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Abstract

Mellin transform is an ill-posed problem. These problems arise in many branches of science and engineering. In the typical situation one is interested in recovering the original function, given a finite number of noisy measurements of data. In this paper, we shall convert Mellin transform to Laplace transform and then to an integral equation of the first kind of convolution type. We solve the integral equation using Tikhonov regularization with splines as basis function. The method is applied to various test examples available in the literature and results are shown in the table.

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1 Introduction

The regularization method for solving the Fredholm integral equations of the first kind (including Mellin transforms) was proposed independently by Phillips [23], and in a more general form by Tikhonov [30]. In this method, instead of an exact solution, one searches for an approximate solution.

One should note that despite the apparent simplicity of the Tikhonov regularization, it requires rather complicated computational algorithms to accomodate numerical integration, selection of the regularization parameter and minimization technique.

The Mellin transform defined by

$$M(\Phi) = g(s) = \int_0^\infty p^{s-1}\Phi(p)dp \qquad s\epsilon c \tag{1.1}$$

is one of the most important integral transforms. It can be proved that the relationship $g = M(\Phi)$ defines a one-one correspondence between g and Φ.

In the present paper we study that equation (1.1) has the form of a Fredholm integral equation of the first kind and it is well known that the problem of solving such equations is basically ill-posed. Ill-posed problems have become a recurrent theme in modern sciences, see for example in crystllography (Grunbaum [16]), Geophysics (Aki and Richards [1]), Medical electrocardiogram (Franzone [13]), meteorology (Smith [26]), radio astronomy (Jaynes [17]), reservoir engineering (Karavaris [18]), and tomography (Budigner [3]).

Basically the smoother the Kernel, the faster the decay rate and hence more ill-posed the problem (De Hoog [9]). Many physicists have discovered after much wasted effort that it is essential to understand this feature before attempting to compute solutions.

The ill-posedness of Laplace transform inversion, in the case where $\Phi \,\epsilon\, L^2(R_+)$ can be investigated by means of Mellin transform (McWhirter [20]). In practice, however, the case of an infinite set of equidistant points was investigated by papoulis [21]. Several other methods have been proposed and a detailed review and comparison is given in Davies [8], Talbot [28], Engl [11], Mastorakis [19].

The above mentioned methods do not include regularization techniques. Regularization methods have been discussed by Brianzi [4], Essa and Delves [12], Ang [2], Gelfat [15]. Varah [31], Wahba [32, 33], D'Amore [5,6,7] and Dong [10].

In particular the theory is used to tackle the Laplace transform inversion in a well-conditioned (regularized) manner. The difficult numerical problem which is frequently encountered by physicists and engineers, is still the subject of much attention in the literature. Finally we include some specific numerical examples available in the literature to illustrate and demonstrate clearly the need to consider information content in order to avoid obtaining meaningless results.

2 Conversion to the Laplace Transform

We will consider an approach to reduce the inversion of the Mellin transform to inversion of Laplace transform.

Denote the Mellin transform

$$g(s = \int_0^\infty \Phi(p) p^{s-1} dp \tag{2.1}$$

using $\Phi(p) = \int_0^\infty f(t) e^{-pt} dt \qquad \mathcal{R}(p) > 0.$

We get

$$\left.\begin{array}{c} g(s) = \Gamma(s)\int_0^\infty f(t)t^{-s}dt \\ \text{where } \Gamma(s) = \int_0^\infty e^{-u}u^{s-1}du \end{array}\right\} \tag{2.2}$$

where $\Gamma(s)$ is the Gamma function.

From (2.2) we get

$$\left.\begin{array}{c} q(1-s) = \frac{g(s)}{\Gamma(s)} \\ q(s) = \frac{g(1-s)}{\Gamma(1-s)} \end{array}\right\} \tag{2.3}$$

The inversion formula for the Mellin transform, applied to (2.3), yields

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{g(1-s)}{\Gamma(1-s)}\right)t^{-s}\,ds. \tag{2.4}$$

