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M. Iqbal

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

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M. IQBAL
Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia

Internet e-mail address: miqbal@kfupm.edu.sa

Abstract

In this paper, a method is described for inverting the Mellin transform which uses an expansion in Laguerre polynomials and converts Mellin transform to Laplace transform, then a regularization method is used to recover the original function of the Mellin transform. The performance of the method is illustrated by the inversion of the test functions available in the literature. Results are shown in the table.

1. Introduction

We study the problem of finding the numerical solution of

$$\int_0^\infty \phi(t)t^{s-1}dt = p(s). \tag{1.1}$$

The Mellin transform defined by formula [5] as (1.1) is one of the most important integral transforms. This arises in a natural way in the solution of boundary value problems concerning an infinite wedge.

If $p(s)$ and $\phi(t)$ in (1.1) satisfy appropriate conditions (see Sneddon [14]) one can prove that the relationship $p(s) = M\phi(t)$ defines a one-one correspondence between $\phi(t)$ and $p(s)$. The properties of Mellin transform and the inversion theorem is given in [14], [9] and [10].

Equation (1.1) has the form of a Fredholm integral equation of the first kind and it is well known that the problem of solving such equations is basically ill-conditioned. Many physicists have discovered after much wasted effort that it is essential to understand this feature before attempting to compute solutions.

The ill-posedness of Laplace transform inversion in the case where $\phi(t) \in L^2(\mathbb{R})$ and $g(s)$ is known for all real and positive values of s, can be investigated by means of the Mellin transform [11, 13]. Several other methods have been proposed and a review and comparison is given in Davis [2] and Talbot [16]. Regularization methods have been discussed by Varah [19] and Essa and Delves [7], and [3], [4], [6].

In particular the theory is used to tackle the Laplace transform inversion in a well-conditioned (regularized) manner. This difficult numerical problem which is frequently encountered by physicists and engineers, is still the subject of much attention in the literature.

In [19] the author published an extremely pessimistic paper. He concluded that methods involving free parameters can be relatively stable with respect to perturbations in the data. However, solutions obtained may differ from one another so that it becomes difficult to see which solution is the best. This is a direct consequence of the ill-conditioning of the discrete problem. More precisely, the proposed method stems from an attempt to apply the complex inversion schemes to the real inversion problems.

Finally, we include some specific numerical examples to illustrate and demonstrate clearly the need to consider information content in order to avoid obtaining meaningless results.

2. Conversion to the Laplace Transform

Since the Mellin transform is essentially the two-sided Laplace transform, we will consider an approach to reduce the inversion of Mellin transforms to the inversion of Laplace transforms. The approach is based on the series expansion of the inversion of Mellin transform, this had been proposed in (Theocaris [18]).

For the numerical inversion of the Mellin transform, we consider the following expansion of Laguarre polynomials (Szego [15]), [12].

$$\phi(t) = \frac{1}{2}qte^{-\frac{qt}{2}} \sum_{k=0}^{\infty} A_k(q)\ell_k\left(\frac{1}{2}qt\right) ., \tag{2.1}$$

where

$$A_k(q) = \sum_{m=0}^{k} \left(\begin{array}{c} k \\ m \end{array}\right) \left(-\frac{1}{2}q\right)^m b_m \tag{2.2}$$

with

$$b_m = \frac{p(m+1)}{m!}, \quad q \text{ being a free parameter .} \tag{2.3}$$

We consider the function $g(s)$ which is represented by

$$g(s) = \sum_{m=}^{\infty} (-1)^m b_m s^m \qquad s \in C \text{ with } \mathrm{Re}(s) > 0, \tag{2.4}$$

where b_m is given by (2.3).

By using Equation (1.1) in (2.4), one can easily see that $g(s)$ is a Laplace transform. In fact, if we set

$$\mathcal{L}[\phi; s] = \int_0^{\infty} e^{-st}\phi(t)dt$$

it follows that

$$g(s) = \mathcal{L}[\phi; -s]. \tag{2.5}$$

The integral in (2.5) may not exist; however, we may always provide values of α such that the Laplace transform related to $p(s+\alpha)$ is

