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Existence of Solutions of Periodic Boundary Value Problems

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Abstract. In this paper we consider the problem of existence of solutions of periodic boundary value problems for second order differential equations. We provide a new sufficient conditions on the nonlinearity in order to obtain a priori bound on solutions. We shall rely on the topological transversality theorem and Ascoli-Arzela theorem to prove the existence of at least one solution.

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1 Introduction

Our aim in this paper is to establish the existence of solutions for the following periodic boundary value problems

$$\begin{cases} y''(t) = f(t, y(t), y'(t)) \qquad 0 < t < 1 \\ y(0) = y(1) \\ y'(0) = y'(1) \end{cases} \tag{1}$$

under fairly simple and quite general conditions on the nonlinearity f, which is assumed to be an L^1-Caratheodory function; i.e. f satisfies

(i) $f(\cdot, y, z)$ is measurable for all $(y, z) \in \mathbb{R}^2$.

ii) $f(t, \cdot, \cdot)$ is continuous for almost all $t \in [0, 1]$.

(iii) for each $\rho > 0$ there exists $h_\rho \in L^1(0, 1)$ such that $|y| + |z| \leq \rho$ implies that $|f(t, y, z)| \leq h_\rho(t)$ for almost all $t \in [0, 1]$.

The method of a priori bound on solutions combined with the topological transversality theorem or the topological degree has been very effective in proving existence of solutions of problem (1) (see for instance [2], [3], [4], [6], [7], [8] and the references therein). Also, the method of upper and lower solutions and the monotone method have been successfully used to establish the existence of at least one solution of problem (1) (see [10], [11] and the references therein).

Our work is motivated by the results in the papers [7] and [8], where the authors assume that f satisfies a sign condition and a Nagumo-Bernstein condition. We shall provide sufficient conditions of f that are more general than the conditions imposed in [7] and [8], and more general then the assumption of the existence of constant upper and lower solutions in the case when these are well ordered and in the case of reverse order. Also, our technique is different from those used in [4], [9], [10] and [11]. In fact, our main result is an improvement of the results in [2], [3], [6], [7] and [8].

2 Preliminaries

Let I denote the real interval [0,1]. $X = AC^1(I)$ denotes the Banach space of absolutely continuous real-valued functions together with their first derivatives on I, equipped with the norm

$$||y|| = \max\{|y(t)| + |y'(t)|; \quad t \in I\} \text{ for any } y \in X.$$

$\mathrm{Car}(I \times \mathbb{R}^2)$ is the set of all real-valued functions satisfying the Caratheodory conditions (i), (ii), (iii).

By a solution of (1) we mean a function

$$y \in X_0 = \{u \in X; \quad u(0) - u(1) = u'(0) - u'(1) = 0\}$$

satisfying the differential equation in (1) almost every where on I.

Since the homogeneous problem $y'' = 0,\ y(0) - y(1) = y'(0) - y'(1) = 0$ has nontrivial solutions, we shall deal with the following problem, for $m > 1$

$$\begin{cases} y''(t) = \frac{1}{m}y(t) + f(t, y(t), y'(t)) \qquad 0 < t < 1 \\ y(0) = y(1) \\ y'(0) = y'(1) \end{cases} \tag{1_m}$$

Problem (1) is considered as a limiting case of (1_m) as $m \to +\infty$.

Lemma 1. The problem $y'' = \frac{1}{m}y,\ y(0) - y(1) = y'(0) - y'(1) = 0$ has only the trivial solution. The Green's function $G_m(t, s)$ exists and there exists a constant $\gamma_m > 0$ such that

$$|G_m(t, s)| + \left|\frac{\partial G_m}{\partial t}(t, s)\right| \leq \gamma_m \text{ for all } (t, s) \in I^2.$$

Proof.

• Suppose, on the contrary that the homogeneous problem has a nontrivial solution y_0. Then

$$\int_0^1 y_0''(t)y_0(t)dt = \frac{1}{m}\int_0^1 y_0(t)^2\,dt$$

A simple integration by parts of the left side and the boundary conditions lead to

$$-\int_0^1 y_0'(t)^2 dt = \frac{1}{m}\int_0^1 y_0(t)^2\,dt$$

which is impossible.

