King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 310

October 2003

ASYMPTOTIC RESULTS FOR SEMI GROUPS OF ORDER-PRESERVING PARTIAL TRANSFORMATIONS

A. Laradji and A.Umar

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

ASYMPTOTIC RESULTS FOR SEMIGROUPS OF ORDER-PRESERVING PARTIAL TRANSFORMATIONS

A. Laradji and A. Umar

Abstract

Let $\mathcal{PC}_n$ be the semigroup of all decreasing and order-preserving partial transformations of an n-element chain, and let $E(\mathcal{PC}_n)$ be its set of idempotents. Then it is shown that for large n, $|\mathcal{PC}_n| \sim (\sqrt{2}+1)^{2n+1}/(2^{3/4}n\sqrt{\pi n})$ and $\frac{|\mathcal{PC}_n|}{|E(\mathcal{PC}_n)|} \sim (\sqrt{2}+1)^{2n+1}/(3^n 2^{-1/4} n\sqrt{\pi n})$. Similar results for $\mathcal{PO}_n$ the (larger) semigroup of all order-preserving partial transformations of an n-element chain are obtained. We also obtained the generating functions for $|\mathcal{PC}_n|$ and $|\mathcal{PO}_n|$ as well as their integral representations.

1 Introduction

Arguably, one of the earliest most fascinating and useful asymptotic formulae in mathematics is Stirling's extraordinary asymptotic formula: for large $n, n! \sim \sqrt{2\pi n}(n/e)^n$. And, even for modest values of n the approximation is quite good: for $n = 10$ the error is only 0.8%, and for $n = 100$ the error drops to 0.08% [6]. There are now several asymptotic formulae, see for example [4, 9]. In this paper we investigate asymptotic formulae associated with the cardinalities and number of idempotents of certain classes of semigroups of order-preserving partial transformations.

Let $\mathcal{PC}_n$ be the semigroup of all decreasing and order-preserving partial transformations of $X_n = \{1, 2, \ldots, n\}$ and let $\mathcal{PO}_n$ be the semigroup of all order-preserving partial transformations of X_n. Higgins [3] contains some nice asymptotic results concerning a certain semigroup of transformations and references to other similar works. After this introductory section, we quote the main results of [7] and [8] in Section 2. In Section 3 we obtain among other things asymptotic formulae for r_n and c_n. In Section

4 we obtain the generating functions for r_n and c_n, while in Section 5 we obtain their integral representations.

2 Combinatorial Results

Let $X_n = \{1, 2, \ldots, n\}$ be a finite chain, and let $\alpha : X_n \to X_n$ be a partial transformation. The set of all partial transformations is denoted by P_n and called the partial transformation semigroup (under composition) or the partial symmetric semigroup. We shall call α in P_n *order-decreasing* (*order-increasing*) or simply *decreasing* (*increasing*) if $x\alpha \leq x$ ($x\alpha \geq x$) for all x in Dom α, and α is *order-preserving* if $x \leq y$ implies $x\alpha \leq y\alpha$ for x, y in Dom α. Subsemigroups of P_n that have been investigated recently by the authors are:

$$\mathcal{PC}_n = \{\alpha \in P_n : (\forall\, x, y \in \text{Dom } \alpha)\, x\alpha \leq x \wedge (x \leq y \Rightarrow x\alpha \leq y\alpha)\} \tag{2.1}$$

the semigroup of all decreasing and order-preserving partial transformations of X_n, and

$$\mathcal{PO}_n = \{\alpha \in P_n : (\forall\, x, y \in \text{Dom } \alpha)\, x \leq y \Rightarrow x\alpha \leq y\alpha\} \tag{2.2}$$

the semigroup of all order-preserving partial transformations of X_n. In [7], the authors showed among other things the following results.

Theorem 2.1 [7, Theorem 2.12]. *Let $\mathcal{PC}_n$ be as defined in (2.1). Then $|\mathcal{PC}_n| = r_n$, the double Schröder number where*

$$\begin{aligned} r_n &= \frac{1}{n+1}\sum_{r=0}^{n}\binom{n+1}{n-r}\binom{n+1}{r} = \sum_{r=0}^{n}\frac{1}{r+1}\binom{n}{r}\binom{n+r}{n} \\ &= \sum_{r=0}^{n}\frac{1}{r+1}\binom{n+r}{2r}\binom{2r}{r}. \end{aligned} \tag{2.3}$$

Theorem 2.2 [7, Proposition 3.5]. *Let $\mathcal{PC}_n$ be as defined in (2.1) and let $E(\mathcal{PC}_n)$ be its set of idempotents. Then $|E(\mathcal{PC}_n)| = (3^n + 1)/2$.*

