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On 3-Normal Operators

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Abstract

In this paper, we introduce the class of 3-normal operators and give some characterizations of 3-normal operators. We study the relations between the class of 3-normal operators and some other classes of operators.

1

Let H be a Hilbert space and let $L(H)$ denote the algebra of all bounded linear operators acting on H. In [3], the author introduces the class of 2-normal operators: $T \in L(H)$ is 2-normal if $AB^2 = B^2A$ and $BA^2 = A^2B$ where A and B are the real and the imaginary parts of T. Several characterizations of 2-normal operators are given in [3] among which: (i) $T \in L(H)$ is 2-normal if and only if $T^2T^* = T^*T^2$ (ii) $T \in L(H)$ is 2-normal if and only if T^2 is normal. The class of all 2-normal operators is denoted by $[2N]$.

In this paper we introduce the class of 3-normal operators: $T \in L(H)$ is 3-normal if $T^3T^* = T^*T^3$. The class of all 3-normal operators will be denoted by $[3N]$.

In Section 2 we prove some general results about a 3-normal operator. In Section 3 we give some characterizations of a 3-normal operator. In Section 4 we study the relation between the two classes $[2N]$ and $[3N]$. We show that the two classes are independent and we give some conditions under which an operator $T \in [2N] \cap [3N]$ becomes normal.

2 In this section we prove some general results about a 3-normal operator.

Proposition 2.1 *Let $T \in [3N]$, then*

(i) $T^* \in [3N]$.

(ii) If T^{-1} exists, then $T^{-1} \in [3N]$.

(iii) If $A \in L(H)$ and A is unitarily equivalent to T, then $A \in [3N]$.

(iv) $\alpha T \in [3N]$ for all complex numbers α.

(v) If M is a closed subspace of H that reduces T, then $T|M$ – the restriction of T to M – is in $[3N]$.

Proof. (i) Since $T \in [3N], T^3T^* = T^*T^3$. Thus $(T^*)^*(T^3)^* = (T^3)^*(T^*)^*$ which implies that $(T^*)^*(T^*)^3 = (T^*)^3(T^*)^*$. Thus $T^* \in [3N]$

(ii) Suppose that T^{-1} exists. Then $(T^{-1})^3(T^{-1})^* = (T^3)^{-1}(T^*)^{-1} = (T^*T^3)^{-1} = (T^3T^*)^{-1} = (T^*)^{-1}(T^3)^{-1} = (T^{-1})^*(T^{-1})^3$. Hence $T^{-1} \in [3N]$.

(ii) Let $A \in L(H)$ such that A is unitarily equivalent to T, then there is a unitary operator $U \in L(H)$ such that $T = U^*AU$ which implies that $T^* = U^*A^*U$. Thus we have $T^3T^* = U^*A^3A^*U$ and $T^*T^3 = U^*A^*A^3U$. Since $T^3T^* = T^*T^3$, we have $U^*A^3A^*U = U^*A^*A^3U$ which implies that $A^3A^* = A^*A^3$. Thus $A \in [3N]$.

(iv) Suppose that M is a closed subspace of H that reduces T, then since $T^3T^* = T^*T^3$ and by using ([1], pp. 158, 159), we get

$$\begin{aligned} (T|M)^3(T|M)^* &= (T^3|M)(T^*|M) \\ &= (T^3T^*|M) \\ &= (T^*T^3|M) \\ &= (T^*|M)(T^3|M) \\ &= (T|M)^*(T|M)^3. \end{aligned}$$

Thus $T|M \in [3N]$.

(v) $(\alpha T)^3(\alpha T)^* = \alpha^3 T^3 \overline{\alpha} T^* = \alpha^3 \overline{\alpha} T^3 T^* = \alpha^3 \overline{\alpha} T^* T^3 = \overline{\alpha} T^*. \alpha^3 T^3 = (\alpha T)^*(\alpha T)^3$. Thus $\alpha T \in [3N]$. ■

Remark 2.1 *Unitary equivalence in Proposition 1(iii), can't be replaced by similarity as shown by the following example.*

Example 2.1 *Consider the operators* $T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $S = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ *acting on* R^2*, then direct calculation shows that* T *is 3-normal but* S *is not 3-normal. Now let* $X = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$*, then* $X^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ *and direct computation shows that* $S = X^{-1}TX$.

Proposition 2.2 *The class* $[3N]$ *is strongly closed i.e.* $\overline{[3N]}^{\text{SOT}}$ *– the closure of* $[3N]$ *in the strong operator topology – equals* $[3N]$.

