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Random sums of random vectors and multitype families of productive individuals

I. Rahimov[*], H. Muttlak [†]

Abstract

We prove limit theorems for a family of random vectors whose coordinates are a special form random sums of Bernoulli random variables. Applying these limit theorems we study the number of productive individuals in n-type indecomposable critical branching stochastic processes with types of individuals $T_1, ..., T_n$. Let τ and $t, \tau < t$, be two observation times and $\theta(\mathbf{t})$ be n-dimensional vector of nonnegative functions. We define n-variate process $\mathbf{X}(\tau, \mathbf{t}) = (X_1(\tau, t), ..., X_n(\tau, t))$, where $X_i(\tau, t)$ is the number of type T_i individuals at time τ whose number of descendants at time t at least of one type is more than corresponding level given by vector $\theta(\mathbf{t} - \tau)$. In the paper limit distributions for $\mathbf{X}(\tau, \mathbf{t})$ are obtained as $\tau, t \to \infty$ in different cases of relationship between times τ and t.

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1 Introduction

We consider a sequence of random vectors which is defined as following. Let $\{\xi_{ij}(k,m), j \geq 1\}, i = 1,2,...,n,$ for any pair $(k,m) \in N_0^2, N = \{1,2,...\}, N_0 = \{0\} \bigcup N$, be n independent sequences of random variables and $\{\nu_{ik}, k \in N_0\}, i = 1,2,...,n$, be n sequences of (not necessarily independent) random variables taking values $0,1,...$ and independent of family $\{\xi_{ij}(k,m)\}$. We consider the family of random vectors

$$\mathbf{W}(k,m) = (W_1(k,m),...,W_n(k,m)), W_i(k,m) = \sum_{j=1}^{\nu_{ik}} \xi_{ij}(k,m). \quad (1)$$

Assume that $\xi_{ij}(k,m), j = 1,2,...$ for any fixed k,m and i are independent and identically distributed Bernoulli random variables with parameter $P_{km}^{(i)}$ (i.e. have distribution $b(1, P_{km}^{(i)})$).

We shall study the asymptotic behavior of $\mathbf{W}(k,m)$ as $k,m \to \infty$ under some assumptions on random variables ν_{ik} and $\xi_{ij}(k,m)$ in different cases of relationship between parameters k and m.

Random sums of independent random variables or random vectors have been considered by many authors. First it is because of the interest in extending classic limit theorems of the probability theory to a more general situation and to discover new properties of the random sums caused by "randomness" of the number of summands. On the other hand many problems in different areas of probability can be connected with a sum of random number of random variables. Rather full list of publications on random sums can be found in recent monograph by Gnedenko and Korolev (1996). Transfer theorems for the random sum of independent random variables can also be seen in Gnedenko(1997).

The relationship between random sums and branching stochastic processes is well known. Starting from early studies (see Harris (1966), for example) including the recent publications the fact that the number of particles in a model of branching process can be represented as a random sum have been mentioned. Some of investigations show that using this relationship in study of branching models makes possible to investigate new variables related to the genealogy of the process, to study more general modifications of branching processes and to consider different characteristics of the process from a unique point of view. So, limit distributions for the number of pairs of individuals at time τ having the same number of descendants at time $t, t > \tau$ are found in [7]. A more general variable of this kind, describing the number of individual pairs having "relatively close" number of descendants considered in the paper [8] (see also [9], Ch IV). Using this relationship limit theorems for different models of branching processes with immigration which may depend on the reproduction processes of particles are also proved. This kind problems are systematically studied in the mentioned above monograph [9]. Investigations of the maximum family size in a population by Arnold and Villasenor (1996), Rahimov and Yanev (1999) and by Yanev and Tsokos (2000) are also based on this kind of a relationship.

Here we discuss the relationship of the random sum of random vectors and multitype branching processes. Although $\mathbf{X}(t)$ the number of individuals of different types at time t is the main object of investigation in the theory of multitype branching processes, there are many other variables related to the population which are of interest as well. One example of such a variable is the time to the closest common ancestor of the entire population observed at certain time. For a single-type Galton-Watson process this variable was considered by Zubkov (1975), who proved that, if the process is critical, the time is asymptotically uniformly distributed. Later, it turned out that the time to the closest common ancestor may be treated as a functional of so called reduced branching processes. This a process was introduced by Fleischman and Siegmund-Schultze(1977) as a process that counts only individuals at a given time τ having descendants at time $t, t > \tau$. They demonstrated that in the critical single-type case the reduced process can well be approximated by a non-homogeneous pure birth process. Later a number of studies extended their results to general single and multitype models of branching processes (see [14], [15] and [12], for example).

In present paper we show that, if one uses theorems proved for the random sums defined in (1), one may study a generalized model of multitype reduced processes. Let $\theta(t) = (\theta_1(t), ..., \theta_n(t))$ be a vector of non-negative functions, τ and $t, \tau < t$ be two times of observation. We define process $\mathbf{X}(\tau, t) = (X_1(\tau, t), ..., X_n(\tau, t))$, where $X_i(\tau, t)$ is the number of type T_i individuals at time τ, whose number of descendants at time t of at least one type is greater than corresponding level, given by vector $\theta(t - \tau)$. It is clear that $\mathbf{X}(\tau, t)$ counts only "relatively productive" individuals at time τ. We also note that $\mathbf{X}(\tau, t)$ is usual n-type reduced process, if $\theta(t) = \mathbf{0}$ for all $t \in N_0$. In the paper we obtain limit distributions for process $\mathbf{X}(\tau, t)$ as $t, \tau \to \infty$ in different cases of relationship between observation times τ and t for critical processes. It must be noted that the generalized reduced single type process was introduced and studied in Rahimov(2003).

In Section 2 we prove several limit theorems for the random sum of random vectors $\mathbf{W}(k,m)$ under some natural assumptions on parameters. Section 3 is devoted to construction and definition of the generalized multitype reduced process. Applications of theorems of Section 2 to reduced branching processes are given in Section 4. Section 5 contains proofs of theorems from Section 4. In Section 6 possible applications of theorems on random sum (1) in study of the number of productive ancestors in large populations are discussed.