In [3], Gomes and Howie obtained

Theorem 2.3 [3, Theorem 3.1]. *Let $\mathcal{PO}_n$ be as defined in (2.2). Then*

$$|\mathcal{PO}_n| = c_n = \sum_{r=0}^{n} \binom{n}{r} \binom{n+r-1}{r}.$$

More recently, the authors in [8] obtained the above result for $\mathcal{PO}_n$ and

Theorem 2.4 [8, Theorem 3.8]. *Let $\mathcal{PO}_n$ be as defined in (2.2), and let $E(\mathcal{PO}_n)$ be its set of idempotents. Then*

$$|E(\mathcal{PO}_n)| = (\sqrt{5})^{n-1} \left[\left(\frac{\sqrt{5}+1}{2} \right)^n - \left(\frac{\sqrt{5}-1}{2} \right)^n \right] + 1.$$

3 Asymptotic Results

Let $P_n(x)$ be the n-th degree Legendre polynomial. Then it is known that [4, p. 404] for large n

$$P_n(x) \sim \frac{(x+\sqrt{x^2-1})^{n+1/2}}{\sqrt{2\pi n\sqrt{x^2-1}}}. \tag{3.1}$$

Now consider the polynomial

$$Q_n(x) = \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k} x^k$$

then

$$r_n(x) = \int_0^x Q_n(t)dt = \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k} \frac{1}{k+1} x^{k+1} \tag{3.2}$$

and so using (2.3) we have

$$r_n(1) = \sum_{k=0}^{n} \frac{1}{k+1} \binom{n+k}{2k} \binom{2k}{k} = |\mathcal{PC}_n| = r_n.$$

From [10, p. 78] we have

$$r_n'(x) = Q_n(x) = P_n(1+2x).$$

Also integrating the well-known recurrence

$$P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x)$$

we obtain

$$P_{n+1}(x) - P_{n-1}(x) = 2(2n+1) r_{n(\frac{x-1}{2})} \tag{3.3}$$

which implies

$$r_n(x) = \frac{P_{n+1}(2x+1) - P_{n-1}(2x+1)}{2(2n+1)}$$

and so

$$r_n(1) = \frac{P_{n+1}(3) - P_{n-1}(3)}{2(2n+1)}. \tag{3.4}$$

On the other hand using the ordinary occurrence

$$(n+1)P_{n+1}(z) - 2(n+1)zP_n(z) + nP_{n-1}(z) = 0,$$

we get

$$(n+1)P_{n+1}(3) - 3(2n+1)P_n(3) - nP_{n-1}(3) = 0. \tag{3.5}$$

And from (3.4) and (3.5) we get

$$r_n(1) = \frac{3P_n(3) - P_{n-1}(3)}{2(n+1)}. \tag{3.6}$$

Thus we now have

Proposition 3.1 *Let $r_n(1)$ be as defined in (3.2). Then for large n*

$$r_n(1) = |\mathcal{PC}_n| \sim (\sqrt{2}+1)^{2n+1}/(2^{3/4} n\sqrt{\pi n}).$$

Proof. From (3.6) and (3.1) successively we have

$$\begin{aligned} r_n(1) &= \frac{3P_n(3) - P_{n-1}(3)}{2(n+1)} \sim \frac{\frac{3(\sqrt{2}+1)^{2n+1}}{2^{5/4}\sqrt{\pi n}} - \frac{(\sqrt{2}+1)^{2n-1}}{2^{5/4}\sqrt{\pi n}}}{2n} \\ &= \frac{(\sqrt{2}+1)^{2n-1}[3(\sqrt{2}+1)^2 - 1]}{2^{9/4} n\sqrt{\pi n}} = \frac{(\sqrt{2}+1)^{2n-1}(8+6\sqrt{2})}{2^{9/4} n\sqrt{\pi n}} \\ &= \frac{(\sqrt{2}+1)^{2n-1} \cdot 2^{3/2}(3+2\sqrt{2})}{2^{9/4} n\sqrt{\pi n}} = \frac{(\sqrt{2}+1)^{2n+1}}{2^{3/4} n\sqrt{\pi n}} \end{aligned}$$

as required. ■

Theorem 3.2 *Let $\mathcal{PC}_n$ be as defined in (2.1) and let $E(\mathcal{PC}_n)$ be its set of idempotents. Then for large n*

$$\frac{|\mathcal{PC}_n|}{|E(\mathcal{PC}_n)|} = \frac{2r_n(1)}{3^n+1} \sim 2^{1/4}(\sqrt{2}+1)^{2n+1}/3^n n\sqrt{\pi n}.$$

Proof. It follows directly from Proposition 3.1, Theorem 2.2 and the fact that $(3^n+1)/2 \sim 3^n/2$ for large n. ■

We also deduce (from Proposition 3.1) the following lemma.