It is known (see Erdelyi [34]) that

$$\left.\begin{array}{c} |\Gamma(x+iy)| = \sqrt{2\pi}|y|^{x-\frac{1}{2}}\,e^{-\frac{\pi}{2}|y|} \\ \text{as } y\to\infty,\quad -\pi < \ \arg\,(x+iy) < \pi \end{array}\right\} \tag{2.5}$$

where x and y are real numbers. Consider $z = x+iy$ and (2.5) becomes,

$$\left.\begin{array}{c} \Gamma(z) = \sqrt{2\pi}\,e^{-z}\,e^{(z-\frac{1}{2})\log z}\left[1+0\left(\frac{1}{|z|}\right)\right] \\ \text{as } |z|\to\infty,\ |arg(z)| < \pi \end{array}\right\} \tag{2.6}$$

Integral (2.4) converges if, for example,

$$|g(1-\sigma+iy)| \le ce^{-\frac{\pi}{2}|y|},\quad 0<\sigma<\frac{1}{2} \tag{2.7}$$

The basic idea of the inversion method based on formula (2.4), consists of using $\Phi(p)$ given for $p>0$, in order to calculate $q(s)$ on the line $\mathcal{R}(s)=\sigma>0$. One of the important applications of the numerical inversion of the Laplace transform from the real axis is related to a problem of the numerical analytic continuation of a function analytic in the complex half plane $\mathcal{R}z)>0$, given on the positive semiaxis to complex half plane $\mathcal{R}(z)>0$. Such numerical analytic continuation is an important step for solving some ill-posed problems of practical interest (see Ramm [24]). Assume that $\Phi(p)$ is a Laplace image of some function $f(t)$, such that $\Phi(p)$ can be continued analytically to the half plane $\mathcal{R}(z) > 0$. In order to compute this analytic continuation numerically one can compute Laplace preimage $f(t)$ of $\Phi(p)$ first and then compute the integral.

$$\Phi(z) = \int_0^\infty e^{-zt} f(t)dt, \quad \mathcal{R}(z) > 0 \tag{2.8}$$

3 Conversion to the Integral Equation of Convolution Type

We are interested in solving (2.8). We make the following substitution. Let

$$z = a^x \text{ and } t = a^{-y}, \qquad a > 1 \tag{3.1}$$

Then (2.8) becomes

$$\Phi(a^x) = \int_{-\infty}^{\infty} (\log a) e^{-a^{x-y}} f(a^{-y})\, a^{-y} dy \tag{3.2}$$

multiplying both sides by a^x, we obtain the convolution equation

$$\int_{-\infty}^{\infty} K(x-y)F(y)dy = G(x) \tag{3.3}$$

where

$$\left.\begin{array}{c} G(x) = a^x \Phi(a^x) = z\Phi(z) \\ K(x) = (\log a)e^{-a^x} \cdot a^x = \log(a) z e^{-z} \\ F(y) = f(a^{-y}) = f(t) \end{array}\right\} \tag{3.4}$$

In order that we can apply our deconvolution method to equation (3.3), $G(x)$ has usually compact support that is

$$G(x) \to 0 \text{ as } x \to \pm\infty.$$

4 Description of the Method

Let $B_j(H;x)$ be the n-the order cardinal B-spline (n-even) with knots $(j-\frac{1}{2}n)H,\ldots,(j+\frac{1}{2}n)H, i.e., B_j(H;x) = Q_n(\frac{x}{H} - j + \frac{n}{2})$ where

$$Q_n(x) = \frac{1}{(n-1)!}\sum_{j=0}^{n}(-1)^j \left(\begin{array}{c} n \\ j \end{array} \right) (x-j)_+^{n-1}. \tag{4.1}$$

In addition let $MH = T$ where $M \leq N$ s an integral power of 2. We assume that $B_j(H;x)$ s periodically continued outside the interval $(0,T)$, with period T. Then $B_j(H;x)$ has a Fourier series

$$B_j(H;x) = \sum_{q=-\infty}^{\infty} \hat{B}_{jq} \exp\,(iw_q x) \tag{4.2}$$

where

$$\hat{B}_{jq} = \int_0^T B_j(H;x)\,\exp(-iw_q x)dx \tag{4.3}$$

and $w_q = \frac{2\pi q}{T}$.