$$g(s) = \mathcal{L}]\phi(t)t^{\alpha}; -s] \text{ which does exist.}$$

3. Regularization Method to solve Laplace Transform

Now we are interested in solving (2.5). We make the following substitution. Let

$$s = a^x \text{ and } t = a^{-y}, \quad a > 1. \tag{3.1}$$

Then

$$g(a^x) = \int_{-\infty}^{\infty} \log a e^{-a^{x-y}} \phi(a^{-y}) a^{-y} dy.$$

Multiplying both sides by a^x, we obtain the convolution equation

$$\int_{-\infty}^{\infty} K(x-y)F(y)dy = G(x). \tag{3.2}$$

where

$$\left.\begin{array}{l} G(x) = a^x g(a^x) = sg(s) \\ K(x) = (\log a)a^x e^{-a^x} = (\log a)se^{-s} \\ F(y) = \phi(a^{-y}) = \phi(t). \end{array}\right\} \tag{3.3}$$

In order that we can apply our deconvolution and regularization method to equation (3.2), it is necessary that $G(x)$ has usually compact support, that is $G(x) \to 0$ as $x \to \pm\infty$.

In Tikhonov [18] regularization, the approximate solution F_λ in (3.2) is defined as

$$C(F;\lambda) = [\|KF - G\|^2 + \lambda\Omega(f)], \tag{3.4}$$

which is minimized over the subspace $H^p \in L_2$ and $\lambda > 0$ is a regularization parameter.

Here Ω is some non-negative 'stabilizing' functional which controls the sensitivity of the regularized solution F_λ to perturbation in G.

We shall restrict our attention to p-th order (In this paper $p = 2$) regularization of the form

$$C(F;\lambda) = \|KF - G\|_2^2 + \lambda\|F^p\|_2^2, \tag{3.5}$$

which is minimized over the subspace $H^p \in L^2$.

Both norms in (3.5) are L_2, F^p denotes the p-th derivative of F and λ the regularization parameter.

3.1 p-th Order Regularization

Consider the smoothing functional $C(F;\lambda)$ of equation (3.5) with $\Omega(F) = \|F^{(2)}\|_2^2$. In the case of convolution equation (3.2) can be written as

$$C(F;\lambda) = \|K(x) * F(x) - G(x)\|_2^2 + \lambda\|F^{(2)}\|_2^2, \tag{3.6}$$

where $*$ denotes convolution and both norms in (3.6) are L^2.

Using Plancherel's identity, the convolution theorem for Fts (Fourier transforms) and the property $(F^{(p)})\widehat{\ } = (i\omega)^p\widehat{F}$ where $\wedge$ denotes Fourier transform and $\vee$ inverse Fourier transform and $p = 2$.

We can write (3.6) as

$$C(F;\lambda) = \frac{1}{2\pi}\|\widehat{K}(\omega)\widehat{F}(\omega) - \widehat{G}(\omega)\|_2^2 + \lambda\|(i\omega)^p\widehat{F}(\omega)\|_2^2 \tag{3.7}$$

(3.7) is minimized with respect to F, where

$$\widehat{F}(\omega) = \frac{\overline{\widehat{K}}\widehat{G}}{|\widehat{K}|^2 + \lambda\omega^{2p}} = z(\omega;\lambda)\frac{\widehat{G}(\omega)}{\widehat{K}(\omega)} \tag{3.8}$$

and

$$z(\omega;\lambda) = \frac{|\widehat{K}|^2}{|\widehat{K}|^2 + \lambda\omega^{2p}}. \tag{3.9}$$

$z(\omega;\lambda)$ is called the p-th order filter function or stabilizer. Therefore, (3.8) can be written as

$$F_\lambda(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} z(\omega;\lambda)\frac{\widehat{G}(\omega)}{\widehat{K}(\omega)}\exp(i\omega y)d\omega. \tag{3.10}$$

$F_\lambda(y)$ in (3.10) can be approximated by

$$F_{N,\lambda}(x) = \sum_{q=0}^{N-1}\frac{\widehat{G}_{N,q}}{\widehat{K}_{N,q}} z_{q;\lambda}$$

where $z_{q;\lambda} = \dfrac{|\widehat{K}_{N,q}|^2}{|\widehat{K}_{N,q}|^2 + \lambda\omega_q^{2p}}$ is a filter function dependent on a parameter λ and $\widetilde{\omega}_q = 2\pi q$ (cut-off frequency)

$$\omega_q = \begin{cases} \widetilde{\omega}_q\ , & 0 \le q < \frac{1}{2}N \\ \widetilde{\omega}_{N-1}\ , & \frac{1}{2}N \le q \le N-1 \end{cases}.$$

The optimal value of λ in (3.9) is still to be determined and N is the number of data points.