• As a consequence, the Green's function, $G_m(t,s)$, exists and has the following representation (see[1]),

$$G_m(t,s) = \begin{cases} -\dfrac{\sqrt{m}}{2}\,\dfrac{\cos h\,\frac{1}{\sqrt{m}}\left(t-s+\frac{1}{2}\right)}{\sin h\,\frac{1}{2\sqrt{m}}} & 0 \le t \le s \\ -\dfrac{\sqrt{m}}{2}\,\dfrac{\cos h\,\frac{1}{\sqrt{m}}\left(t-s-\frac{1}{2}\right)}{\sin h\frac{1}{2\sqrt{m}}} & s < t \le 1 \end{cases}$$

The properties of the hyperbolic cosine function and the fact that $\sin h\theta > \theta$ for all $\theta > 0$ imply that

$$|G_m(t,s)| \le m\cos h\,\frac{1}{2\sqrt{m}} \;\text{ and }\; \left|\frac{\partial G_m}{\partial t}(t,s)\right| \le \frac{1}{2}$$

for all $(t,s) \in I^2$.

Letting $\gamma_m = \dfrac{1}{2} + m\,\cos h\,\dfrac{1}{2\sqrt{m}}$ we get the desired inequality, and the lemma is proved.

Lemma 2. Assume there exists $h \in L^1(I)$ such that

$$|f(t,y,z)| \le h(t) \text{ for almost all } t \in I.$$

Then problem (1_m) has at least one solution.

Proof. It follows from Lemma 1 that problem (1) is equivalent to the nonlinear integral equation

$$y(t) = \int_0^1 G_m(t,s) f(s, y(s), y'(s))\, ds \ \text{ a.e. } \ t \in I$$

So that

$$y'(t) = \int_0^1 \frac{\partial G_m(t,s)}{\partial t} f(s, y(s),\, y'(s))\, ds$$

Therefore

$$||y|| \leq \gamma_m ||h||_{L^1}$$

Define a nonlinear operator $T : X \longrightarrow X_0$ by

$$(Ty)(t) = \int_0^1 G_m(t,s) f(s, y(s),\, y'(s))\, ds$$

Let $D := \{y \in X_0;\ ||y|| \leq \gamma_m ||h||_{L^1}\}$.

One can easily show that T is completely continuous and maps the closed convex set D into itself. By the Schauder fixed point theorem T has a fixed point in D, which is a solution of problem (1_m).

3 Main Results

In this section we shall state a new sufficient condition in order to obtain a priori bound on solutions of a one-parameter family of problems related to (1_m). We then prove that for each $m > 1$, problem (1_m) has at least one solution y_m such that $||y_m||$ is uniformly bounded, and moreover the bound does not depend on m. Going to subsequences, if necessary, we see that $y = \lim_{m \to +\infty} y_m$ is a solution of problem (1).

Since our arguments are based on the topological transversality theorem (see [5], [6] for definitions and properties) we shall consider the following one-parameter family of problems

$$\begin{cases} y''(t) = \frac{1}{m}\, y(t) + \lambda f(t, y(t),\, y'(t)) & 0 < t < 1 \\ y(0) = y(1) & \\ y'(0) = y'(1) & \end{cases} \qquad (1_m \cdot \lambda)$$

where $0 \leq \lambda \leq 1$.

Notice that $(1_m \cdot 1)$ is exactly (1_m) and $(1_m \cdot 0)$ has only the trivial solution. Moreover $(1_m \cdot \lambda)$ is equivalent to the abstract equation.

$$y = H_m(\lambda, y) \qquad (2_m \cdot \lambda)$$

where $H_m(\lambda, y)(t) = \lambda \int_0^1 G_m(t, s) f(s, y(s),\, y'(s) ds$ and for each λ, $H_m(\lambda, \cdot)$ is a compact operator. We now state and prove our main results.

