0,τ_M}. Then $u(x, y_0+\epsilon) = y_0+\epsilon \ \forall x \in \pi_x(D_{y_0,\tau_M})$. By Theorem 1.2, $u(x,y) = y$ in $D_{y_0,\tau_M} \cap [v = y]$. □

Lemma 4.4. *Under assumptions of Lemma 4.3, there exists* $\omega_1', \tau_M', y_0' \in \mathbb{R}$ *satisfying*

$$\omega_1' \in (\omega_0, \omega_0+\eta_1), \quad \tau_M' \geqslant \tau_{y_0+\epsilon}(\omega_0), \quad y_0+\epsilon < y_0' < y_0+2\epsilon$$

such that

$$\widetilde{p}_h = 0 \qquad in \ \Big((\omega_1,\omega_1')\times(\tau_M',+\infty)\Big) \cap \mathcal{T}_h^{-1}\Big([y > y_0+\epsilon]\Big).$$

Proof. By Theorem 2.2 ii), we deduce that there exists $(\omega_1', \tau_1') \in S_h(D_h)$ such that

$$\begin{array}{l} \omega_0 < \omega_1' < \omega_0+\eta, \quad y_0+\epsilon < X_2 o S_h^{-1}(\omega_1',\tau_1') < y_0+2\epsilon \\ \text{and} \quad \widetilde{u}_h(\omega_1',\tau_1') = T_h^2 o S_h^{-1}(\omega_1',\tau_1'). \end{array}$$

Arguing as above, with $h_0' = y_0+\epsilon$, there exists $t_{h_0'}(\omega_0) \in (\alpha_-(\omega_0), \alpha_+(\omega_0))$ such that

$$X_2(t_{h_0'}(\omega_0),\omega_0) = h_0' \quad \text{ and } \quad \tau_{h_0'}(\omega_0) = \int_{\alpha_-(\omega_0)}^{t_{h_0'}(\omega_0)} |\mathcal{A}(X(s,\omega_0))| ds.$$

Since $\widetilde{u}_h(\omega_0, \tau_{h_0'}(\omega_0)) = T_h^2 o S_h^{-1}(\omega_0,\tau_{h_0'}(\omega_0))$ and $\widetilde{u}_h$ is continuous at $(\omega_0, \tau_{h_0'}(\omega_0))$, there exists $\eta_1', 0 < \eta_1' < \eta_1$ such that

$$\begin{array}{l} \widetilde{u}_h(\omega,\tau) \leq T_h^2 o S_h^{-1}(\omega,\tau) + \displaystyle\int_0^\epsilon \Big(1-\exp\Big(-\frac{M}{q-1}s\Big)\Big)ds \quad \forall(\omega,\tau) \in B_{\eta_1'}(\omega_0,\tau_{h_0'}(\omega_0)) \\ X(\alpha_+(\omega),\omega) \in Int(S_3) \qquad \forall \omega \in (\omega_0-\eta_1', \omega_0+\eta_1'). \end{array}$$

Set $\tau_M' = \max(\tau_1', \tau_{h_0'}(\omega_0))$, $\qquad Z_{\tau_M'} = \mathcal{T}_h\Big(\big((\omega_0,\omega_1')\times(\tau_M',+\infty)\big)\cap S_h(D_h)\Big)$.
Let $y_0' \in \mathbb{R}$ such that $[y = y_0'] \cap \big[\mathcal{T}_h(B_{\eta_1'}(\omega_0, \tau_{h_0'}(\omega_0)))\big] \neq \emptyset$.
Set

$$\left| \begin{array}{l} D_{y_0',\tau_M'} = [y > y_0'] \cap Z_{\tau_M'} \neq \emptyset, \qquad S_3^{D'} = S_3 \cap \bar{D}_{y_0',\tau_M'} \\ \bar{v}(y) = \begin{cases} y + \int_y^{y_0'+\epsilon} \bar{\kappa}(s)ds & \text{if } y_0' \leq y < y_0'+\epsilon \\ y & \text{if } y_0 \geq y_0'+\epsilon \end{cases} \\ \bar{\kappa}(s) = 1 - \bar{\lambda}(s)^{\frac{1}{q-1}} \quad \text{ with } \quad \bar{\lambda}(s) = \exp(-M(s-y_0')). \end{array} \right.$$