Lemma 3.3 $\lim_{n\to\infty} \frac{r_{n+1}}{r_n} = 3 + 2\sqrt{2} = (\sqrt{2}+1)^2$.

Next we obtain similar results for $\mathcal{PO}_n$, however, first we quote from [8] the following lemma:

Lemma 3.4 [8, Lemma 2.11]. *For all $n > 0$, we have*

$$2c_n = (n+1)r_n - (n-1)r_{n-1}.$$

Proposition 3.5 *Let $\mathcal{PO}_n$ be as defined in (2.2). Then for large n*

$$|\mathcal{PO}_n| = c_n \sim (\sqrt{2}+1)^{2n}/(2^{3/4}\sqrt{\pi n}).$$

Proof. The result follows easily from Lemma 3.4, Proposition 3.1 and same techniques as in the proof of Proposition 3.1. ■

Theorem 3.6 *Let $\mathcal{PO}_n$ be as defined in (2.2) and let $E(\mathcal{PO}_n)$ be its set of idempotents. Then for large n*

$$|E(\mathcal{PO}_n)| = e_n \sim \frac{1}{\sqrt{5}}\left(\frac{5+\sqrt{5}}{2}\right)^n.$$

Proof. It follows directly from Theorem 2.4 and the fact that $\left(\frac{\sqrt{5}-1}{2}\right)^n \sim 0$, for large n. ■

Theorem 3.7 *Let $\mathcal{PO}_n$ be as defined in (2.2) and let $E(\mathcal{PO}_n)$ be its set of idempotents. Then for large n*

$$\frac{|\mathcal{PO}_n|}{|E(\mathcal{PO}_n)|} = \frac{c_n}{e_n} \sim 2^{n-3/4}\sqrt{\frac{5}{\pi n}}\left(\frac{3+2\sqrt{2}}{5+\sqrt{5}}\right)^n.$$

Proof. It follows directly from Proposition 3.5 and Theorem 3.6. ■

Also from Proposition 3.5 we deduce

Lemma 3.8 $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 3 + 2\sqrt{2} = (\sqrt{2}+1)^2$.

Two further results associating r_n and c_n, whose proofs follow directly from Propositions 3.1 and 3.5 are:

Lemma 3.9 $\lim_{n\to\infty} \frac{r_n}{c_n} = 0.$

Lemma 3.10 $\lim_{n\to\infty} \frac{nr_n}{c_n} = \sqrt{2}+1.$

We conclude the section with the following result (from [1]) and some of its consequences.

Lemma 3.11 [1, p. 292]. $\binom{(a+b)n}{an} \sim \frac{(a+b)^{n(a+b)+1/2}}{a^{an+1/2}b^{bn+1/2}\sqrt{2\pi n}}.$

Thus, with $a = b = 1$, we have $\binom{2n}{n} \sim \frac{2^{2n+1/2}}{\sqrt{2\pi n}} = \frac{4^n}{\sqrt{\pi n}}$. Hence we have

Theorem 3.12 *Let $\mathcal{C}_n$ be the semigroup of all decreasing and order-preserving full transformations of X_n. Then for large n, we have* $|\mathcal{C}_n| = \frac{1}{n+1}\binom{2n}{n} \sim \frac{4^n}{n\sqrt{\pi n}}.$

Theorem 3.13 [5, Theorem 3.19]. *Let $\mathcal{C}_n$ be the semigroup of all decreasing and order-preserving full transformations of X_n, and let $E(\mathcal{C}_n)$ be its set of idempotents. Then for large n, we have* $\frac{|E(\mathcal{C}_n)|}{|\mathcal{C}_n|} \sim \frac{2^{n-1}n\sqrt{\pi n}}{2^{2n}} = \frac{n\sqrt{\pi n}}{2^{n+1}}.$

4 Generating Functions

Recall from [7] that the small Schröder number is usually denoted by s_n and defined as $s_0 = 1, s_n = r_n/2$ $(n \geq 1)$. (Note that our s_n is s_{n+1} in [11].) Thus from (2) on page 349 of [11] we easily deduce that

$$\sum_{n\geq 1} s_n x^n = \frac{1}{4x}(1+3x-\sqrt{1-6x+x^2}),$$

from which it follows that

$$\sum_{n\geq 1} r_n x^n = \frac{1}{2x}(1+3x-\sqrt{1-6x+x^2}). \tag{4.1}$$

Hence we deduce from (4.1)