Since $B_j(H;x)$ is simply a translation of $B_o(H;x)$ by an amount jH, we have

$$\hat{B}_{jq} = \hat{B}_{oq}\,\exp\,(-iw_q H)$$

where

$$\hat{B}_{0q} = H \left[\frac{\sin \frac{w_q H}{2}}{\frac{w_q H}{2}} \right]^4 \tag{4.4}$$

4.1 Tikhonov Regularization Using Cardinal Cubic B-Splines

We shall approximate the convolution equation (3.3) by

$$\int_0^T K_N(x-y)F_M(y)dy = G_N(x) \tag{4.5}$$

where

$$G_N(x) = \frac{1}{N}\sum_{q=0}^{N-1} \hat{G}_{N,q}\,\exp\,(iw_q x)$$

and

$$K_N(x) = \frac{1}{N}\sum_{q=0}^{N-1} \hat{K}_{N,q} \exp\,(iw_q x)$$

where we assume that F, G and K in (3.4) have essentially finite support in $[0, T]$, F_M is a cubic spline ($n = 4$) of the form

$$F_M(x) = \sum_{j=1}^{M-1} a_j B_j(H; x), \qquad M \leq N \tag{4.6}$$

where $N = 2^n$ (data points.

The real M dimensional vector

$$\underline{\alpha} = (\alpha_0, \ldots, \alpha_{M-1})^T$$

of unknown coefficients will be determined where $B_j(H; x)$ is the n-th order cardinal B-spline.

$$B_j(H; x) = Q_n\left(\frac{x}{H} - j + \frac{1}{2}n\right)$$

and

$$Q_n(x) = \frac{1}{(n-1)!}\sum_{j=0}^{n}(-1)^j \begin{pmatrix} n \\ j \end{pmatrix} (x-j)_+^{n-1}$$

The spline in equation (4.6) has the Fourier series

$$F_M(x) = \sum_{q=-\infty}^{\infty} \hat{F}_{M,q} \exp\,(iw_q x) \tag{4.7}$$

where

$$\hat{F}_{M,q} = \sum_{j=0}^{M-1} \alpha_j \hat{B}_{jq} \tag{4.8}$$

$$= \hat{B}_{0q} \sum_{j=0}^{M-1} \alpha_j \exp\left(-\frac{2\pi i}{M} j_q\right) \tag{4.9}$$

$$= \sqrt{M}\hat{B}_{0q}\hat{\alpha}_s, \qquad s \equiv q(mod\, M),$$

where ψ is the unitary matrix with elements

$$\psi_{rs} = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi}{N} s\right), \qquad r, s = 0, 1, 2, \ldots, N-1$$

and

$$\underline{\hat{\alpha}} = \psi_M^H \underline{\alpha} \tag{4.10}$$

The matrix ψ_M is a circulant matrix.

We find it advantageous to determine $\underline{\hat{\alpha}}$ rather than $\underline{\alpha}$, because of the simple properties available in discrete Fourier spaces. The vector $\underline{\alpha}$ in equation (4.6) may then be determined from the inverse M-dimensional FFT (Fast Fourier transform)

$$\underline{\alpha} = \psi_M \underline{\hat{\alpha}} \tag{4.11}$$

4.2 p-th Order Tikhonov Regularization [30]

Consider the smoothing functional

$$C(F_M; \lambda) = C(\underline{\alpha}, \lambda) = ||K_N * F_M - G_N||_2^2 + \lambda ||F_M^{(p)}||_2^2 \tag{4.12}$$

$F_M^{(p)}$ is the p-th derivative.

Using Plancherel's theorem we have

$$||K_N * F_M - G_N||_2^2 = \frac{1}{N^2} \sum_{q=-\frac{1}{2}N}^{\frac{1}{2}N} |\hat{K}_{N,q} \hat{F}_{M,q} - \hat{G}_{N,q}|^2$$

Hence using equation (4.9)

$$||K_N * F_M - G_N||_2^2 = \frac{1}{N^2} \sum_{q=-\frac{1}{2}N}^{\frac{1}{2}} \left[\left(\sqrt{M} \hat{B}_{0,q} \hat{\alpha}_s \hat{K}_{N,q} - \hat{G}_{N,q}\right) \left(\sqrt{M} \hat{B}_{0q} \overline{\hat{B}}_{0q} \overline{\hat{K}}_{N,q} \overline{\hat{\alpha}}_s - \overline{\hat{G}}_{N,q}\right) \right] \tag{4.13}$$

where

$$s \equiv q(\bmod M)$$

Plancherel's theorem applied to the regularizing functional in equation (4.12) gives

$$\begin{aligned} ||F_M^{(p)}||^2 &= \sum_{q=-\infty}^{\infty} w_q^{2p} \left|\hat{F}_{M,q}\right|^2 = 2\sum_{q=1}^{\infty} w_q^{2j} \left|\hat{F}_{M,q}\right|^2 \\ &= 2M \sum_{q=1}^{\infty} w_q^{2p} \hat{B}_{0q}^2 \left|\hat{\alpha}_2\right|^2 \text{ where } s \equiv q(\bmod M) \end{aligned} \tag{4.14}$$