4. Choice of Regularization Parameter λ

The idea of generalized cross-validation (GCV) method is simple to understand. Suppose we ignore the j-th data point G_j, and define filtered solution $F^j_{N,\lambda}(x) \in T_N$ (T_N stands for trigonometric polynomial of degree N) as the minimizer of $\sum_{\substack{n=0\\ n\neq j}}^{N-1}[(K_N * F)(x_n) - G(x_n)]^2 + \lambda\|F^{(p)}(x)\|_2^2$. Then we get a vector $G^{(j)}_{N,\lambda,j} \in R_n$ defined by

$$G^{(j)}_{N,\lambda} = KF^{(j)}_{N,\lambda}. \tag{4.1}$$

Now the j-th element $G^{(j)}_{N,\lambda,j}$ in (4.1) should predict the missing value G_j. We may thus construct the weighted mean square prediction error over all j, i.e.,

$$V(\lambda) = \frac{1}{N}\sum_{j=0}^{N-1} Q_j(\lambda)\left[G^{(j)}_{N,\lambda,j} - G_j\right]^2. \tag{4.2}$$

The filtered solution to the problem should minimize the mean square prediction error in (4.2). The minimizer $V(\lambda)$ in the form given by (4.2) is a time-consuming process. The alternative expression which depends on particular choice of weights resulting in considerable simplification. Let

$$F_{N,\lambda}\left(F_{N,\lambda}(x_0) + F_{N,\lambda}(x_1) + \cdots + F_{N,\lambda}(x_{N-1})\right)^T \tag{4.3}$$

and define $G_{N,\lambda} = KF_{N,\lambda}$. Then there exists a matrix $A(\lambda)$, called an influence matrix such that

$$G_{N,\lambda} = A(\lambda)G_N. \tag{4.4}$$

Let $K = \operatorname{diag}(\widehat{K}_{N,q})$ and $\widehat{z} = \operatorname{diag}(z_{q,\lambda})$, then from (4.1), we have $F_{N,\lambda} = \psi(\widehat{K})^{-1}\widehat{z}\widehat{G}_N$, where

$$\widehat{G}_N = \psi^H G_n \text{ and so } A(\lambda) = \psi\widehat{z}\psi^H. \tag{4.5}$$

Also $K = \psi\widehat{K}\psi^H$ where ψ is the unitary matrix with elements

$$\psi_{rs} = \frac{1}{\sqrt{N}}\exp\left(\frac{2\pi i}{N}rs\right), \quad r, s = 0, 1, \ldots, N-1. \tag{4.6}$$

Now the choice of weights,

$$Q_j(\lambda) = \left(\frac{(1 - a_{jj}(\lambda))}{(1/N)\operatorname{Trace}(I - A(\lambda)))}\right)^2, \quad j = 0, 1, 2, \ldots, N-1, \tag{4.7}$$

where $A(\lambda)$ is the influence matrix in (4.4).

The expression (4.2) to be written as

$$V(\lambda) = \frac{(1/N)\|I - A(\lambda)G_N\|_2^2}{[(1/N)\operatorname{Trace}(I - A(\lambda))]^2}, \tag{4.8}$$

and using (4.5), we get

$$V(\lambda) = \frac{(1/N)\|(I-\widehat{z})\widehat{G}_N\|_2^2}{[(1/N)\text{Trace}(I-\widehat{z})]^2},$$

i.e.

$$V(\lambda) = \frac{1/N \sum_{q=0}^{N-1} (I - z_{q;\lambda})^2 |\widehat{G}_{N,q}|^2}{\left(1/N \sum_{q=0}^{N-1} (1 - z_{q;\lambda})\right)^2}. \tag{4.9}$$

The expression in (4.9) is minimized using the quadratic interpolation technique to obtain a minimum.

5. The Convergence of the Method

In this section we give the proof of convergence of the method. Let X be a separable Hilbert space, K be a compact linear operator from X to X, the subspaces $\{X_n\}$ are ultimately dense in X, i.e. $\bigcup_{n=1}^{\infty} X_n = X$.

Step I. By use of Tikhonov regularization, the problem of finding the best approximate solution (an element of minimal norm that minimizes the residual $\|KF-G\|$ of the equation

$$KF = G \tag{5.1}$$

is converted into solving the following equation

$$(\lambda I + K^*K)F = K^*G, \tag{5.2}$$

where λ is the regularization parameter.