Theorem 4.1 *Let r_n be as defined in (3.2). Then the generating function for r_n is given by*

$$\sum_{n\geq 0} r_n x^n = \frac{1}{2x}(1 - x - \sqrt{1 - 6x + x^2}).$$

To deduce the generating function for c_n, first we establish

Lemma 4.2 *For all $n \geq 0$, we have $c_{n+1} - c_n = (2n+1)r_n$.*

Proof. This could be verified routinely by using the first expression for r_n in Theorem 2.1 and Theorem 2.3. ■

Lemma 4.3 *For all $n \geq 1$, we have $c_n = \dfrac{P_n(3) + P_{n-1}(3)}{2}$.*

Proof. First observe that from Lemma 4.2 and (3.3) we have

$$c_{n+1} - c_n = \frac{P_{n+1}(3) - P_{n-1}(3)}{2}$$

and so

$$c_n - c_{n-1} = \frac{P_n(3) - P_{n-2}(3)}{2}, \ldots, c_2 - c_1 = \frac{P_2(3) - P_0(3)}{2}.$$

Hence

$$c_n = \frac{P_n(3) + P_{n-1}(3) - P_1(3) - P_0(3)}{2} + c_1 = \frac{P_n(3) + P_{n-1}(3)}{2}$$

since $P_0(3) = 1, P_1(3) = 3$ and $c_1 = 2$. ■

Now from Ex. 11 on page 78 of [10] the generating function for $P_n(3)$ is given by

$$g(t) = \sum_{n\geq 0} P_n(3)t^n = (1 - 6t + t^2)^{-1/2}.$$

This together with Lemma 4.3 yield the generating function for c_n.

Theorem 4.4 *Let c_n be as defined in Theorem 2.3. Then the generating function for c_n is given by*

$$\sum_{n\geq 0} c_n t^n = \frac{1 + (1+t)g(t)}{2} = \frac{1 + (1+t)(1 - 6t + t^2)^{-1/2}}{2}.$$

5 Integral Representations for r_n and c_n

The integral representation for the n-th degree Legendre polynomials [9, p. 172] is given by

$$P_n(z) = \frac{1}{\pi}\int_0^\pi (z + \sqrt{z^2-1}\cos\theta)^n d\theta. \tag{5.1}$$

Now from (3.6) and (5.1) we deduce

Theorem 5.1 *Let r_n be as defined in Theorem 2.1. Then*

$$r_n = \frac{1}{\pi(n+1)}\int_0^\pi (4+3\sqrt{2}\cos\theta)(3+2\sqrt{2}\cos\theta)^{n-1}d\theta.$$

Similarly, from Lemma 4.3 and (5.1) we deduce

Theorem 5.2 *Let c_n be as defined in Theorem 2.3. Then*

$$c_n = \frac{1}{\pi}\int_0^\pi (2+\sqrt{2}\cos\theta)(3+2\sqrt{2}\cos\theta)^{n-1}d\theta.$$

These representations are much faster to work with when computing large values of r_n and c_n than the combinatorial identities and recurrences. Moreover, they are more accurate than the asymptotic formulae.

Acknowledgment. We would like to gratefully acknowledge support from the King Fahd University of Petroleum and Minerals.

REFERENCES

1. L. Comtet, "Advanced Combinatorics: the art of finite and infinite expansions," (D. Reidel Publishing Company, Dordrecht, Holland: 1974).

2. G. U. Garba, On the idempotent ranks of certain semigroups of order-preserving transformations, *Portugaliae Mathematica* **51** (1994), 185-204.

3. G. M. S. Gomes and J. M. Howie, On the ranks of certain semigroups of order-preserving transformations, *Semigroup Forum*, **45** (1992), 272-282.

4. P. Henrici, *Applied and Computational Complex Analysis, Vol. 2* (Wiley, 1977).

5. P. M. Higgins, Combinatorial results for semigroups of order-preserving mappings *Math. Proc. Camb. Phil. Soc.* **113** (1993), 281-296.

6. J. M. Howie, *Real Analysis* (Springer, 2001).

7. A. Laradji and A. Umar, Combinatorial results for semigroups of order-decreasing partial transformations, *submitted*.

8. A. Laradji and A. Umar, Combinatorial results for semigroups of order-preserving partial transformations, *Journal of Algebra* (to appear).

9. N. N. Lebedev, *Special functions and their applications*, (Dover, 1972); English translation, translated by R. A. Silverman.

10. J. Riordan, *Combinatorial Identities*, (John Wiley and Sons, New York, 1968).

11. R. P. Stanley, Hipparchus, Plutarch, Schröder and Hough, *Amer. Math. Monthly* **104** (1997), 344-350.