The simplification of expression (4.14) requires the use of an attenuation factor τ_q. For cubic cardinal splines ($n = 4$) it is shown by Stoer [27] and Gautschi [14] that

$$\tau_q = \left[\frac{\sin \frac{\pi q}{M}}{\frac{\pi q}{M}}\right]^4 \left(\frac{3}{1 + 2\cos^2\left(\frac{\pi q}{M}\right)}\right). \tag{4.15}$$

In expression (4.14) we wish to arrange the summation over q to summation over s, where $s \equiv q(\bmod M)$. Define the $(N \times M)$ matrix,

$$W^{(1)} = \begin{bmatrix} \text{diag } \sqrt{M}\hat{B}_{0,s}\hat{K}_{N,s} \\ \cdots \\ \text{diag } \sqrt{M}\hat{B}_{0,M-s}\overline{\hat{K}}_{N,M-s} \end{bmatrix}; s = 0, 1, \ldots, M-1 \tag{4.16}$$

From the property $\hat{K}_{N,q} = \overline{\hat{K}}_{N,N-q}$ of discrete FTs it then follows that expression (4.13) simplifies to

$$||K_N * F_M - G_N||_2^2 = ||W^{(1)}\underline{\hat{\alpha}} - \underline{\hat{G}}_N||_2^2 \tag{4.17}$$

and (4.14) can be written as

$$\begin{aligned} ||F_M^{(p)}||^2 &= 2M \sum_{s=1}^{M-1} \left\{ |\hat{\alpha}_s|^2 \sum_{n=0}^{\infty} w_{Mn+s}^{2p} \hat{B}_{0,Mn+s} \right\} \\ &= 2M \sum_{s=1}^{M-1} \tau_s |\hat{\alpha}_s|^2 \end{aligned} \tag{4.18}$$

where

$$\tau_s = \sum_{n=0}^{\infty} w_{Mn+s}^{2p} \hat{B}_{0,Mn+s}^2 \tag{4.19}$$

$$= (2\pi)^{2p} \sum_{n=0}^{\infty} (M_{n+s})^{2p} H^2 \left[\frac{\sin \frac{\pi(M_{n+s})}{M}}{\frac{\pi(M_{n+s})}{M}} \; hat^8 \right. \tag{4.20}$$

$$\tau_s = (2\pi)^{2p} H^2 s^8 \left[\frac{\sin \frac{\pi s}{M}}{\frac{\pi s}{M}} \right]^8 \sum_{n=0}^{\infty} (M_{n+s})^{2j-8}$$

$$= (2\pi)^{2p} s^8 \hat{B}_{0,s}^2 \sum_{n=0}^{\infty} (M_{n+s})^{2j-8}$$

From the simple property of discrete Fourier transforms it then follows that $\hat{\alpha}_s = \overline{\hat{\alpha}_{M-s}}$, the equality (4.18) further simplifies to

$$||F_M^{(p)}||_2^2 = 2M \sum_{s=1}^{\frac{1}{2}M} (\tau_s - \tau_{M-s}) |\hat{\alpha}_s|^2 \tag{4.21}$$

In particular, when $p = 2$, from (4.20) it follows that

$$\tau_s = (2\pi)^4 s^4 \hat{B}_{0,s}^2 \sum_{n=0}^{\infty} \left(\frac{s}{M_{n+s}} \right)^4$$

using $\hat{B}_{0,M-s} = \left(\frac{s}{M_{n-s}} \right)^4 \hat{B}_{0,s}$

$$\tau_{M-s} = (2\pi)^4 s^4 \hat{B}_{0,s}^2 \left(\frac{s}{M_{n-s}} \right)^4$$

and

$$\tau_s + \tau_{M-s} = (2\pi)^4 s^4 \hat{B}_{0,s}^2 \sum_{n=-\infty}^{\infty} \left(\frac{s}{M_{n+s}} \right)^4$$

$$= \frac{(2\pi)^4 s^4 \hat{B}_{0,s}^2 \left[1 + 2\cos^2\left(\frac{\pi s}{M}\right) \right]}{3 \left[\frac{\sin \frac{\pi s}{M}}{\frac{\pi s}{M}} \right]^4} \tag{4.22}$$