2 Convergence of the random sum

For n -dimensional vectors $\mathbf{x} = (x_1, ..., x_n), \mathbf{y} = (y_1, ..., y_n)$ we denote $\mathbf{x} \oplus \mathbf{y} = (x_1 y_1, ..., x_n y_n), \mathbf{x}^{\mathbf{y}} = (x_1^{y_1}, ..., x_n^{y_n}), \mathbf{x}/\mathbf{y} = (\frac{x_1}{y_1}, ..., \frac{x_n}{y_n}), (\mathbf{x}, \mathbf{y}) = x_1 y_1 + ... + x_n y_n, \sqrt{\mathbf{x}} = (\sqrt{x_1}, ..., \sqrt{x_n})$, and $\mathbf{x} \geq \mathbf{y}$ or $\mathbf{x} > \mathbf{y}$ if $x_i \geq y_i$ or $x_i > y_i$ respectively.

First theorem concerning the vector (1) covers the case when normalized vector $\nu_k = (\nu_{ik}, i = 1, ..., n)$ has a limit distribution. Namely we assume that there exists a sequence of positive vectors $\mathbf{A_k} = (A_{ik}, i = 1, ..., n)$ such that $A_{ik} \to \infty, k \to \infty$,

$$\{\frac{\nu_{\mathbf{k}}}{\mathbf{A_k}} | \nu_{\mathbf{k}} \neq \mathbf{0}\} \to \mathbf{Y} = (Y_1, ..., Y_n) \tag{2}$$

in distribution and for the vector $\mathbf{P}(k,m) = (P_{km}^{(i)}, i = 1, ..., n)$

$$\mathbf{P}(k,m) \oplus \mathbf{A_k} \to \mathbf{a} = (a_1, ..., a_n), \tag{3}$$

where the components of the vector $\mathbf{a}$ may be $+\infty$.

Theorem 1. *If conditions (2) and (3) are satisfied, then*

$$\{\frac{\mathbf{W}(k,m)}{\mathbf{P}(k,m) \oplus \mathbf{A}_k} | \nu_k \neq \mathbf{0}\} \to \mathbf{W}$$

in distribution and $Ee^{(\lambda, \mathbf{W})} = \varphi(\lambda^*)$, *where* $\varphi(\lambda)$ *is the Laplace transform of the vector* $\mathbf{Y}, \lambda^* = (\lambda_1^*, ..., \lambda_n^*)$ *and* $\lambda_i^* = \lambda_i$, *if* $a_i = \infty$ *and* $\lambda_i^* = a_i(1 - e^{-\lambda_i / a_i})$, *if* $a_i < \infty$.

Proof. First we consider the case when $a_i < \infty, i = 1, ...n$. Since variables $\xi_{ij}(k,m), j = 1, 2, ...$ are independent and identically distributed, by total probability arguments we find for any $\mathbf{S} = (S_1, ..., S_n), \mathbf{0} < \mathbf{S} < \mathbf{1}$

$$E[\prod_{i=1}^{n} S_i^{W_i(k,m)}] = E[E[\prod_{i=1}^{n}\prod_{j=1}^{\nu_{ik}} S_i^{\xi_{ij}(k,m)}|\nu_{\mathbf{k}}]] = F(k, \mathbf{G}(k,m,\mathbf{S})), \tag{4}$$

where

$$\mathbf{G}(k,m,\mathbf{S}) = (G_i(k,m,S_i), i = 1, ..., n), G_i(k,m,S_i) = ES_i^{\xi_{ij}(k,m)}.$$

and $F(k,\mathbf{S})$ is the probability generating function of the vector ν_k. Note here that, since $\xi_{ij}(k,m)$ are Bernoulli random variables with parameter $P_{km}^{(i)}$,

$$G_i(k,m,S_i) = 1 - P_{km}^{(i)}(1 - S_i). \tag{5}$$

It follows from condition (2) that for any $\mathbf{0} < \mathbf{S} < \mathbf{1}$

$$\frac{1 - F(k, \mathbf{e}^{-\lambda_0/\mathbf{A_k}})}{P\{\nu_{\mathbf{k}} \neq \mathbf{0}\}} \to 1 - \varphi(\lambda_0), \tag{6}$$

where $\lambda_0 = \mathbf{a} \oplus (\mathbf{1} - \mathbf{S})$.

Now we consider

$$\varepsilon(k,m,\mathbf{S}) = \frac{F(k, \mathbf{G}(k,m,\mathbf{S})) - F(k, \mathbf{e}^{-\lambda_0/\mathbf{A_k}})}{P\{\nu_{\mathbf{k}} \neq \mathbf{0}\}} = E[B(k,m,\nu_k)|\nu_k \neq \mathbf{0}], \tag{7}$$

where

$$B(k,m,\nu_k) = \prod_{i=1}^{n} G_i^{\nu_{ik}}(k,m,S_i) - \prod_{i=1}^{n} e^{-\nu_{ik}a_i(1-S_i)/A_{ik}}.$$

Let Δ be a positive number. Introducing the event

$$C(\Delta, \nu_k) = \{\nu_{ik}/A_{ik} < \Delta, i = 1, ..., n\}$$

we write $\varepsilon(k,m,\mathbf{S})$ as following

$$\varepsilon(k,m,\mathbf{S}) = E[B(k,m,\nu_{\mathbf{k}})\chi|\nu_k \neq \mathbf{0}] + E[B(k,m,\nu_{\mathbf{k}})(1-\chi)|\nu_k \neq \mathbf{0}] \tag{8}$$

with $\chi = \chi\{C(\Delta, \nu_k)\}$. If we use inequality

$$|\prod_{i=1}^n a_i - \prod_{i=1}^n b_i| \leq \sum_{i=1}^n |a_i - b_i|, \tag{9}$$

which holds for any sets of numbers a_i, b_i such, that $|a_i| \leq 1, |b_i| \leq 1, i = 1, ..., n$, we obtain that the absolute value of the first expectation on the right side of (8) is not greater than

$$\sum_{i=1}^n E[|G_i^{\nu_{ik}} - e^{-\nu_{ik} a_i (1-S_i)/A_{ik}}|\chi|\nu_k \neq \mathbf{0}] = \sum_{i=1}^n E[|\exp\{\nu_{ik}\delta_i(k,m)/A_{ik}\} - 1|\chi|\nu_k \neq \mathbf{0}],$$

where

$$\delta_i(k,m) = A_{ik} \ln G_i(k, m, S_i) + a_i(1 - S_i).$$

Taking into account the definition of event $C(\Delta, \nu_k)$ we obtain that the last sum can be estimated by

$$\sum_{i=1}^n \max_{l \in D} |\exp\{(l/A_{ik})\delta_i(k,m)\} - 1|, \tag{10}$$

where $D = \{l \in N_0 : l/A_{ik} < \Delta\}$. It follows from (5) that $1 - G_i(k, m, \mathbf{S_i}) \to 0$ (since $A_{ik} \to \infty$). Therefore $\ln G_i \sim -(1 - G_i)$ and we conclude from condition (3) that $\delta_i(k,m) \to 0, i = 1, ..., n$. Hence the first expectation on the right side of (8) tends to zero.