Note that we have $h'_0 < y'_0 \le h'_0 + \epsilon$ and $\forall x \in \pi_x(D_{y'_0,\tau'_M})$

$$\bar{v}(y'_0) = y'_0 + \int_{y'_0}^{y'_0+\epsilon} \Big(1 - \exp\Big(-\frac{M(s-y'_0)}{q-1}\Big)\Big) ds = y'_0 + \int_0^{\epsilon} \Big(1 - \exp\Big(-\frac{M}{q-1}s\Big)\Big) ds \ge u(x, y'_0).$$

So $\pm\xi = \pm(u-\bar{v})^+\chi(D_{y'_0,\tau'_M})$ are test functions for (P) and we argue as before to deduce that $u \le \bar{v}$ in $D_{y'_0,\tau'_M} \cap [\bar{v} > y]$ which leads to $u(x, y'_0+\epsilon) = y'_0+\epsilon < y_0+2\epsilon$. By Theorem 1.2, $u(x,y) = y \ \forall (x,y) \in D_{y'_0,\tau'_M} \cap [\bar{v} = y]$. Hence

$$\widetilde{p}_h(\omega,\tau) = 0 \qquad \forall(\omega,\tau) \in \Big((\omega_1,\omega'_1) \times (\tau'_M, +\infty)\Big) \cap \mathcal{T}_h^{-1}([y > y_0 + 2\epsilon]).$$

□

Proof of Theorem 4.2. Now, let $N \in \mathbb{N}$ such that $N > \dfrac{2(y_0 - X_2 o S_h^{-1}(\omega_0,\tau_0))}{\epsilon}$ and set

$$\begin{cases} k_0 = y_0 + 2\epsilon, & k_N = X_2 o S_h^{-1}(\omega_0,\tau_0) + 2\epsilon \\ k_{i+1} = k_i - \frac{k_0 - k_N}{N} & \text{for } 0 \le i \le N-1 \end{cases}$$

Let $\epsilon_1 \in (0, \delta_*/8)$ with $\delta_* = \dfrac{y_0 - X_2 o S_h^{-1}(\omega_0,\tau_0)}{N} = \dfrac{k_0 - k_N}{N}$.
We have $k_1 = k_0 - \delta_*$, $k'_1 = k_1 - \delta_*/2 = k_0 - 3\delta_*/2$. There exists $t_{k'_1}(\omega_0) \in (\alpha_-(\omega_0), \alpha_+(\omega_0))$ such that

$$X_2(t_{k'_1}(\omega_0), \omega_0) = k'_1 \quad \text{and} \quad \tau_{k'_1}(\omega_0) = \int_{\alpha_-(\omega_0)}^{t_{k'_1}(\omega_0)} |\mathcal{A}(X(s,\omega_0))| ds.$$

Since $\widetilde{u}_h(\omega_0, \tau_{k'_1}(\omega_0)) = T_h^2 o S_h^{-1}(\omega_0, \tau_{k'_1}(\omega_0))$ and $\widetilde{u}_h$ is continuous at $(\omega_0, \tau_{k'_1}(\omega_0))$, there exists $\epsilon'_1 > 0, \epsilon'_1 < \min(\frac{c_0}{c_1}\epsilon_1, \frac{\omega'_1 - \omega_1}{2})$ such that

$$\begin{aligned} &\widetilde{u}_h(\omega,\tau) \le T_h^2 o S_h^{-1}(\omega,\tau) + \int_0^{\epsilon_1} \Big(1 - \exp\Big(-\frac{M}{q-1}s\Big)\Big) ds \quad \forall(\omega,\tau) \in B_{\epsilon'_1}(\omega_0, \tau_{k'_1}(\omega_0)) \\ &X(\alpha_+(\omega), \omega) \in Int(S_3) \qquad \forall \omega \in (\omega_0 - \epsilon'_1, \omega_0 + \epsilon'_1). \end{aligned}$$