In order to solve equation (5.2) we use Galerkin method. For the sake of simplicity of the proof we consider $X_n = Y_n$ (The case of $X_n \neq Y_n$ is similar). Let

U_n be an orthonormal projection from X to X_n, the equation (5.2) is equivalent to finding $F_{n,\lambda} \in X_n$ such that

$$\lambda F_{n,\lambda} + U_n K^* K F_{n,\lambda} = U_n K^* G. \tag{5.3}$$

Step II. Using (R.H. Li [25]), suppose X be a separable Hilbert space, K be a compact linear operator from X to X, the subspaces $\{X_n\}$ are ultimately dense in X, F_λ be the solution of (5.2), then for any given constant $\lambda > 0$, the Galerkin equation (5.3) has a unique solution $F_{n,\lambda} \in X_n$ satisfying

$$\lim_{n\to\infty} F_{n,\lambda} = F_\lambda. \tag{5.4}$$

Step III. Suppose that the integral operator K is defined by Equation (5.1), the kernel $K(x-y) \in L^2$. Choose $X_n = S^k_{\mu^n}$ $(k \geq 1, \mu > 1)$.

Let U_n denote an orthogonal projection from X to X_n. $F_\lambda(y) \in L^2[0,T]$, be the unique solution of the equation (5.2), then for any given $G \in L^2$, there exists an N_0 such that $n > N_0$, the equation (5.3) has a unique solution $F_{n,\lambda} \in X_n$ satisfying

$$\lim_{n+\infty} F_{n,\lambda} = F_\lambda,$$

particularly, we have

$$\|F_\lambda - F_{n,\lambda}\| \leq 2\lambda \|(\lambda I + \overline{K})^{-1}\| \inf\{\|F_\lambda - F_n\|, U_n \in X_n\}. \tag{5.5}$$

where $\overline{K} = K^*K$, $\lambda > 0$ is a regularization parameter.

Step IV. Since K is a compact operator, $\overline{K} = K^*K$ is also compact and has non-negative eigenvalues. Therefore for each parameter $\lambda > 0$, $(\lambda I + K)^{-1}$ exists and is bounded. Let U_n be an orthogonal projection from X to X_n, from (5.2), we have

$$\lambda U_n F_\lambda + U_n K F_\lambda = U_n \overline{G},$$

where $\overline{G} = K^*G$. Comparing with (5.3), we have

$$\lambda(U_nF_\lambda - F_{n,\lambda}) + U_n\overline{K}(F_\lambda - F_{n,\lambda}) = 0$$
$$(\lambda I + U_n\overline{K})(F_\lambda - F_{n,\lambda}) = \lambda(F_\lambda - U_nF_\lambda). \quad (5.6)$$

On the other hand,

$$\begin{aligned} \lambda I + U_nK &= \lambda I + \overline{K} + U_n\overline{K} - \overline{K} \\ &= (\lambda I + \overline{K})[I + (\lambda I + \overline{K})^{-1}(U_n\overline{K} - \overline{K})]. \end{aligned} \quad (5.7)$$

From the construction of the space X_n, we know the subspaces $\{X_n\}$ are ultimately dense in XR, then for $n \to \infty$, $\|U_n\overline{K} - \overline{K}\| \to 0$ and then $\exists$ an N, such that for $n > n$, we have

$$\|[I + (\lambda I + \overline{K})^{-1}(U_n\overline{K} - \overline{K})^{-1}]\| \leq \frac{1}{1 - 1/2} = 2. \quad (5.8)$$

Hence, $(\lambda I + U_n\overline{K})^{-1} = [I + (\lambda I + \overline{K})^{-1}(U_n\overline{K} - \overline{K})]^{-1}(\lambda I + \overline{K})^{-1}$ exists and is bounded. Therefore, from (5.6) $\sim$ (5.8) we have

$$\begin{aligned} \|F_\lambda - F_{n,\lambda}\| &\leq \|(\lambda I + U_n\overline{K})^{-1}\|(\lambda\|F_\lambda - U_nF_\lambda)\| \\ &\leq 2\lambda\|(\lambda I + \overline{K})^{-1}\|\inf\{F_\lambda - F_n\|, \quad F_n \in X_n\}. \end{aligned}$$

6. The Numerical Procedure

In order to solve (3.2) we need to choose two numbers $x_{\min}$ and $x_{\max}$ such that $|G(x)| < \epsilon$, whenever $x < x_{\max}$ and $x > x_{\min}$. In what follows we choose $\epsilon = (10)^{-4} \max|G(x)|$.