Now we consider the second term. Since $|B(k, m, \nu_k)| \leq 1$ for all sample points, we obtain that the absolute value of the second expectation is not greater than

$$1 - P\{C(\Delta, \nu_k)|\nu_k \neq \mathbf{0}\}$$

which, according to the condition (2) and definition of $C(\Delta, \nu_k)$, tends to

$$1 - P\{Y_1 \leq \Delta, Y_2 \leq \Delta, ..., Y_n \leq \Delta\}.$$

This estimation shows that the second expectation on the right side of (8) can be made arbitrarily small by choosing sufficiently large Δ. Thus we conclude that under conditions (2) and (3) $\varepsilon(k, m, \mathbf{S}) \to 0$. This along with (4), (6) and the fact that

$$E[\prod_{i=1}^n S_i^{W_i(k,m)}|\nu_k \neq \mathbf{0}] = 1 - \frac{1 - E[\prod_{i=1}^n S_i^{W_i(k,m)}]}{P\{\nu_k \neq \mathbf{0}\}} \tag{11}$$

gives the assertion of the theorem in the case $a_i < \infty$.

Now we consider the case, when the limit in condition (3) is not finite i.e. $a_i = \infty, i = 1, ..., n$. In this case we use the following notation $\mathbf{M}(k,m) = \mathbf{A}_k \oplus \mathbf{P}(k,m) = (M_i(k,m), i = 1, ..., n)$. The proof is the same as of that of the first case and, therefore, we only provide some important points .

We consider relation (11) with $\mathbf{S} = (S_1, ..., S_n)$ and $S_i = \exp\{-\lambda_i / M_i(k,m)\}$ where $\lambda_i > 0, i = 1, ..., n$. Using (5) we obtain this time that

$$1 - G_i(k,m,S_i) \sim A_{ik}\lambda_i, k, m \to \infty, i = 1, ..., n. \tag{12}$$

Therefore in relation (6) we have λ in place of λ_0. Since $A_{ik} \to \infty$, again $\ln G_i \sim -(1 - G_i)$ and

$$\frac{\ln G_i(k,m,S_i)}{A_{ik}} \to -\lambda_i \tag{13}$$

as $k, m \to \infty$.

We consider again $\varepsilon(k,m,\mathbf{S})$ from (7) replacing λ_0 by λ and putting $S_i = \exp\{-\lambda_i / M_i(k,m)\}$. By the same arguments as in the proof of the first case we obtain that absolute value of $\varepsilon(k,m,\mathbf{S})$ can be estimated by sum (10) with

$$\delta_i(k,m) = \frac{\ln G_i(k,m,S_i)}{A_{ik}} + \lambda_i$$

and $\delta_i(k,m) \to 0, i = 1, ..., n$ due to (13). Hence $\varepsilon(k,m,\mathbf{S}) \to 0$ as $k, m \to \infty$. Again appealing to relation (11) we obtain the assertion of the theorem when $a_i = \infty$. It is now clear that when some of a_i are finite and the others are infinite, the limit random variable has the Laplace transform $\varphi(\lambda^*)$. Theorem 1 is proved.

The family of vectors (1) is eventually a sum of independent vectors, if vectors $\nu_k = (\nu_{ik}, i = 1, ...n)$ have degenerate distributions. Therefore one may expect to obtain a normal limit distribution under some natural assumptions. The next theorem obtains the conditions under which the limit of vector $\mathbf{W}(k,m)$ is a mixture of the normal and a given distribution. Assume

C1. For a given sequence of positive vectors $\mathbf{A}_k$ there exists a sequence $\mathbf{l}_k = (l_{ik}, i = 1, ..., n), k \geq 1$ such that $A_{ik}/l_{ik} \to \infty, k \to \infty, i = 1, ..., n$ and

$$\mathbf{l}_k \oplus \mathbf{P}(k,m) \oplus (1 - \mathbf{P}(k,m)) \to \mathbf{C}, i = 1, ..., n$$

as $k, m \to \infty$, where $\mathbf{C} = (C_i, i = 1, ..., n)$ is a positive vector of constants.

Theorem 2. *If conditions (2) and C1 are satisfied, then*

$$\left\{ \frac{\mathbf{W}(k,m) - \nu_{ik} \oplus \mathbf{P}(k,m)}{\sqrt{\mathbf{A}_k \oplus \mathbf{C}/\mathbf{l}_k}} | \nu_k \neq \mathbf{0} \right\} \to \mathbf{W}$$

as $k, m \to \infty$ *, where*

$$P\{\mathbf{W} \leq \mathbf{x}\} = \int_0^\infty ... \int_0^\infty \prod_{i=1}^n \Phi(\frac{x_i}{\sqrt{y_i}}) dT(y_1, ..., y_n),$$

$\Phi(x)$ *is the standard normal distribution and* $T(x_1, ..., x_n)$ *is the distribution of the vector* $\mathbf{Y}$ *in (2).*

Proof. Let $\mathbf{W}^*(k,m) = (W_i^*(k,m), i = 1, ..., n)$ with

$$W_i^*(k,m) = \sum_{j=1}^{A_{ik}} \xi_{ij}, i = 1, ..., n.$$

In the proof we use the following proposition.

Proposition 1. Assume that there exist sequences $\{l_{ik}, k \geq 1\}, i = 1, ..., n$ for which condition C1 satisfied. Then for each $i = 1, ..., n$ variable $(W_i^*(k,m) - A_{ik} P_{km}^i)/\sqrt{A_{ik} C_i / l_{ik}}$ is asymptotically normal as $k, m \to \infty$.