(see Pennisi [22])

$$= \frac{16}{3} M^2 \sin^4 \left(\frac{\pi s}{M} \right) \left[1 + 2\cos^2 \left(\frac{\pi s}{M} \right) \right]$$

Defining the $M \times M$ matrix

$$W^{(2)} = \text{ diag } \left\{ [M(\tau_s + \tau_{M-s})]^{1/2} \right\} \tag{4.23}$$

it follows from (4.21) that

$$||F_M^{(p)}||_2^2 = ||W^{(2)}\underline{\hat{\alpha}}||^2 \tag{4.24}$$

Thus, from equations (4.17) and (4.24) we may express the smoothing functional (4.12) as

$$C(\underline{\alpha}, \lambda) = ||W^{(1)}\underline{\hat{\alpha}} - \underline{\hat{G}}_N||_2^2 + \lambda||W^{(2)}\underline{\hat{\alpha}}||_2^2 \tag{4.25}$$

The minimizer of (4.25) is clearly

$$\underline{\hat{\alpha}} = (W + \lambda V)^{-1} \ \ W^{(1)^H}\underline{\hat{G}}_N \tag{4.26}$$

where

$$\left.\begin{array}{c} W = W^{(1)H}W^{(1)} \\ V = W^{(2)H}W^{(2)} \end{array}\right\} \tag{4.27}$$

The matrix $W + \lambda V$ is diagonal, so from equations (4.16), (4.23), (4.26) and (4.27) it follows that

$$\begin{aligned} \hat{\alpha}_s &= \frac{1}{\sqrt{M}} \frac{\hat{B}_{0,s}\overline{\hat{K}}_{N,s}\hat{G}_{N,s} + \hat{B}_{0,M-s}\hat{K}_{N,M-s}\overline{\hat{G}}_{N,M+s}}{\left[\hat{B}_{0,s}^2\left[\left|\hat{K}_{N,s}\right|^2 + \hat{B}_{0,M-s}^2\left|\hat{K}_{N,M-s}\right|^2\right] + N^2\lambda(\tau_s + \tau_{M-s})\right]} \qquad (4.28) \\ &= \frac{1}{\sqrt{M}} \frac{\hat{B}_{0,s}\left[\overline{\hat{K}}_{N,s}\hat{G}_{N,s} + \left(\frac{s}{M-s}\right)^4\hat{K}_{N,M-s}\overline{\hat{G}}_{N,M+s}\right]}{\left[\left|\hat{K}_{N,s}\right|^2 + \left(\frac{s}{M-s}\right)^8\left|\hat{K}_{N,M-s}\right|^2\right] + N^2\lambda(\tau_s + \tau_{M-s})} \end{aligned}$$

since

$$\hat{B}_{0,M-s} = \left(\frac{s}{Mn - s}\right)^4 \hat{B}_{0,s} \tag{4.29}$$

we can easily verify that $\hat{\alpha}_s = \psi_{M-s}\overline{\hat{\alpha}_s}$ so that the inverse FFT gives $\underline{\alpha} = \psi_{M-s}\underline{\hat{\alpha}}$ is a real vector as required.

4.3 The Filter for Cardinal B-Spline Regularization

The Fourier coefficiens of the regularized (filtered) solution $F_M \in B_M(0,T)$, a basis function of cardinal B-splines, clearly depends on λ through equations (4.8), (4.10) and (4.28). In equatioin (4.28), we denote the dependence of $\hat{\alpha}_s$ and λ by writing $\hat{\alpha}_s = \hat{\alpha}_s(\lambda)$. Thus the Fourier coefficients of the filtered solution are

$$\hat{F}_{M,q}(\lambda) = \sqrt{M}\hat{B}_{0,q}\hat{\alpha}_s(\lambda), \qquad s \equiv q(\mod M)$$

where as those of the unregularized (unfiltered) solutions are

$$\hat{F}_{M,q}(0) = \sqrt{M}\hat{B}_{0,q}\hat{\alpha}_s(0).$$

Clearly the underlying filter $Z_{q;\lambda}$ must satisfy

$$\hat{F}_{M,q}(\lambda) = Z_{q;\lambda}\hat{F}_{M,q}(0)$$

so that we can deduce

$$Z_{q;\lambda} = \frac{\hat{\alpha}_s(\lambda)}{\hat{\alpha}_s(0)} \tag{4.30}$$

$$= \frac{\hat{B}^2_{0,s}[|\hat{K}_{N,s}|^2 + (\frac{s}{M_{n-s}})^8|\hat{K}_{N,M-s}|^2]}{\hat{B}^2_{0,s}[|\hat{K}_{N,s}|^2 + (\frac{s}{M_{n-s}})^8|\hat{K}_{N,M-s}|^2] + N^2\lambda(\tau_s + \tau_{M-s})} \tag{4.31}$$