There exists, for example, $(\omega_2,\tau_2) \in B_{\epsilon'_1}(\omega_0, \tau_{k'_1}(\omega_0))$ such that $\omega_2 < \omega_0$ and $\widetilde{p}_h(\omega_2,\tau_2) = 0$.
Set $\tau_M^1 = \max(\tau_2, \tau_{k'_1}(\omega_0))$, $Z_{\tau_M^1} = \mathcal{T}_h\Big(\big((\omega_2,\omega_0) \times (\tau_M^1, +\infty)\big) \cap S_h(D_h)\Big)$.
Let $y_1 \in \mathbb{R}$ such that $[y = y_1] \cap \big[\mathcal{T}_h(B_{\epsilon'_1}(\omega_0, \tau_{k'_1}(\omega_0)))\big] \ne \emptyset$.
Set

$$\left|\begin{aligned} &D_{y_1,\tau_M^1} = [y > y_1] \cap Z_{\tau_M^1} \ne \emptyset, \qquad S_3^{D^1} = S_3 \cap \bar{D}_{y_1,\tau_M^1} \\ &v_1(y) = \begin{cases} y + \int_y^{y_1+\epsilon_1} \kappa_1(s) ds & \text{if } y_1 \le y < y_1 + \epsilon_1 \\ y & \text{if } y \ge y_1 + \epsilon_1 \end{cases} \\ &\kappa_1(s) = 1 - \lambda_1(s)^{\frac{1}{q-1}} \quad \text{with} \quad \lambda_1(s) = \exp(-M(s - y_1)). \end{aligned}\right.$$

Note that we have $k_1' < y_1 \le k_1' + \epsilon_1$, $k_1' + \epsilon_1 < k_0$ since $\epsilon_1 < 3\delta_*/2$, and $\forall x \in \pi_x(D_{y_1,\tau_M^1})$,

$$\begin{aligned} v_1(y_1) &= y_1 + \int_{y_1}^{y_1+\epsilon_1} \Big(1 - \exp\Big(-\frac{M(s-y_1)}{q-1}\Big)\Big)ds \\ &= y_1 + \int_0^{\epsilon_1} \Big(1 - \exp\Big(-\frac{M}{q-1}t\Big)\Big)dt \ge u(x, y_1). \end{aligned}$$

So $\pm\xi = \pm(u - v_1)^+\chi(D_{y_1,\tau_M^1})$ are test functions for (P). We argue as before and we deduce that $u \le v_1$ in $D_{y_1,\tau_M^1} \cap [v_1 > y]$ which leads to $u(x, y_1 + \epsilon_1) = y_1 + \epsilon_1$. By Theorem 1.2, $u(x, y) = y$ $\forall (x, y) \in D_{y_1}, \tau_M^1 \cap [v_1 = y]$. Hence

$$\widetilde{p}_h(\omega, \tau) = 0 \qquad \forall(\omega, \tau) \in \big((\omega_2, \omega_0) \times (\tau_M^1, +\infty)\big) \cap \mathcal{T}_h^{-1}([y > y_1 + \epsilon_1]).$$

In the same way one can establish a similar result at the right side of ω_0 and obtain

$$\widetilde{p}_h(\omega, \tau) = 0 \qquad \forall(\omega, \tau) \in \big((\omega_2, \omega_2') \times (\tau_M^{1\prime}, +\infty)\big) \cap \mathcal{T}_h^{-1}([y > k_1]).$$

If we repeat the above procedure, one gets at the end

$$\widetilde{p}_h(\omega, \tau) = 0 \qquad \forall(\omega, \tau) \in \big((\omega_{N+1}, \omega_{N+1}') \times (\tau_M^{N\prime}, +\infty)\big) \cap \mathcal{T}_h^{-1}([y > k_N]).$$

Let $H_0 = X_2 o S_h^{-1}(\omega_0, \tau_0) = k_N - 2\epsilon$. Since $\widetilde{p}_h(\omega, \tau_{k_N}(\omega)) = 0$ $\forall \omega \in (\omega_0 - \eta, \omega_0 + \eta)$, $\eta < (\omega_{N+1}' - \omega_{N+1})/4$, we have $\phi_h(\omega) \le \tau_{k_N}(\omega) = \tau_{H_0+2\epsilon}(\omega)$. It follows that

$$\tau_{H_0+2\epsilon}(\omega) = \tau_{H_0}(\omega) + \int_{t_{H_0}(\omega)}^{t_{H_0+2\epsilon}(\omega)} |\mathcal{A}(X(s, \omega), e)| ds \le \tau_{H_0}(\omega) + c_1(t_{H_0+2\epsilon}(\omega) - t_{H_0}(\omega)).$$