Proof. The assertion follows directly from central limit theorem and from trivial identities:

$$EW_i^*(k,m) = A_{ik} P_{km}^{(i)}, var W_i^*(k,m) = A_{ik} P_{km}^{(i)} (1 - P_{km}^{(i)}).$$

Now we continue the proof of Theorem 2. Let $L(k, m, \mathbf{x})$ be the conditional distribution in Theorem 2 and for $i = 1, ..., n$

$$F_i(k, m, t_i, x_i) = P\left\{ \frac{V_i(k,m) - t_i P_{km}^{(i)}}{\sqrt{A_{ik} C_i / l_{ik}}} \leq x_i \right\},$$

where $t_i \in N_0$ and $V_i(k,m) = \xi_{i1}(k,m) + ... + \xi_{it_i}(k,m)$. Using independence of sequences $\{\xi_{ij}(k,m), j \geq 1\}$ by total probability arguments we obtain for any $q > 0$

$$L(k,m,\mathbf{x}) = \sum_{\mathbf{h} \in N_0^n} \prod_{i=1}^{n} \sum_{t_i \in \Delta} F_i(k,m,t_i,x_i) P_k(\mathbf{t}), \tag{14}$$

where $\mathbf{h} = (h_1, ..., h_n), \mathbf{t} = (t_1, ..., t_n), P_k(\mathbf{t}) = P\{\nu_k = \mathbf{t} | \nu_k \neq \mathbf{0}\}$ and

$$\Delta_i = \{t_i \in N_0 : \frac{h_i}{q} \leq \frac{t_i}{A_{ik}} < \frac{h_i + 1}{q}\}.$$

Let now $p > 0$ be such that pq is an integer. The sum on the right side of (14) we partition as following

$$L(k,m,\mathbf{x}) = \Sigma^{'} + \Sigma^{''} = I_1 + I_2, \tag{15}$$

where $\Sigma^{'}$ is the sum over all vectors $\mathbf{h} \in N_0^n$ such that $h_i \leq pq, i = 1, ..., n$ and $\Sigma^{''}$ is the sum over all such vectors that at least one of coordinates is greater than pq.

First we consider I_1. Using the monotonicity of the distribution function we obtain the following estimate for I_1.

$$I_1 \leq \Sigma^{'} \prod_{i=1}^{n} \sum_{t_i \in \Delta_i} P\{\frac{V_i(k,m) - t_i P_{km}^{(i)}}{\sqrt{t_i C_i / l_{ik}}} \leq x_i \alpha(i,q)\} P_k(\mathbf{t}),$$

where $\alpha(i,q) = \sqrt{q/h_i}$, if $x_i > 0$ and it is equal to $\sqrt{q/(h_i+1)}$, if $x_i < 0$. We denote $\mathbf{Y} = (Y_1, ..., Y_n)$ a random vector having distribution $T(x_1, ..., x_n)$. Since $h_i A_{ik}/q \leq t_i < (h_i+1) A_{ik}/q$ for $t_i \in \Delta_i$, if $A_{ik} \to \infty$, then so does t_i and $t_i / l_{ik} \to \infty$. Consequently, if we use Proposition 1 and condition (2), we get

$$\limsup_{k,m \to \infty} I_1 \leq \Sigma^{'} \prod_{i=1}^{n} \Phi(x_i \alpha(i,q)) P\{\frac{h_i}{q} \leq Y_i < \frac{h_i+1}{q}, i = 1, ..., n\}. \tag{16}$$

Repeating similar arguments we obtain that

$$\liminf_{k,m \to \infty} I_1 \geq \Sigma^{'} \prod_{i=1}^{n} \Phi(x_i \beta(i,q)) P\{\frac{h_i}{q} \leq Y_i < \frac{h_i+1}{q}, i = 1, ..., n\}, \tag{17}$$

where $\beta(i,q) = \sqrt{q/(h_i+1)}$,if $x_i > 0$ and it is equal to $\sqrt{q/h_i}$ otherwise. Since for each fixed p and $q \to \infty$ right sides of (16) and (17) have the same limit, we conclude that

$$\lim_{k,m\to\infty} I_1 = \int_0^p \dots \int_0^p \prod_{i=1}^n \Phi(\frac{x_i}{\sqrt{y_i}}) dT(y_1,\dots,y_n). \tag{18}$$

Now we consider I_2. Recall that Σ'' is the sum over all vectors $\mathbf{h} \in N_0^n$ such that at least one of coordinates is greater than pq. Let h_j be the coordinate of $\mathbf{h}$ which is greater than pq. Then it is not difficult to see that

$$I_2 \le \sum_{h_j=pq+1}^{\infty} P\{\frac{h_j}{q} \le \frac{\nu_{jk}}{A_{jk}} < \frac{h_j+1}{q} | \nu_k \ne \mathbf{0}\} \le P\{\frac{\nu_{jk}}{A_{jk}} > p | \nu_k \ne \mathbf{0}\}.$$

From here due to condition (2) we obtain that

$$\limsup_{k,m\to\infty} I_2 \le 1 - T(p), \tag{19}$$

where $T(p) = P\{Y_i < \infty, i \ne j, Y_j \le p\}$. It is clear that the difference on the right side of (19) can be made arbitrarily small by choosing p sufficiently large. Therefore $I_2 \to 0$ as $k,m \to \infty$. Theorem 2 is proved.

3 Generalized reduced processes

Now we give a rigorous definition of the generalized reduced process $\mathbf{X}(\tau,t)$. We use the following notation for individuals participating in the process. Let the process starts with a single ancestor at time $t=0$ of type $T_i, i = 1,\dots,n$. We denote it by T_i and consider as zeroth generation. The direct offspring of the initial ancestor we denote as (T_i, T_j, m_j), where $T_j, j = 1,\dots,n$ is the type of the direct descendant and $m_j \in N, N = \{1,2,\dots\}$ is the label (the number) of the descendant in the set of all immediate descendants of T_i. Thus m_{k+1}th direct descendant of the type $T_{i_{k+1}}$ of the individual $\alpha = (T_i, T_{i_1}, m_1, \dots, T_{i_k}, m_k)$ will be denoted as $\alpha' = (\alpha, T_{i_{k+1}}, m_{k+1})$. Here and later on for any two vectors $\alpha = (i_1,\dots,i_k)$ and $\beta = (j_1,\dots,j_m)$ the ordered pair (α,β) we will understand as $k+m$ dimensional vector $(i_1,\dots,i_k,j_1,\dots,j_m)$.