Obviously, the filter will be applied to every Fourier coefficient $q = 0, \pm 1, \pm 2, \ldots$. It assumes only M possible values depending on q modulo M. The regularization parameter λ is still to be determined.

4.4 Determination of Regularization Parameter λ

Let the filtered solution $F_M(x) \in B_M(0,T)$, which minimizes the form $||K_N * F_M - G_N||_2^2 + \lambda||F''_M||_2^2$, be given by (we have $p = 2$)

$$F_M(x) = \sum_{q=-\infty}^{\infty} \hat{F}_{M,q} \exp\,(iw_q x) \tag{4.32}$$

Consider

$$\hat{G}_{N,\lambda,q} = \hat{K}_{N,q}\hat{F}_{M,q}, \quad q = 0, 1, \ldots, N-1 \tag{4.33}$$

$$= \begin{cases} \sqrt{M}\hat{B}_{0,q}\hat{K}_{N,q}\hat{\alpha}_s & , \quad s \equiv q(\mod M), \quad q = 0, 1, \ldots, N-1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

We now introduce the $N \times N$ influence matrix

$$A(\lambda) = \psi_N \hat{A}(\lambda) \psi_N^H$$

where

$$\underline{\hat{G}}_{N,\lambda} = \hat{A}(\lambda) \underline{\hat{G}}_N \tag{4.34}$$

$\hat{A}(\lambda)$ is a block diagonal matrix with the following structure

$$\hat{A}(\lambda) = \left[\begin{array}{c|c} \text{diag } \underline{\alpha}_1 & \text{diag } \underline{\alpha}_2 \\ \hline \text{diag } \underline{\alpha}_3 & \text{diag } \underline{\alpha}_4 \end{array} \right] \tag{4.35}$$

where $[\alpha_{k,s}] \in \mathcal{C}^M, \;\; K = 1, 2, 3, 4$ and

$$\alpha_{1,s} \;\; = \;\; \begin{cases} \frac{\sqrt{M}(\hat{B}_{0,s})^2 |\hat{K}_s|^2}{D_s}, & s = 0 \\ \frac{\sqrt{M}(\hat{B}_{0,s})^2 |\hat{K}_s|^2}{2D_s}, & 1 \leq s \leq M - 1 \end{cases}$$

$$\alpha_{2,s} \;\; = \;\; \begin{cases} \frac{\sqrt{M}(\hat{B}_{0,s})^2 (\frac{s}{M-s})^4 \hat{K}_s \overline{\hat{K}}_{M+s}}{2D_s}, & 1 \leq s \leq M - s \\ 0, & 1 \leq s \leq M - s \end{cases}$$

$$\alpha_{3,s} \;\; = \;\; \begin{cases} \frac{\sqrt{M}\hat{K}_{M+s}\hat{B}_{0,M+s}\hat{B}_{0,s}\overline{\hat{K}}_s}{D_s}, & s = 0 \\ \frac{\sqrt{M}\hat{K}_{M+s}\hat{B}_{0,M+s}\hat{B}_{0,s}\overline{\hat{K}}_s}{2d_s} & 1 \leq s \leq M - s \end{cases}$$

$$\alpha_{4,s} \;\; = \;\; \begin{cases} 0, & s = 0 \\ \frac{\sqrt{M}\hat{B}_{0,M+s}\hat{B}_{0,s}\left(\frac{s}{M+1}\right)^4 |\hat{K}_{M+s}|^2}{2D_s}, & 1 \leq s \leq M - s \end{cases}$$

where

$$D_s = M\hat{B}_{0,s}^2 \left[|\hat{K}_s|^2 + \left(\frac{s}{M - s} \right)^8 |\hat{K}_{M-s}|^2 \right] + \lambda N^2 \left(\tau_s + \tau_{M-s} \right).$$