Proof. Let $\{S_n\}$ be a sequence of operators in $[3N]$ such that $\{S_n\}$ converges strongly to $S \in L(H)$. Since the product of operators is sequentially continuous in the strong operator topology ([2], p. 62), one concludes that $\{S_n^3\}$ converges strongly to S^3. Now since $\{S_n\}$ converges strongly to S, we have $\|S_n x - Sx\| \to 0$ as $n \to \infty$ for each $x \in H$. Thus $\|S_n^* x - S^* x\| = \|(S_n^* - S_n^*)x\| = \|(S_n - S)^* x\| \leq \|(S_n - S)^*\| \, \|x\| = \|S_n - S\| \, \|x\| \to 0$ as $n \to \infty$. Thus $\{S_n^*\}$ converges strongly to S^*. By the above argument we have $S_n^3 S_n^*$ converges strongly to $S^3 S^*$ i.e. $S_n^3 S_n^* \xrightarrow{S} S^3 S^*$. Similarly $S_n^* S_n^3 \xrightarrow{S} S^* S^3$. Since $\{S_n^3 S_n^*\} = \{S_n^* S_n^3\}$, we must have $S^3 S^* = S^* S^3$. Thus $S \in [3N]$. ■

3

In this section we give some characterizations of a 3-normal operator. Also we study the sum and the product of two 3-normal operators.

Proposition 3.1 *Let* $T \in L(H)$ *and let* $T = A + iB$ *be the Cartesian decomposition of* T*, then* T *is 3-normal if and only if*

$$A(AB + BA)B = B(AB + BA)A.$$

Proof. Suppose first that T is 3-normal, then $T^3T^* = T^*T^3$. Thus T^3T^* and T^*T^3 have the same real parts which implies – by direct calculation – that

$$A^4 - B^2A^2 - AB^2A - BABA + A^2B^2 - B^4 + ABAB + BA^2B$$

$$= A^4 - AB^2A - A^2B^2 - ABAB + BA^2B - B^4 + BABA + B^2A^2 \qquad (*)$$

Canceling equal terms on both sides of $(*)$, we get $A^2B^2 + ABAB = B^2A^2 + BABA$ which implies that $A(AB + BA)B = B(AB + BA)A$.

Direct calculations show that the imaginary parts of T^3T^* and T^*T^3 are equal. Suppose now that $A(AB + BA)B = B(AB + BA)A$, then $A^2B^2 + ABAB = B^2A^2 + BABA$ from which we get $(*)$ which means that the real parts of T^3T^* and T^*T^3 are equal. Since the imaginary parts of T^3T^* and T^*T^3 are equal, $T^3T^* = T^*T^3$. Thus $T \in [3N]$. ■

Proposition 3.2 *Let $T \in L(H)$, then $T \in [3N]$ if and only if T^3 is normal.*

Proof. Let $T \in [3N]$, then $T^3T^* = T^*T^3$. Multiplying the last equation on the right by T^{*^2} we get $T^3T^{*^3} = T^*T^3T^{*^2}$. Thus we have

$$\begin{aligned} T^3(T^3)^* &= T^*T^3T^*T^* \\ &= T^*T^*T^3T^* \\ &= (T^*)^3T^3 \\ &= (T^3)^*T^3. \end{aligned}$$

Thus T^3 is normal.

Suppose now that T^3 is normal. Since $T^3T = TT^3$, therefore by Fuglede theorem, $T^{3^*}T = TT^{3^*}$ which implies that $T^*T^3 = T^3T^*$. Thus $T \in [3N]$. ■

Corollary 3.1 *Let $T \in L(H)$, then T is 3-normal if and only if $\|T^3x\| = \|T^{3^*}x\|$ for all $x \in H$.*

Proof. The proof follows immediately from Proposition 3.2 and from the fact that $T \in L(H)$ is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in H$. ([2], p. 154). ■

Proposition 3.3 *If $T, F \in [3N]$ are commuting, then $TF \in [3N]$.*

Proof. Let $T, F \in [3N]$, then by Proposition 3.2, T^3 and F^3 are normal operators. Since $TF = FT$, therefore $T^3F = FT^3$ and $F^3T = TF^3$. Since T^3 and F^3 are normal, therefore by Fugelede theorem, $T^3F^* = F^*T^3$ and $F^3T^* = T^*F^3$.