If we use the above notation, the set $\Re_n \in E$, where E is the space of all finite subsets of

$$\bigcup_{k=1}^{\infty} N_1^k, N_1^k = N_1^{k-1} \times N_1, N_1 = \{T_i\} \times \{T_1, ..., T_n\} \times N$$

corresponds to the population of the nth generation. It is clear that $\Re_n$ can be decomposed as $\Re_n = \cup_{i=1}^n \Re_n^{(i)}$, where $\Re_n^{(i)}$ is the population of the type T_i individuals of the nth generation. Consequently components of the process $\mathbf{X}(t)$ are found as $X_i(t) = card\{\Re_t^{(i)}\}, t \in N_0$ and for any τ and t such that $\tau < t$ we have

$$\mathbf{X}(t) = \sum_{i=1}^{n} \sum_{\alpha \in \Re_\tau^{(i)}} \mathbf{X}^{(\alpha)}(t - \tau),$$

where $\mathbf{X}^{(\alpha)}(t) = (X_1^{(\alpha)}(t), ..., X_n^{(\alpha)}(t))$ is the n-type branching process generated by individual α.

Let $\Im_i([\theta], \tau, t)$ be the set of individuals in $\Re_\tau^{(i)}$ having at least one type of descendants at time t more than corresponding component of $\theta(t - \tau)$. It is not difficult to see that it can be described as following.

$$\Im_i([\theta], \tau, t) = \{\alpha \in \Re_\tau^{(i)} : \text{for at least one } j \ \exists \ \text{ more than } \theta_j(t - \tau)$$

$$\beta \text{ -sets such that } (\alpha, \beta) \in \Re_t^{(i)}\},$$

where $\alpha \in N_1^\tau, \beta \in N_1^{t-\tau}$. Thus the generalized reduced process is defined as $\mathbf{X}(\tau, t) = (X_i(\tau, t), i = 1, ..., n)$ with $X_i(\tau, t) = \text{card}\{\Im_i([\theta], \tau, t)\}$.

In particular, if $\theta(t) = \mathbf{0}$ for all t, then $\Im_i([\mathbf{0}], \tau, t)$ contains all individuals of type T_i only living in τth generation and having descendants (at least of one type) in generations $\tau + 1, \tau + 2, ..., t$. Consequently in this case $\mathbf{X}(\tau, t), 0 < \tau < t$, is the n-type usual reduced branching process.

4 Limit behavior of the reduced process

We denote by $P_\alpha^i, \alpha = (\alpha_1, ..., \alpha_n) \in N_0^n$, the offspring distribution of the process $\mathbf{X}(t)$, i.e.

$$P_\alpha^i = P\{\mathbf{X}(1) = \alpha | \mathbf{X}(0) = \delta_i\}$$

is the probability that an individual of type T_i generates the total number α of new individuals. Here $\delta_i = (\delta_{ij}, j = 1, ..., n), \delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$. We also denote

$$F^i(\mathbf{S}) = \sum_{\alpha \in N_0^n} P_\alpha^i S_1^{\alpha_1} ... S_n^{\alpha_n}, \quad \mathbf{F}(\mathbf{S}) = (F^1(\mathbf{S}), ..., F^n(\mathbf{S})),$$

$$Q^i(t) = P\{\mathbf{X}(t) \neq \mathbf{0} | \mathbf{X}(0) = \delta_i\}, \mathbf{Q}(\mathbf{t}) = (Q^1(t), ..., Q^n(t)).$$

Let for $i, j, k = 1, 2, \cdots, n$

$$a_i^j = \frac{\partial F^j(\mathbf{S})}{\partial S_i} |_{\mathbf{S}=\mathbf{1}} , b_{ik}^j = \frac{\partial^2 F^j(\mathbf{S})}{\partial S_i \partial S_k} |_{\mathbf{S}=\mathbf{1}} ,$$

$\mathbf{A} = \left\| a_i^j \right\|$ be the matrix of expectations, ρ be its Peron root and the right and the left eigenvectors $\mathbf{U} = (u_1, u_2, ..., u_n)$ and $\mathbf{V} = (v_1, v_2, ..., v_n)$ corresponding to the Peron root be such that

$$\mathbf{AU} = \rho \mathbf{U}, \quad \mathbf{VA} = \rho \mathbf{V}, \qquad (\mathbf{U}, \mathbf{V}) = 1, \quad (\mathbf{U}, \mathbf{1}) = 1.$$

If $\mathbf{A}$ is indecomposable, aperiodic and $\rho = 1$, the process $\mathbf{X}(t)$ is called critical indecomposable multitype branching process. We assume that the generating function $\mathbf{F}(\mathbf{S})$ satisfies the following representation

$$x - \sum_{j=1}^n v_j (\mathbf{1} - F^j(\mathbf{1} - \mathbf{U}x)) = x^{1+\alpha} L(x), \tag{20}$$

where $0 < x \leq 1, \alpha \in (0, 1]$, and $L(x)$ is a slowly varying function as $x \downarrow 0$. Note that in this case $\rho = 1$, i.e. the process is critical and the second moments of the offspring distribution $b_{ik}^j, i, j, k = 1, ..., n$, may not be finite. Under this assumption the following limit theorem for the process $\mathbf{X}(t)$ holds (see Vatutin (1977)).

Proposition 2. *If the offspring generating function* $\mathbf{F}(\mathbf{S})$ *satisfies representation (20) then we have*
a)

$$Q^i(t) \sim u_j t^{-1/\alpha} L_1(t)$$

as $t \to \infty$, *where* $L_1(t)$ *is a slowly varying as* $t \to \infty$ *function;*
b)

$$\lim_{t \to \infty} P \left\{ \mathbf{X}(t) q(t) \leq \mathbf{x} \oplus \mathbf{V} \mid \mathbf{X}(t) \neq \mathbf{0}, \mathbf{X}(0) = \delta_i \right\} = \pi(\mathbf{x}),$$

where $q(t) = \sum_{j=1}^n v_j Q^j(t)$ *and* $\pi(\mathbf{x}) = \pi(x_1, x_2, ..., x_n)$ *a distribution having the Laplace transform*

$$\phi(\lambda) = \int_{R_+^n} e^{-(\mathbf{x},\lambda)} d\pi(\mathbf{x}) = 1 - (1 + \bar{\lambda}^{-\alpha})^{-1/\alpha}, \quad \bar{\lambda} = (\lambda, \mathbf{1}). \tag{21}$$

Now we are in a position to state our first result about $\mathbf{X}(\tau, t)$. Let $\theta = (\theta_1, ..., \theta_n) \in R_+^n, R_+ = [0, \infty), \mathbf{C} = (C_1, ..., C_n) \in R_+^n$ be some nonnegative vectors.