For simplicity of notation we have written $\hat{K}_s$ for $\hat{K}_{N,s}$ in $a_{1,s}, a_{2,s}, a_{3,s}, a_{4,s},$ and D_s. The optimal λ as defined by GCV method may be found in Wahba [32, 33]. Now we minimize the expression

$$V(\lambda) = \frac{\frac{1}{N} ||(I - \hat{A}(\lambda))\underline{\hat{G}}_N||^2}{[\frac{1}{N} \text{ Trace } (I - \hat{A}))]^2} \tag{4.36}$$

By equation (2.38) we can simplify it as

$$V(\lambda) = \frac{\frac{1}{N}\sum_{s=0}^{M-1}(1-a_{1,s})\hat{G}_s - a_{2,s}\overline{\hat{G}}_{M-s}|^2 + \sum_{s=0}^{M-1}|(1-a_{4,s})\overline{\hat{G}}_{M-s} - a_{3,s}\hat{G}_s|^2\}}{[1-\frac{1}{N}\sum_{s=0}^{M-s}(a_{1,s}+a_{4,s})]^2} \tag{4.37}$$

In order to minimize $V)\lambda)$ in equation (4.37) we have used a subroutine which uses a quadratic interpolation technique to obtain a minimum.

5 The Convergence of the Method

In this section we give the proof of convergence of the method. Let X be a separable Hilbert space, K be a compact linear operator from X to X, the subspaces $\{X_n\}$ are ultimately dense in X, i.e. $\bigcup_{n=1}^{\infty} X_n = X$.

Step I. By use of Tikhonov regularization, the problem of finding the best approximate solution (an element of minimal norm that minimizes the residual $||KF-G||$ of the equation is converted into solving the following equation

$$KF = G \tag{5.1}$$

is converted into solving the following equation

$$(\lambda I + K^*K)F = K^*G \tag{5.2}$$

where λ is the regularization parameter.

In order to solve equation (5.2) we use Galerkin method. For the sake of simplicity of the proof we consider $X_n = Y_n$ (the case of $X_n \neq Y_n$ is similar). Let U_n be an orthonormal projection from X to X_n, the equation (5.2) is equivalent to finding $F_{n,\lambda} \in X_n$ such that

$$\lambda F_{n,\lambda} + U_nK^*KF_{n,\lambda} = U_nK^*G \tag{5.3}$$

Step II. Using (R.H.Li [25]), suppose X be a separable Hilbert space, K be a compact linear operator from X to X, the subspaces $\{X_n\}$ are ultimately dense in X, F_λ be the solution of (5.2), then for any given constant $\lambda > 0$, the Galerkin equation (5.3) has a unique solution $F_{n,\lambda} \in X_n$ satisfying

$$\lim_{n\to\infty} F_{n,\lambda} = F_\lambda \tag{5.4}$$

Step III. Suppose that the integral operator K is defined by equation (5.1), the Kernel $K(x-y) \in L^2$, choose $X_n = s^K_{\mu^n}(K \geq 1,\ \ \mu > 1)$.

Let U_n denote an orthonormal projection from X to X_n. $F_\lambda(y) \in L^2[0,T]$, be the unique solution of the equation (5.2), then for any given $G \in L^2$, there exists an N_0, such that $n > N_0$, the equation (5.3) has a unique solution $F_{n,\lambda} \in X_n$ satisfying

$$\lim_{n\to\infty} F_{n,\lambda} = F_\lambda$$

particularly we have

$$||F_\lambda - F_{n,\lambda}|| \leq 2\lambda||\lambda I + \overline{K})^{-1}||\ \inf\ \{||F_\lambda - F_n||, U_n \in X_n\} \tag{5.5}$$

where $\overline{K} = K^*K,\ \ \lambda > 0$ is a regularization parameter

Step IV Since K is a compact operator, $\overline{K} = K^*K$ is also compact and has non-negative eigenvalues.