Now

$$\begin{aligned}(TF)^3(TF)^* &= T^3F^3F^*T^* \\ &= T^3F^*F^3T^* \\ &= F^*T^3T^*F^3 \\ &= F^*T^*T^3F^3 \\ &= (TF)^*(TF)^3.\end{aligned}$$

Thus $TF \in [3N]$. ■

Corollary 3.2 *If $T \in [3N]$, then $T^n \in [3N]$ for any positive integer n.*

Proof. The proof follows from Proposition 3.3 and by using induction. ■

In Proposition 3.3, if $TF \neq FT$, then it is not necessary that $TF \in [3N]$ as shown by the following example:

Example 3.1 *Consider the operator $S = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ acting on R^2, then direct computations show that:*

(i) $SS^* = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = S^*S$. *Thus S is normal which implies that $S \in [3N]$.*

(ii) $T^3T^* = T^*T = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$. *Thus $T \in [3N]$.*

(iii) $(ST)^3(ST)^* = \begin{pmatrix} -4 & 6 \\ 0 & -4 \end{pmatrix} \neq \begin{pmatrix} -7 & 3 \\ -3 & -1 \end{pmatrix} = (ST)^*(ST)^3$. *Thus $ST \notin [3N]$.*

(iv) $ST = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix} \neq TS = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$.

In the following example we show that the sum of two commuting 3-normal operators is not necessarily 3-normal.

Example 3.2 *Consider the operators* $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *and* $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *acting on* R^2*, then direct computation shows that both* S *and* T *are 3-normal operators. Consider the operator* $S+T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$*, then again by direct computations, we see that* $(S+T)^3(S+T)^* = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$ *while* $(S+T)^*(S+T)^3 = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$*. Thus* $S+T \notin [3N]$*. Clearly* $ST = TS$*.*

4 In this section we study the relation between the two classes $[2N]$ and $3N]$.

Proposition 4.1 *If* $T \in L(H)$ *is normal, then* $T \in [3N]$*.*

Proof. Since T is normal, $TT^* = T^*T$. Multiplying the last equation on the left by T^2, we get $T^3T^* = T^2T^*T$ which implies that $T^3T^* = T^*T^3$. Thus $T \in [3N]$. ■

The converse of Proposition 4.1 is not in general true as the following example shows.

Example 4.1 *Consider the operator* $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *acting on* R^2*, then direct calculation shows that* $TT^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *while* $T^*T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$*. Thus* T *is not normal. However we can – by direct calculation – show that* $T^3T^* = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = T^*T^3$ *which means that* $T \in [3N]$*.*

In the following we give a condition under which a 3-normal operator becomes normal.

Proposition 4.2 *If* $T \in [3N]$ *is unitarily equivalent to* T^**, then* T *is normal.*

Proof. Let $T \in [3N]$, then $T^3T^* = T^*T^3$. Multiplying the last equation on left by ${T^*}^2$, we get $T^3{T^*}^3 = T^*T^3T^*T^* = T^*T^*T^3T^* = {T^*}^3T^3$. Now since T^3 is unitarily equivalent to T^*, there is a unitary operator $U \in L(H)$ such that $T^3 = U^*T^*U$. Thus $T^* = UT^3U^*$ which implies that $T = UT^{3^*}U^*$. So $TT^* = UT^{3^*}U^*UT^3U^* = UT^{3^*}T^3U^*$ and $T^*T = UT^3U^*UT^{3^*}U^* = UT^3T^{3^*}U^*$. Since $T^3T^{3^*} = T^{3^*}T^3$, we have $TT^* = T^*T$ i.e. T is normal. ■

Definition 4.1 *Let $T = A+iB \in L(H)$, then T is called skew-normal if $AB = -BA$.* ([4]).

Proposition 4.3 *Let $T = A + iB \in L(H)$ be skew-normal, then $T \in [3N]$.*

Proof. Since $T = A + iB$ is skew-normal, $AB = -BA$. Thus $AB + BA = 0$ which implies that $A(AB+BA)B = 0 = B(AB+BA)A$. Thus by Proposition 3.1, $T \in [3N]$. ■

In the following example we show that there is a 2-normal operator which is not 3-normal.