Theorem 3. *If condition (20) is satisfied,* $\theta(t) = \theta \oplus \mathbf{V}/q(t)$ *and* $t, \tau \to \infty, t - \tau \to \infty$ *such that* $\mathbf{Q}(t - \tau)/\mathbf{Q}(\tau) \to \mathbf{C}$, *then*

$$P\{\mathbf{X}(\tau, t) = \mathbf{k} | \mathbf{X}(\tau) \neq \mathbf{0}, \mathbf{X}(0) = \delta_i\} \to P_{\mathbf{k}}^*,$$

where $\mathbf{k} = (k_1, ..., k_n) \in N_0^n$ *and the probability distribution* $\{P_{\mathbf{k}}^*, \mathbf{k} \in N_0^n\}$ *has the generating function* $\phi^*(\mathbf{S}) = \phi(\mathbf{a})$ *with* $\mathbf{a} = b\mathbf{C} \oplus \mathbf{U} \oplus \mathbf{V} \oplus (\mathbf{1} - \mathbf{S}), b = 1 - \pi(\theta), \mathbf{S} = (S_1, ..., S_n)$ *and* $\phi(\lambda)$ *is the Laplace transform defined in (21).*

Remark. It is clear that vector $\mathbf{C}$ in the condition $\mathbf{Q}(t - \tau)/\mathbf{Q}(\tau) \to \mathbf{C}$ necessarily has the form $\mathbf{C} = C\mathbf{1}$, where $C \geq 0$ is some constant.

Example 1. Let $\mathbf{F}(\mathbf{S})$ satisfies condition (20) with $\alpha = 1$. We shall note here that in this case the second moments of the offspring distribution still may be infinite. For this kind of a process the limit distribution $\pi(\theta)$ is exponential and the generating function $\phi^*(\mathbf{S})$ has the form $\phi^*(\mathbf{S}) = (1 + d)^{-1}$, where $d = bC \sum_{j=1}^n u_j v_j (1 - S_j), b = e^{-\theta^*}, \theta^* = \min\{\theta_1, ..., \theta_n\}$. We represent it as following

$$\phi^*(\mathbf{S}) = \frac{1}{1 + Ce^{-\theta^*}} \left(1 - \frac{Ce^{-\theta^*}}{1 + Ce^{-\theta^*}} \sum_{i=1}^n u_i v_i S_i\right)^{-1}. \tag{22}$$

What is the distribution having the last probability generating function? To answer this question we consider a sequence of independent random variables $X_1, X_2, ...$ such that $P\{X_i = j\} = p_j, \quad j = 0, 1, 2, ..., n, \quad \sum_{j=0}^n p_j = 1$, where $p_0 = (1 + Ce^{-\theta^*})^{-1}, \quad p_j = Ce^{-\theta^*} u_j v_j / (1 + Ce^{-\theta^*}), \quad j = 1, 2, ..., n$. Let Δ_1 be the number of $1's$, Δ_2 be the number of $2's$ and so on Δ_n be the number of $n's$ observed in the sequence $X_1, X_2, ...$ before the first zero is obtained. Then it follows from the formula for the generating function of generalized multivariate geometric distribution in Ch. 36.9 of Johnson at. all.(1997) that the vector $(\Delta_1, ..., \Delta_n)$ has the probability generating function given by(22) i. e.

$$E\left(S_1^{\Delta_1} S_2^{\Delta_2} ... S_n^{\Delta_n}\right) = \phi^*(\mathbf{S}).$$

Hence we have the following result.

Corollary 1. If assumptions of Theorem 3 are satisfied with $\alpha = 1$, then the probability distribution $\{P^*_{\mathbf{k}}, \mathbf{k} \in N_0^n\}$ is multivariate geometric distribution defined by generating function (22) such that

$$P^*_{\mathbf{k}} = P\{\Delta_i = k_i, i = 1, ..., n\}.$$

It is clear that, if $n = 1$, the distribution is geometric, i. e. $P^*_k = pq^k, k = 0, 1, ...$ with $p = (1 + Ce^{-\theta_1})^{-1}, q = Ce^{-\theta_1}(1 + Ce^{-\theta_1})^{-1}$.

Example 2. Let assumptions of Theorem 3 be satisfied and $\tau = [\varepsilon t], 0 < \varepsilon < 1$. Using asymptotic behavior of $\mathbf{Q}(\mathbf{t})$ and uniform convergence theorem for the slowly varying functions (see Seneta (1985), for example) we obtain that as $t \to \infty$

$$\frac{\mathbf{Q}(t-\tau)}{\mathbf{Q}(\tau)} \to \left(\frac{\varepsilon}{1+\varepsilon}\right)^{1/\alpha} \mathbf{1}.$$

Consequently in this case the limit distribution has the generating function $\phi^*(\mathbf{S})$ with $C = (\varepsilon/(1+\varepsilon))^{1/\alpha}$. In particular we have the following result.

Corollary 2. If assumptions of Theorem 3 are satisfied and $\tau = o(t)$, then

$$\lim_{t\to\infty} P\{\mathbf{X}(\tau, t) = \mathbf{k} | \mathbf{X}(\tau) \neq \mathbf{0}, \mathbf{X}(0) = \delta_i\} = 0$$

for all $\mathbf{k} \in N_0^n$ and $\mathbf{k} \neq \mathbf{0}$.

It is known that in the critical case the process $\mathbf{X}(t)$ goes to extinction with probability 1. Corollary 2 shows that, if $\tau = o(t)$, even conditioned process $\mathbf{X}(\tau, t)$ given $\mathbf{X}(\tau) \neq \mathbf{0}$ vanishes with a probability approaching 1.

Theorem 3 gives a limit distribution for $\mathbf{X}(\tau, t)$ when the times of observation $\tau \to \infty$ and $t \to \infty$ such that $\mathbf{Q}(t-\tau)/\mathbf{Q}(\tau)$ has a finite limit. Now we consider the case when this limit is not finite. Let $T_i(\tau, t) = Q^i(t-\tau)/Q^i(\tau)$

and $\mathbf{T}(\tau, t) = (T_1(\tau, t), ..., T_n(\tau, t))$.

Theorem 4. *If condition (20) holds, $\theta(t) = \theta \oplus \mathbf{V}/q(t)$ and $t, \tau \to \infty, t - \tau \to \infty$ such that $T_i(\tau, t) \to \infty, i = 1, 2, ..., n$, then*

$$P\left\{\frac{\mathbf{X}(\tau, \mathbf{t})}{\mathbf{T}(\tau, \mathbf{t})} \leq \mathbf{x} | \mathbf{X}(\tau) \neq \mathbf{0}, \mathbf{X}(0) = \delta_i\right\} \to \pi(\frac{1}{b}\mathbf{x}),$$

where $\pi(\mathbf{x}), \mathbf{x} \in R_+^n$, is the distribution from Proposition 2 and $b = 1 - \pi(\theta)$.