On the other hand, we have

$$\begin{aligned} 2\epsilon &= X_2(t_{H_0+2\epsilon}(\omega), \omega) - X_2(t_{H_0}(\omega), \omega) \\ &= \int_{t_{H_0}(\omega)}^{t_{H_0+2\epsilon}(\omega)} a^2(X(s, \omega) ds \ge c_0(t_{H_0+2\epsilon}(\omega) - t_{H_0}(\omega)) \end{aligned}$$

then $\phi_h(\omega) \le \tau_{H_0}(\omega) + (2\epsilon)c_1/c_0$. Now by continuity of $\tau_{H_0}(\omega)$, there exists $0 < \eta_0 < \eta$ such that $\forall \omega \in (\omega_0 - \eta_0, \omega_0 + \eta_0)$, $\tau_{H_0}(\omega) - \tau_{H_0}(\omega_0) < \epsilon$. Therefore

$$\phi_h(\omega) < \tau_{H_0}(\omega_0) + \big(1 + 2\frac{c_1}{c_0}\big)\epsilon = \phi_h(\omega_0) + \big(1 + 2\frac{c_1}{c_0}\big)\epsilon$$

which proves the u.s.c of ϕ at ω_0. □

Theorem 4.3. *Let $h \in \pi_y(\Omega)$ and $\omega_0 \in Int(\pi_\omega o S_h o T_h^{-1}(S_3))$*

i) If $\dfrac{\beta(.,\varphi(.))}{\mathcal{A}(.,e).\nu(.)}(X(\alpha_+(\omega_0),\omega_0)) \geq 1$, *then the region below a neighborhood of ω_0, in the sense of Remark 1.4, is completely saturated.*

ii) If $0 < \dfrac{\beta(.,\varphi(.))}{\mathcal{A}(.,e).\nu(.)}(X(\alpha_+(\omega_0),\omega_0)) < 1$, *then* $0 \leq \widetilde{\gamma}_h(\omega_0,\phi_h(\omega_0)) \leq 1$ *and*

- *if* $0 \leq \gamma_h o T_h o S_h^{-1}(\omega_0,\phi_h(\omega_0)) < 1$, *then ϕ_h is continuous at ω_0,*

- *if* $\gamma_h o T_h o S_h^{-1}(\omega_0,\phi_h(\omega_0)) = 1$ *and* $div\big(\mathcal{A}(X o S_h^{-1}(\omega_0,\phi_h(\omega_0)),e)\big) > 0$, *then ϕ_h is continuous at ω_0.*

Proof. i) see Corollary 3.1.
ii) Assume that $\widetilde{\gamma}_h(\omega_0,\phi_h(\omega_0)) > 1$. Then by continuity, there exists $\epsilon_0 > 0$ such that $\widetilde{\gamma}_h(\omega,\tau) > 1\ \forall(\omega,\tau) \in B_{\epsilon_0}(\omega_0,\phi_h(\omega_0))$. Since $\widetilde{\gamma}_h$ is nondecreasing, we deduce that $\widetilde{\gamma}_h(\omega,\tau) > 1, \quad \forall(\omega,\tau) \in \Big(\big((\omega_0-\epsilon_0,\omega_0+\epsilon_0)\times(-\infty,\phi_h(\omega_0))\big)\cap S_h(D)\Big) \cup B_{\epsilon_0}(\omega_0,\phi_h(\omega_0)) = D_{\epsilon_0}$. Then $g = 0$ a.e. in $\mathcal{T}_h(D_{\epsilon_0})$ and by (1.6) we get $div(\mathcal{A}(X,\nabla u)) = 0$ in $\mathcal{T}_h(D_{\epsilon_0})$. By the strong maximum principle we obtain $u > y$ or $u = y$ in $\mathcal{T}_h(D_{\epsilon_0})$.
Assuming that $u = y$ and choosing in $(P)iii)$ test functions vanishing on $\partial(\mathcal{T}_h(D_{\epsilon_0}))\cap\Omega$, we get by Green's formula $\mathcal{A}(X,e).\nu = 0$ on $T_h(\{(\alpha_-(\omega),\omega),\omega \in \pi_\omega(D_h)\}) \cap \partial(\mathcal{T}_h(D_{\epsilon_0}))$ which contradicts (1.10). Hence $u > y$ in $\mathcal{T}_h(D_{\epsilon_0})$ and we get a contradiction.
So necessarily, we have $0 \leq \widetilde{\gamma}_h(\omega_0,\phi_h(\omega_0)) \leq 1$. Now,