Consider the modified problem

$$\begin{cases} y''(t) = \dfrac{1}{m}\, y(t) + \lambda f_1(t, y(t),\, y'(t)) & 0 < t < 1 \\ y(0) = y(1) & \\ y'(0) = y'(1) & \end{cases} \qquad (3_m \cdot \lambda)$$

where $f_1 : I \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is given by

$$f_1(t, y, z) = \begin{cases} \max\left\{ f(t, y, z), -\dfrac{k_0}{m} + \displaystyle\int_0^1 f(t, k_0, 0) dt \right\} & y > k_0 \\ f(t, y, z) & -k_0 \leq y \leq k_0 \\ \min\left\{ f(t, y, z), \dfrac{k_0}{m} + \displaystyle\int_0^1 f(t, -k_0, 0) dt \right\} & y < -k_0 \end{cases}$$

Proposition 1. Assume $f \in Car\,(I \times \mathbb{R}^2)$ satisfies

(H1) there exists $k_0 > 0$ such that

$$\left(\int_0^1 f(t, k_0, 0) dt \right) \left(\int_0^1 f(t, -k_0, 0) dt \right) < 0$$

Then, any solution y of $(3_m \cdot \lambda)$ satisfies the estimate

$$|y(t)| \leq k_0 \ \text{ for all } t \in I \qquad (* \cdot 1)$$

Proof. The proof follows closely that of Lemma 3.6 in [8] with some modifications. Without loss of generality, we shall assume that

$$\int_0^1 f(t, k_0, 0)dt > 0 \text{ and } \int_0^1 f(t, -k_0, 0)dt < 0.$$

Notice that any solution y of $(3_m \cdot \lambda)$ satisfying $(* \cdot 1)$ is a solution of $(1_m \cdot \lambda)$ and conversely any solution y of $(1_m \cdot \lambda)$ satisfying $(* \cdot 1)$ is a solution of $(3_m \cdot \lambda)$ since $f(t, y, z) \equiv f_1(t, y, z)$ when $|y| \leq k_0$.

We show that any solution y of $(3_m \cdot \lambda)$ satisfies $(* \cdot 1)$.

For $\lambda = 0$, problem $(3_m \cdot \lambda)$ has only the trivial solution, which, clearly, satisfies $(* \cdot 1)$.

Let y be a possible solution of $(3_m \cdot \lambda)$ for $0 < \lambda \leq 1$.

Let $t_0 \in I$ be a such that y takes on its positive maximum at t_0. Then $y'(t_0) = 0$.

Suppose $y(t_0) > k_0$ and $t_0 \in (0, 1)$. Then there exists $a > 0$ such that $y(t) > k_0$ for all $t \in [t_0, t_0 + a]$. It follows from the differential equation in $(3_m \cdot \lambda)$ and the definition of f_1, that

$$y''(t) \geq \frac{y(t)}{m} - \frac{k_0}{m} + \int_0^1 f(t, k_0, 0)dt \qquad t_0 \leq t \leq t_0 + a.$$

Hence $y''(t) > 0$ for all $t \in [t_0, t_0 + a]$.

Since $y(t) - y(t_0) = \int_{t_0}^t (t - s)y''(s)ds$ for all $t \geq t_0$ we see that $y(t) > y(t_0)$ and this contradicts the fact that $y(t_0)$ is the maximum value of y on I.

If, it happens that $t_0 = 0$, then assuming $y(0) > k_0$ we also arrive at a contradiction. By periodicity, we have $y(1) = y(0)$ and so, $y(1) > k_0$ will also lead to a contradiction.

Therefore,

$$y(t) \le k_0 \text{ for all } t \in I$$

Next, if y takes on a negative minimum at $t = \tau_0$ such that $y(\tau_0) < -k_0$ then we can find $b > 0$ such that $y(t) < -k_0$ for all $t \in [\tau_0, \tau_0 + b]$.

Hence

$$y''(t) \le \frac{y(t)}{m} + \frac{k_0}{m} + \int_0^1 f(s, -k_0, 0)ds < 0 \qquad \tau_0 \le t \le \tau_0 + b$$

This implies that

$$y(t) - y(\tau_0) = \int_{\tau_0}^t (t-s)y''(s)ds < 0 \qquad t \ge \tau_0$$

which contradicts the fact that $y(\tau_0)$ is the minimum value of y on I.

Hence

$$y(t) \ge -k_0 \text{ for all } t \in I$$

Therefore, we have proved that the estimate $(* \cdot 1)$ holds for any solution y of $(3_m \cdot \lambda)$.

Remark 1. The authors in [7] and [8] assume the existence of $M > 0$ such that $|y| \ge M$ implies that $yf(t, y, 0) > 0$ for almost all $t \in I$. It is clear that our assumption is much more general than this sign condition.