We find $x_{\min}$ and $x_{\max}$ as the smallest and largest solutions of the nonlinear equation $G(x) = \epsilon$. We may then pose the deconvolution problem (3.2) on the interval $[0, T]$, where $T = x_{\max} - x_{\min}$. Since the size of the essential support of

$G(x)$ depends on 'a', we write $T = T_a$ for a fixed number N of equidistant data points $\{x_n\}$, the spacing $h = \dfrac{Ta}{N}$.

We have minimized (4.9) with respect to λ for values of $a > e$ and computed the L_∞ error of the resulting solution with the values of true solution. We found that the minimum value of $V(\lambda)$ for optimal value of λ for which the L_∞ error of the regularized solution is the least.

7. Error Analysis

Lemma 1. *Let F_λ denote the solution of*

$$\min \|KF - G\|_2^2 + \lambda^2 \|F\|_2^2 \qquad \text{([8] and [13])}$$

and let $\overline{F}_\lambda$ denote the solution to

$$\min \|KF - \overline{G}\|_2^2 + \lambda^2 \|F\|_2^2.$$

Then

$$\frac{\|F_\lambda - \overline{F}_\lambda\|_2^2}{\|F_\lambda\|_2^2} \leq \frac{\sigma \|e\|_2^2}{2\lambda \|G_\lambda\|_2^2},$$

where

$$G_\lambda = KF_\lambda \text{ and } e = G - \overline{G}.$$

8. Numerical Examples

Example 1. This example has been taken from Theocaris [(17, Case 5)],

$$\begin{aligned} p(s) &= \alpha^{-s}\Gamma(s) \qquad \mathrm{Re}(s) > 0 \\ \phi(t) &= e^{-\alpha t} \\ g(s) &= \frac{1}{s+\alpha}, \qquad \text{where } \alpha = 1. \end{aligned}$$

The numerical calculations are given in the table.

Example 2. This example has been taken from Theocaris [(17, p. 79)],

$$
\begin{aligned}
p(s) &= (\alpha^2+\beta^2)^{-1/2}\Gamma(s)\sin[s\tan^{-1}(\beta/\alpha)], \quad \alpha=4.0,\ \beta=1.2\\
\phi(t) &= e^{-\alpha t}\sin\beta t\\
g(s) &= \frac{u\sin v}{s^2+2us\cos v+u^2}, \qquad \text{where } u=(\alpha^2+\beta^2)^{-1/2}, v=\tan^{-1}(\beta/\alpha).
\end{aligned}
$$

The numerical calculations are given in the table.

Example 3. This example has been taken from Varah [19].

$$
\begin{aligned}
p(s) &= \frac{\Gamma(\alpha+1)}{(s+\beta)^{\alpha+1}}\\
\phi(t) &= t^{\alpha}e^{-\beta t}\\
g(s) &= (\beta)^{-(s+\alpha)}\Gamma(s+\alpha) \quad \mathrm{Re}(s)>0, \quad \alpha=3.0, \ \ \beta=1.0.
\end{aligned}
$$

The numerical calculations are given in the table.

Example 4. This example has been taken from Brianzi [1] and McWhirter [11].

$$
\begin{aligned}
p(s) &= \frac{\Gamma(\alpha)}{(s+\beta)^{\alpha+1}}\\
\phi(t) &= t^{\alpha}e^{-\beta t}, \quad \alpha=1, \ \ \beta=1\\
g(s) &= (\beta)^{-(s+\alpha)}\Gamma(s+\alpha) \quad \mathrm{Re}(s)>0.
\end{aligned}
$$

The numerical calculations are given in the table.

Table

Example	T The period	H step size	a $(a > e)$	λ The regularization parameter	$V(\lambda)$	error norm $\|F - F_\lambda\|_\infty$
1	10.50	0.16406	10	0.7021×10^{-8}	0.55122	0.0125
2	9.0	0.14063	10	0.213×10^{-10}	0.1074×10^{-8}	0.0004
3	14.50	0.2265	5	0.341×10^{-9}	0.3872×10^{2}	0.0021
4	12.50	0.04883	10	0.213×10^{-11}	0.839×10^{2}	0.0062

9. Conclusion

Our method worked well over all the four test examples. The results obtained are perfect as shown in the table.

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