Example 4.2 *Consider the operator $T = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ acting on R^2, then direct computation shows that $T^2T^* = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} = T^*T^2$. Thus $T \in [2N]$. However and also by direct computation one can show that $T^3T^* = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$ while $T^*T^3 = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$. Thus $T \notin [3N]$.*

Proposition 4.4 *If $T \in [2N]$ and T is partial isometry, then $T \in [3N]$.*

Proof. Since T is partial isometry, therefore–by ([1], p. 153),

$$TT^*T = T. \qquad (*)$$

Multiplying $(*)$ on the right by T, we get $TT^*T^2 = T^2$. Since T is 2-normal, the last equation becomes $T^3T^* = T^2$. Similarly multiplying $(*)$ on the left by T we get $T^2T^*T = T^2$ which implies that $T^*T^3 = T^2$. Thus $T^3T^* = T^*T^3$ which means that $T \in [3N]$. ■

Example 4.3 *Consider the operator* $T = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ *acting on* R^2*, then direct calculation shows that* $T^2T^* = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = T^*T^2$*. Thus* $T \notin [2N]$*. However and again by direct calculation one can show that* $T^3T^* = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = T^*T^3$ *which means that* $T \in 3N]$*. Thus the two classes* $[2N]$ *and* $[3N]$ *are independent.*

Proposition 4.5 *Let* $T \in L(H)$ *such that* $T \in [2N] \cap [3N]$*, then* $T^nT^* = T^*T^n$ *for all positive integers* $n \geq 4$*.*

Proof. We prove the result by induction. First, we prove it for $n = 4$. Since $T \in [2N]$,

$$T^2T^* = T^*T^2, \tag{i}$$

Multiplying (i) on the left by T we get

$$T^3T^* = TT^*T^2. \tag{ii}$$

Multiplying (i) on the right by T we get

$$T^2T^*T = T^*T^3. \tag{iii}$$

Since $T \in [3N], T^3T^* = T^*T^3$. Thus from (ii) and (iii) we get

$$T^2T^*T = TT^*T^2. \tag{iv}$$

Multiplying (iv) on the left by T we get

$$T^3T^*T = T^2T^*T^2$$

which implies that $T^*T^4 = T^4T^*$.

Now suppose the result is true for $n > 4$ i.e. suppose that $T^nT^* = T^*T^n$, then multiplying the last equation on the left by T we get $T^{n+1}T^* = TT^*T^n = TT^*T^2T^{n-2} = T^3T^*T^{n-2} = T^*T^3T^{n-2} = T^*T^{n+1}$. Thus $T^{n+1} \in [3N]$. The result now follows by induction. ■

Proposition 4.6 *Let $T \in [2N] \cap [3N]$ and suppose that ker T = ker T^2, then T is normal.*

Proof. Since $T \in [2N] \cap [3N, T^3T^* = T^*T^3$. Thus $0 = T^3T^* - T^*T^3 = T^3T^* - T^2T^*T = T^2(TT^* - T^*T)$. Since ker $T^2 =$ ker $T, T(TT^* - T^*T) = 0$ which implies that $T^2T^* = TT^*T$. Hence $T^*TT^* = T^{*^2}T$. Now

$$\begin{aligned}(TT^* - T^*T)^2 &= TT^*TT^* - TT^{*^2}T - T^*T^2T^* + T^*TT^*T \\ &= T^2T^*T^* - TT^{*^2}T - T^*T^2T^* + T^{*^2}TT \\ &= T^*T^2T^* - T^{*^2}TT - T^*T^2T^* + T^{*^2}TT \\ &= 0.\end{aligned}$$

Since T^*T and TT^* are self-adjoint, $TT^* - T^*T$ is self-adjoint. Since $(TT^* - T^*T)^2 = 0$, $TT^* - T^*T = 0$ which implies that T is normal. ∎

If T is normal, then $T \in [2N] \cap [3N]$. The converse is not in general true as shown by the following example:

Example 4.4 *Consider the operator $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ acting on R^2, then direct computations show that $TT^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ while $T^*T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Thus T is not normal. However and also by direct computation one shows that $T^2T^* = T^*T^2 = T^3T^* = T^*T^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ which implies that $T \in [2N] \cap [3N]$.*

Proposition 4.7 *Let $T \in [2N] \cap [3N]$ be one-to-one, then T is normal.*

Proof. Since $T \in [2N] \cap [3N]$, then $T^2T^* = T^*T^2$ and $T^3T^* = T^*T^3$ which implies that $T^2(TT^* - T^*T) = T^3T^* - T^2T^*T = T^3T^* - T^*T^3 = 0$. Since T is one-to-one, $TT^* - T^*T = 0$. Thus T is normal. ∎

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