Remark 2. If we assume that problem (1_m) has a lower solution, $-k_0$,and an upper solution k_0, then we see that $f(t, -k_0, 0) < 0$ for all $t \in I$ and $f(t, k_0, 0) > 0$ for all $t \in I$. Obviously, our assumption is much weaker than this condition

Remark 3. Our assumption covers also the case of constant upper and lower solutions which are in the reverse order. To see this, consider the case $\int_0^1 f(t, k_0, 0)dt < 0$ and $\int_0^1 f(t, -k_0, 0)dt > 0$ in the proof of the proposition with an appropriate modified function f_1.

Remark 4. We can add a suitable condition on f in order to develop the monotone iterative method (see [10], [11]), starting with $-k_0$ as a lower solution and k_0 as an upper solution.

Our next result gives an a priori bound on y' for any solution y of $(3_m \cdot \lambda)$ satisfying $(* \cdot 1)$.

Proposition 2. Assume $f \in Car\,(I \times \mathbb{R}^2)$ satisfies.

$(H2)$ there exists $\gamma > 0$, $l \in L^1(I)$ and $\psi : [0, +\infty) \to (0, +\infty)$ with $\dfrac{1}{\psi}$ integrable over bounded intervals and $\displaystyle\int_0^{+\infty} \frac{d\sigma}{\psi(\sigma)} = +\infty$ such that $\displaystyle\sup_{|y| \le k_0} |y + f(t, y, z)| \le (l(t) + \gamma |z|)\psi(|z|)$ for all $(t, z) \in I \times \mathbb{R}$.

Then there exists $C_0 > 0$, independent of m and λ, such that, for any solution y of $(3_m \cdot \lambda)$ satisfying the estimate $(* \cdot 1)$ it holds

$$|y'(t)| \le C_0 \text{ for all } t \in I \qquad (* \cdot 2)$$

Remark 5. Condition (H2) is known as the Nagumo-Wintner condition, which is more general than the Nagumo or Nagumo-Bernstein condition.

Proof. Let y be a solution of $(3_m \cdot \lambda)$ such that

$$|y(t)| \le k_0 \text{ for all } t \in I.$$

Choose

$$C_0 > 0 \text{ such that } \int_0^{C_0} \frac{d\sigma}{\psi(\sigma)} > ||l||_{L^1} + 2k_0\,\gamma$$

We want to show that $|y'(t)| \le C_0$ for all $t \in I$. Suppose, on the contrary, that there exists $\overline{t} \in I$ such that $|y'(\overline{t})| > C_0$. Since $y(1) = y(0)$ there exists $\underline{t} \in I$ such that $y'(\underline{t}) = 0$.

Thus, we have $|y'(\underline{t})| = 0$ and $|y'(\overline{t})| > C_0$. Since $y \in AC^1(I)$, there exists an interval $[\sigma_1, \sigma_2] \subset I$ such that one of the following situation holds

(i) $y'(\sigma_1) = 0,\ \ y'(\sigma_2) = C_0$ and $0 < y'(t) < C_0$ for all $t \in (\sigma_1, \sigma_2)$.

(ii) $y'(\sigma_1) = C_0,\ \ y'(\sigma_2) = 0$ and $0 < y'(t) < C_0$ for all $t \in (\sigma_1, \sigma_2)$.

(iii) $y'(\sigma_1) = 0,\ \ y'(\sigma_2) = -C_0$ and $-C_0 < y'(t) < 0$ for all $t \in (\sigma_1, \sigma_2)$.

(iv) $y'(\sigma_1) = C_0,\ \ y'(\sigma_2) = 0$ and $-C_0 < y'(t) < 0$ for all $t \in (\sigma_1, \sigma_2)$.

We consider only the first case, since the other cases can be handled similarly.

We have

$$\begin{aligned} y''(t) &= \frac{y(t)}{m} + \lambda f_1(t, y(t),\, y'(t)) \\ &= \frac{y(t)}{m} + \lambda f(t, y(t),\, y'(t)) \end{aligned}$$

(Since f and f_1 coincide when $|y| \leq k_0$).