Remark. It follows from the asymptotic behavior of $Q^i(t)$ that, if $T_i(\tau, t) \to \infty$ for at least one i, then it holds for each $i = 1, 2, ..., n$.

Example 3. If matrix $\mathbf{A}$ is indecomposable, aperiodic, $\rho = 1$ and $b_{jk}^i < \infty, i, j, k = 1, ..., n$, then (20) is satisfied with $\alpha = 1, L(x) \to const, x \to 0$. In this case $Q^i(t) \sim 2u_i/\sigma^2 t, \quad i = 1, ..., n$ as $t \to \infty$, where $\sigma^2 = \sum_{j,m,k=1}^n v_j b_{mk}^j u_m u_k$. Consequently

$$q(t) = \sum_{j=1}^n Q^j(t) v_j \sim \frac{2}{\sigma^2 t}, t \to \infty$$

and $\theta(t) \sim \sigma^2 t \theta \oplus \mathbf{V}/2$. On the other hand $b = e^{-\theta^*}, \theta^* = \min\{\theta_1, ..., \theta_n\}$ and $T_j(\tau, t) \sim \tau/(t - \tau), j = 1, ..., n$. Thus $T_j(\tau, t) \to \infty$, if, for example, $\tau \sim t$ and we obtain the following result from Theorem 4.

Corollary 3. If $\rho = 1, 0 < \sigma^2 < \infty$ and $t, \tau \to \infty, t - \tau \to \infty$ such that $\tau \sim t$, then

$$P\left\{\frac{t - \tau}{\tau}\mathbf{X}(\tau, t) \leq \mathbf{x} | \mathbf{X}(\tau) \neq \mathbf{0}, \mathbf{X}(0) = \delta_i\right\} \to 1 - \exp\{-\frac{x^*}{b^*}\},$$

where $\mathbf{x} \in R_+^n, x^* = \min\{x_1, ..., x_n\}, \quad b^* = \exp\{-\min\{\theta_1, ..., \theta_n\}\}$.

The above two theorems describe the asymptotic behavior of $\mathbf{X}(\tau, t)$ when $t - \tau \to \infty$. Now we consider the case $\tau = t - \Delta$, where $\Delta \in (0, \infty)$ is a constant.

Theorem 5. *If condition (20) is satisfied, $t, \tau \to \infty$ such that $t - \tau = \Delta \in (0, \infty)$ and $\theta(\mathbf{t}) = \theta = (\theta_1, ..., \theta_n) \in R_+^n$, then*

$$P\{\mathbf{X}(\tau, t) \oplus \mathbf{Q}(\tau) \le \mathbf{x} | \mathbf{X}(\tau) \ne \mathbf{0}, \mathbf{X}(0) = \delta_i\} \to \pi(\frac{\mathbf{x}}{\mathbf{R}(\Delta)}),$$

where $\mathbf{x} \in R_+^n$ and

$$\mathbf{R}(\Delta) = (R^1(\Delta), ..., R^n(\Delta)), R^i(\Delta) = P\{\bigcup_{j=1}^{n} \{X_j(\Delta) > \theta_j\} | \mathbf{X}(0) = \delta_i\}.$$

Remark. It follows from Proposition 2 that

$$\frac{Q^i(\tau)}{Q^i(t)} \sim \Big(\frac{t}{t - \Delta}\Big)^{1/\alpha} \frac{L_1(t - \Delta)}{L_1(t)}$$

which shows that $Q^i(\tau) \sim Q^i(t)$ as $t, \tau \to \infty, t - \tau = \Delta$, for each $i = 1, ..., n$. Therefore the vector of normalizing functions $\mathbf{Q}(\tau)$ in Theorem 5 can be replaced by $\mathbf{Q}(t)$.

5 Proofs of theorems of Section 4

Proof of Theorem 3. It follows from the definition of $\mathbf{X}(\tau, t)$ in Section 3 that its ith component can be written as

$$X_i(\tau, t) = \sum_{\alpha \in R_\tau^{(i)}} \chi(\bigcup_{j=1}^{n} \{X_j^{(\alpha)}(t - \tau) > \theta_j(t - \tau)\}). \quad (23)$$

Since $card\{R_\tau^{(i)}\} = X_i(\tau)$ from here we can see that it can be presented in the form (1) with $\nu_{i\tau} = X_i(\tau)$ and

$$\xi_{ij}(\tau, t) = \chi(\bigcup_{l=1}^{n} \{X_{il}^j(t - \tau) > \theta_l(t - \tau)\}), \quad (24)$$

where $X_{il}^j(t)$ is the number of individuals of type T_l at time t in the process initiated by jth individual of type T_i. Hence Theorem 1 can be applied. It follows from Proposition 2 that condition (2) satisfied with $\mathbf{A}_\tau = \mathbf{V}/q(\tau)$ and with

$$\varphi(\lambda) = Ee^{-(\mathbf{Y},\lambda)} = \phi(\lambda),$$

where $\phi(\lambda)$ is defined in (21). Further it is not difficult to see that

$$E\xi_{ij}(\tau,t) = P(\bigcup_{l=1}^{n}\{X_{il}^{j}(t-\tau) > \theta_l(t-\tau)\})$$

$$= P(\bigcup_{l=1}^{n}\{X_{il}^{j}(t-\tau)q(t-\tau) > \theta_l v_l\}, \mathbf{X}(t-\tau) \neq \mathbf{0}\mathbf{X}(0) = \delta_i).$$

Hence using the asymptotic behavior of $\mathbf{Q}(\mathbf{t})$ again we obtain that when $\tau, t \to \infty, t-\tau \to \infty$

$$\frac{E\xi_{ij}(\tau,t)}{q(\tau)/v_i} \sim bv_i \frac{Q^i(t-\tau)}{q(\tau)}.$$

From here taking into account condition $\mathbf{Q}(t-\tau)/\mathbf{Q}(\tau) \to \mathbf{C}$ we conclude that

$$\frac{E\xi_{ij}(\tau,t)}{q(\tau)/v_i} \to bC_i u_i v_i,$$

which shows that condition (3) of Theorem 1 is also satisfied with $\mathbf{a} = b\mathbf{C} \oplus \mathbf{U} \oplus \mathbf{V}$. Consequently the assertion of Theorem 3 follows from Theorem 1. Theorem 3 is proved.