- if $0 \leq \widetilde{\gamma}_h(\omega_0,\phi_h(\omega_0)) < 1$, then ϕ_h is continuous at ω_0 (see Theorem 4.2),

- if $\widetilde{\gamma}_h(\omega_0,\phi_h(\omega_0)) = 1$, then $\dfrac{Y_h o S_h^{-1}(\omega_0,\phi_h(\omega_0))}{Y_h o S_h^{-1}(\omega_0,\tau^*(\omega_0))} = \dfrac{\beta(.,\varphi(.)}{\mathcal{A}(.,e).\nu(.)}(X(\alpha_+(\omega_0),\omega_0))$.

Since $Y_h o S_h^{-1}$ is non-increasing, we have $\phi_h(\omega_0) < \tau^*(\omega_0)$ i.e. $(\omega_0,\phi_h(\omega_0)) \in S_h(D_h)$. Moreover $\dfrac{\partial Y_h}{\partial t}(S_h^{-1}(\omega_0,\phi_h(\omega_0)) < 0$ if $div\big(\mathcal{A}(X o S_h^{-1}(\omega_0,\phi_h(\omega_0)),e)\big) > 0$. Remark that

$$\frac{\partial\widetilde{\gamma}_h}{\partial\tau}(\omega_0,\phi_h(\omega_0)) = M_h(\omega_0)\frac{-\partial(Y_h o S_h^{-1})/\partial\tau}{(Y_h o S_h^{-1})^2}(\omega_0,\phi_h(\omega_0))$$

with $M_h(\omega_0) = Y_h o S_h^{-1}(\omega_0,\tau^*(\omega_0))\dfrac{\beta(.,\varphi(.))}{\mathcal{A}(.,e).\nu(.)}(X(\alpha_+(\omega_0),\omega_0)) < 0$, since we have $Y_h o S_h^{-1}(\omega_0,\tau^*(\omega_0)) < 0$ and $\dfrac{\beta(.,\varphi(.))}{\mathcal{A}(.,e).\nu(.)}(X(\alpha_+(\omega_0),\omega_0)) > 0$.

Then $\dfrac{\partial\widetilde{\gamma}_h}{\partial\tau}(\omega_0,\phi_h(\omega_0)) < 0$ and by continuity there exists $\eta_0 > 0$ such that

$$\frac{\partial\widetilde{\gamma}_h}{\partial\tau}(\omega,\tau) < 0 \qquad \forall(\omega,\tau) \in B_{\eta_0}(\omega_0,\phi_h(\omega_0)).$$

Now for $\epsilon \in (0, \eta_0/3)$ we have $(\omega_0, \phi_h(\omega_0)+\epsilon) \in B_{\eta_0}(\omega_0, \phi_h(\omega_0))$ and $\widetilde{\gamma}_h(\omega_0, \phi_h(\omega_0)+\epsilon) < \widetilde{\gamma}_h(\omega_0, \phi_h(\omega_0)) = 1$. Arguing as in the proof of Theorem 4.2, one can prove the existence of $0 < \rho_0 < \eta_0$ such that

$$\begin{aligned} &\widetilde{p}_h(\omega, \tau) = 0 \\ &\forall (\omega, \tau) \in \Big((\omega_0 - \rho_0, \omega_0 + \rho_0) \times \mathbb{R}\Big) \cap \mathcal{T}_h\Big([y > X_2 o S_h^{-1}(\omega_0, \phi_h(\omega_0) + \epsilon) + 2\epsilon]\Big), \end{aligned}$$

then $\phi_h(\omega) \le \tau_{X_2 o S_h^{-1}(\omega_0, \phi_h(\omega_0)+\epsilon)}(\omega) + 2\epsilon c_1/c_0$.
Set $H_\epsilon = X_2 o S_h^{-1}(\omega_0, \phi_h(\omega_0) + \epsilon)$ and $H_0 = X_2 o S_h^{-1}(\omega_0, \phi_h(\omega_0))$. Remark that

$$0 < \epsilon = (\phi_h(\omega_0)+\epsilon) - \phi_h(\omega_0) = \tau_{H_\epsilon}(\omega_0) - \tau_{H_0}(\omega_0) = \int_{t_{H_0}(\omega_0)}^{t_{H_\epsilon}(\omega_0)} |\mathcal{A}(X(s, \omega_0), e)| ds$$