Hence, since $m > 1$ and $0 < \lambda \leq 1$

$$\begin{aligned} y''(t) &\leq |y''(t)| \leq |y(t)| + |f(t, y(t),\, y'(t))| \\ &\leq (l(t) + \gamma |y'(t)| \psi(|y'(t)|) \quad \text{for all } \ t \in I \end{aligned}$$

So that, for $t \in [\sigma_1, \sigma_2]$ we have

$$y''(t) \leq [l(t) + \gamma\, y'(t)]\psi(y'(t))$$

which gives

$$\frac{y''(t)}{\psi(y'(t))} \leq l(t) + \gamma\, y'(t)$$

Thus

$$\begin{aligned}\int_{\sigma_1}^{\sigma_2} \frac{y''(t)dt}{\psi(y'(t))} &\leq \int_{\sigma_1}^{\sigma_2} l(t)dt + \gamma \int_{\sigma_1}^{\sigma_2} y'(t)dt \\ &\leq \int_0^1 l(t)dt + \gamma(y(\sigma_2) - y(\sigma_1)) \\ &\leq ||l||_{L^1} + 2k_0\gamma\end{aligned}$$

A change of variables in the left side gives

$$\int_0^{C_0} \frac{d\sigma}{\psi(\sigma)} \leq ||l||_{L^1} + 2k_0\gamma.$$

This is clearly in contradiction with the definition of C_0.

Therefore,

$$|y'(t)| \leq C_0 \text{ for all } t \in I.$$

This completes the proof of the proposition.

Theorem 3. Assume of $f \in Car(I \times \mathbb{R}^2)$ satisfies $(H1)$ and $(H2)$. Then problem (1) has at least one solution.

Proof. It follows from the above results that any possible solution y of $(3_m \cdot \lambda)$ satisfies the estimates

$$|y(t)| \leq k_0,\ |y'(t)| \leq C_0 \ \text{ for all } t \in I$$

Therefore,

$$||y|| \leq N := k_0 + C_0.$$

On the other hand, solutions of $(3_m \cdot \lambda)$ satisfying $(* \cdot 1)$ are also solutions of $(1_m \cdot \lambda)$ and equivalently, solutions of $(2_m \cdot \lambda)$.

Let $\Omega := \{y \in X_0;\ ||y|| < 1 + N\}$. Then $H_m : [0,1] \times \overline{\Omega} \longrightarrow X_0$ is a compact homotopy without fixed points on $\partial\Omega$, the boundary of Ω. Since $H_m(0, \cdot) \equiv 0$ is essential, the topological transversality theorem (see for instance [5], [6]) implies that $H_m(1,\cdot)$ is essential. Hence, it has a fixed point in Ω, which we denote by y_m. This fixed point y_m is a solution of (1_m) and satisfies (see the previous discussion) the estimates $(*\cdot 1)$ and $(*\cdot 2)$, i.e., $|y_m(t)| \leq k_0$, $|y_m'(t)| \leq C_0$ for all $t \in I$. We deduce that the sequences $\{y_m(t)\}$ and $\{y_m'(t)\}$ are uniformly bounded. To see that these are equicontinuous, let $t_1, t_2 \in I$, and consider

$$y_m'(t_2) - y_m'(t_1) = \int_{t_1}^{t_2} y_m''(s)ds = \frac{1}{m}\int_{t_1}^{t_2} y_m(s)ds + \lambda \int_{t_1}^{t_2} f(s, y_m(s), y_m'(s))ds.$$

Since $m > 1$, $0 \leq \lambda \leq 1$ and $f \in Car(I \times \mathbb{R}^2)$ it follows from $(*\cdot 1)$ and $(*\cdot 2)$ that

$$|y_m'(t_2) - y_m'(t_1)| \leq k_0 |t_2 - t_1| + \int_{t_1}^{t_2} h_{k_0}(s)ds$$

Also,

$$|y_m(t_2) - y_m(t_1)| \leq C_0 |t_2 - t_1|.$$

The last two inequalities show that $\{y_m(t)\}$ and $\{y_m'(t)\}$ are equicontinuous. By the Arzela-Ascoli theorem, we can extract subsequences, which we label the same, and which are uniformly convergent on I.

Let

$$y(t) = \lim_{m\to+\infty} y_m(t) \text{ and } z(t) = \lim_{m\to+\infty} y_m'(t).$$

Since $y_m(t) = y_m(0) + \int_0^t y_m'(s)ds$, by the uniform convergence of $\{y_m(t)\}$ and $\{y_m'(t)\}$ we have that $y(t) = y(0) + \int_0^t z(s)ds$.

Therefore, $z(t) = y'(t)$ i.e. $\lim_{m\to\infty} y_m'(t) = y'(t)$. Consequently, y is a solution of (1). This completes the proof of the theorem.

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