Proof of Theorem 4. We again use Theorem 1. As it was shown in the proof of Theorem 3, condition (2) of Theorem 1 is satisfied with $\mathbf{A}_\tau = \mathbf{V}/q(\tau)$. Now we consider

$$M_i(\tau,t) = \frac{E\xi_{ij}(\tau,t)}{q(\tau)} v_i,$$

where $\xi_{ij}(\tau,t)$ is the same as in (24). Appealing again to the asymptotic behavior of $\mathbf{Q}(\mathbf{t})$, we obtain that

$$M_i(\tau,t) \sim bv_i T_i(\tau,t) \frac{Q^i(\tau)}{q(\tau)} \tag{25}$$

as $t, \tau \to \infty, t - \tau \to \infty$. It follows from (25) that condition $M_i(\tau, t) \to \infty$ of Theorem 1 is also satisfied when $T_i(\tau, t) = Q^i(t - \tau)/Q^i(\tau) \to \infty$. The assertion of Theorem 4 follows now from Theorem 1.

Proof of Theorem 5. We use again Theorem 1. As in the proofs of the preceding theorems condition (2) follows from Proposition 2. If $t-\tau \in (0, \infty)$ we obtain from (24) that

$$M_i(\tau, t) = \frac{R_i(\Delta)}{q(\tau)} v_i.$$

Thus we have that $M_i(\tau, t) \to \infty$ as $t, \tau \to \infty, t - \tau \in (0, \infty)$. We obtain from Theorem 1 that

$$E[\prod_{i=1}^{n} e^{-\lambda_i X_i(\tau,t)Q^i(\tau)} | \mathbf{X}(\tau) \neq \mathbf{0}] \to \varphi(\lambda \oplus \mathbf{R}(\Delta)).$$

This yields the assertion of Theorem 5.

6 The number of productive ancestors

Now we consider a population containing at time $t = 0$ a random number $\nu_i(t), i = 1, ..., n, t \in N_0$ individuals (ancestors) of n different types $T_1, ..., T_n$. Each of these individuals generates a discrete time indecomposable n type branching stochastic process. Let $\theta(t) = (\theta_1(t), ..., \theta_n(t))$ be a vector of non-negative functions. In how many processes generated by these ancestors the number of descendants at time t of at least one type will exceed the corresponding level given by $\theta(t)$? To answer the question we investigate process $\mathbf{Y}(t) = \mathbf{Y}([\theta], t) = (Y_1(t), ..., Y_n(t))$, where $Y_i(t)$ is the number of initial individuals of type T_i whose number of descendants at time t of at least one type is greater than corresponding component of the vector $\theta(t)$. It is clear that $\mathbf{Y}(t)$ takes into account only "relatively productive" ancestors regulated by family of levels $\theta(t), t \in N_0$.

Process $\mathbf{Y}(t)$ may be associated with the following scheme describing growth of n-type trees in a forest. Suppose at time zero we have $\nu_i(t), i = 1, ..., n$, one branch trees of types T_i. Each of these trees will grow and give new branches of types $T_1, ..., T_n$ according to independent, indecomposable n

type branching processes. Then process $\mathbf{Y}(t) = (Y_1(t), ..., Y_n(t))$ will count the number of "big trees": $Y_i(t)$ is the number of big trees of type T_i having more than $\theta_j(t)$ new branches at time t for at least one $j, j = 1, ..., n$.

It is not difficult to see that the components of the process $Y_i(t)$ can be presented as

$$Y_i(t) = \sum_{j=1}^{\nu_i(t)} \xi_{ij}(t), \tag{26}$$

where $\xi_{ij}(t) = \chi(\bigcup_{l=1}^n \{X_{il}^j(t) > \theta_l(t)\})$ and $X_{il}^j(t)$ is, as before, the number of individuals of type T_l at time t in the process initiated by jth ancestor of type T_i. Consequently theorems proved for random sum (1) may be applied to this process.

Let all assumptions from Part 4 on $n-$ type branching process $\mathbf{X}(t), t \in N_0$ be satisfied and the generating function corresponding to probability distribution $P_\alpha^i, \alpha \in N_0^n$ satisfies equation (20).

Theorem 6. *Let condition (20) be satisfied and $\theta(t) = \theta \oplus \mathbf{V}/q(t), \theta \in R_+^n$. If condition (2) is satisfied and for the normalizing coefficients in (2)*

$$A_{it} Q^i(t) \to \infty \tag{27}$$

as $t \to \infty$ for $i = 1, ..., n$, then

$$P\{\frac{Y_i(t) - \nu_{it} a_i(t)}{\sqrt{\nu_{it} a_i(t)}} \leq x_i, i = 1, ..., n | \nu \neq \mathbf{0}\} \to L(\mathbf{x}),$$

where $\mathbf{x} \in R^n, a_i(t) = bQ^i(t), b = 1 - \pi(\theta), \theta \in R_+^n$ and $L(\mathbf{x})$ defined in Theorem 2.

Proof. We demonstrate that conditions of Theorem 2 are satisfied. It is clear that we just need to show that condition C1 holds for the variables defined in (26). As in the proof of Theorem 3 we easily obtain that

$$E\xi_{ij}(t) = P(\bigcup_{l=1}^n \{X_{il}^j(t) > \theta_l(t)\}) =$$

$$P(\bigcup_{l=1}^n \{X_{il}^j(t) q(t) > \theta_l v_l\}, \mathbf{X}(t) \neq \mathbf{0}, \mathbf{X} = \delta_i) \sim bQ^i(t).$$

Consequently, if we take $l_{it} = 1/Q^i(t)$, then for $i = 1, ..., n$ as $t \to \infty$

$$l_{it} E\xi_{ij}(t)(1 - E\xi_{ij}(t)) \to b.$$

On the other hand $A_{it}/l_{it} \to \infty, i = 1, ..., n$ as $t \to \infty$ due to condition (27). Hence condition C1 of Theorem 2 is satisfied and assertion of the theorem follows from Theorem 2. Theorem 6 is proved.

In conclusion we note that according to condition (27) the assertion of Theorem 6 holds when the initial population is large enough. One may obtain limit distributions for $\mathbf{Y}(t)$ when this condition is not satisfied. To do it one needs to apply theorem 1 from Section 2.

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