then $\tau_{H_\epsilon}(\omega_0) > \tau_{H_0}(\omega_0)$ $\forall \omega \in (\omega_0 - \rho_0, \omega_0 + \rho_0)$ and since we have

$$\begin{aligned} &H_\epsilon - H_0 = \int_{t_{H_0}(\omega_0)}^{t_{H_\epsilon}(\omega_0)} a^2(X(s, \omega_0)) ds \ge c_0(t_{H_\epsilon}(\omega_0) - t_{H_0}(\omega_0)), \\ &\tau_{H_\epsilon}(\omega_0) \le \tau_{H_0}(\omega_0) + c_0(t_{H_\epsilon}(\omega_0) - t_{H_0}(\omega_0)), \end{aligned}$$

we obtain $\phi_h(\omega) \le \tau_{H_0}(\omega) + (2\epsilon + H_\epsilon - H_0) c_1/c_0$. We then deduce the u.s.c of ϕ_h. □

Corollary 4.1. *Let (u, g) be a solution of (P). Then we have*

$$\widetilde{g}_h = (1 - \widetilde{\gamma}_h)^+ \chi([\widetilde{p}_h = 0]) \qquad \textit{a.e. in } S_h(D_h), h \in \pi_y(\Omega).$$

Proof. Let $h \in \pi_y(\Omega)$. First by Theorem 1.7, we have $\widetilde{g}_h = 0$ a.e. in $S_h(D_h) \cap [\widetilde{\gamma}_h \ge 1]$ and then $\widetilde{g}_h = \widetilde{g}_h \chi([\widetilde{\gamma}_h < 1])$. Next by Proposition 1.4, we have $[\widetilde{p}_h > 0] = [\tau < \phi_h(\omega)]$. So $\widetilde{g}_h = 0$ a.e. in $[\tau < \phi_h(\omega)]$.
Let $(\omega_0, \tau_0) \in [\tau > \phi_h(\omega)] \cap [\widetilde{\gamma}_h < 1]$. By continuity of γ_h, there exists a ball $B_r(\omega_0, \tau_0) \subset S_h(D)$ such that $\widetilde{\gamma}_h < 1$ in $B_r(\omega_0, \tau_0)$ and $\phi_h(\omega_0) < \tau_0 - r$.
Since $\widetilde{\gamma}_h$ is non-increasing with respect to τ, we have $\widetilde{\gamma}_h(\omega, \tau) < 1$ in $Z = \big((\omega_0 - r/2, \omega_0 + r/2) \times (\tau_0 - r/2, +\infty)\big) \cap S_h(D_h)$. Moreover, we have $\widetilde{p}_h(\omega_0, \tau) = 0\, \forall \tau \ge \tau_0 - r/2$. Then arguing as in the proof of Theorem 2.2, one can show that $\exists r' \in (0, r/2)$ such that

$$\widetilde{p}_h(\omega_0, \tau) = 0 \quad \forall (\omega, \tau) \in \big((\omega_0 - r', \omega_0 + r') \times (\tau_0 - r', +\infty)\big) \cap S_h(D_h).$$

So by Theorem 1.4, we have $\widetilde{g}_h = 1 - \widetilde{\gamma}_h$ a.e. in $\big((\omega_0 - r', \omega_0 + r') \times (\tau_0 - r', +\infty)\big) \cap S_h(D_h)$ from which we deduce that $\widetilde{g}_h = 1 - \widetilde{\gamma}_h$ a.e. in $[\tau > \phi_h(\omega)] \cap [\widetilde{\gamma}_h < 1]$. Now the set $[\tau = \phi_h(\omega)] \cap [\widetilde{\gamma}_h < 1]$ being of measure zero (since ϕ_h is continuous at any point $(\omega_0, \tau_0) \in S_h(D_h)$ where $\widetilde{\gamma}_h(\omega_0, \phi_h(\omega_0)) < 1$), we get

$$\widetilde{g}_h = (1 - \widetilde{\gamma}_h) \chi([\tau > \phi_h(\omega)]).\chi([\widetilde{\gamma}_h < 1])$$

$$= (1 - \widetilde{\gamma}_h)^+ \chi([\widetilde{p}_h = 0]).$$

□

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