Therefore for each parameter $\lambda > 0, (\lambda I + K)^{-1}$ exists and is bounded. Let U_n be an orthogonal projection from X to X_n, from (5.2), we have

$$\lambda U_n F_\lambda + U_n K F_\lambda = U_n \overline{G}$$

where $\overline{G} = K^*G$, comparing with (5.3) we have

$$\lambda(U_n F_\lambda - F_{n,\lambda}) + U_n \overline{K}\,(F_\lambda - F_{n,\lambda}) = 0$$

$$(\lambda I + U_n \overline{K})(F_\lambda - F_{n,\lambda}) = \lambda(F_\lambda - U_n F_\lambda). \tag{5.6}$$

on the other hand

$$\begin{aligned} \lambda I + U_n K &= \lambda I + \overline{K} + U_n \overline{K} - \overline{K} \quad (5.7) \\ &= (\lambda I + \overline{K})[I + (\lambda I + \overline{K})^{-1}(U_n \overline{K} - \overline{K})] \end{aligned}$$

From the construction of the space X_n, we know the subspaces $\{X_n\}$ are ultimately dense in X, then for $n \to \infty$ $||U_n\overline{K} - \overline{K}|| \to 0$ and then $\exists$ an N, such that for $n > N$, we have

$$||[I + (\lambda I + \overline{K})^{-1}(U_n\overline{K} - \overline{K})^{-1}]|| \leq \frac{1}{1 - \frac{1}{2}} = 2 \tag{5.8}$$

Hence, $(\lambda I + U_n\overline{K})^{-1} = [I + (\lambda I + \overline{K})^{-1}(U_n\overline{K} - \overline{K})^{-1}](\lambda I + \overline{K})^{-1}$ exists and is bounded. Therefore from (5.6) $\sim$ (5.8) we have

$$\begin{aligned} ||F_\lambda - F_{n,\lambda}|| &\leq ||(\lambda I + U_n\overline{K})^{-1}||(\lambda||F_\lambda - U_nF_\lambda|| \\ &\leq 2\lambda||(\lambda I + \overline{K})^{-1} \text{ inf } \{F_\lambda - F_n||,\, F_n \in X_n\}\,. \end{aligned}$$

6 Numerical Examples

In this section we calculate the results of the above method applied to test problems over the interval $[0, T]$. All data functions have property $\Phi(z) = 0(z^{-1})$ and so we deal with severely ill-posed problems. Therefore, no noise is added apart from machine rounding error. In all cases we have taken $N = 256$ data points.

Example 1. (Theocaris [29], case 5, page 79)

$$\begin{aligned} f(t) &= e^{-at}, \;\; a = 1.0 \\ \Phi(z) &= \frac{1}{z+a} \\ g(s) &= a^{-s}\Gamma(s), \quad \text{Re } (s) > 0 \end{aligned}$$

The numerical calculations are given in the table.

Example 2. (Theocaris [29], case 4, page 79).

$$\begin{aligned} f(t) &= e^{-at}\sin bt, \;\; a = 5.0, \;\; b = 2.2 \\ \Phi(z) &= \frac{u \sin v}{z^2 + 2uz\cos v + u^2}, \;\; u = \left(a^2+b^2\right)^{-\frac{1}{2}}, \;\; v = \tan^{-1}\left(\frac{b}{a}\right) \\ g(s) &= \left(a^2+b^2\right)^{-\frac{1}{2}}\Gamma(s)\sin\left(s\tan^{-1}\left(\frac{b}{a}\right)\right), \quad \text{Re } (s) > 0 \end{aligned}$$

The numerical calculations are given in the Table.

Example 3. (Brianzi [4], McWhirter [2]),

$$\begin{aligned} f(t) &= t^a e^{-bt}, \;\; a = 1.0, \;\; b = 1.0 \\ \Phi(z) &= \frac{\Gamma(a)}{(z+b)^{a+1}} \\ g(s) &= (b)^{-(s+a)}\;\Gamma(s+a), \quad \text{Re } (s) > 0 \end{aligned}$$

The numerical calculations are given in the Table.

Example 4. (Varah [31])

$$\begin{aligned} f(t) &= t^a\;e^{-bt}, \;\; a = 3.0, \;\; b = 1.0 \\ \Phi(z) &= \frac{\Gamma(a+1)}{(z+1)^{a+1}} \\ g(s) &= (b)^{-(s+a)}\;\Gamma(s+a), \;\; \text{Re } (s) > 0. \end{aligned}$$

The numerical calculations are given in the Table.

Conclusion.

Our method worked well over all the four test examples, the results obtained are shown in the table.

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Table

Problem	T The period	H The step size	a	λ The regularization parameter	$V(\lambda)$	$\|\|F - F_\lambda\|\|_\infty$
1	9.0	0.03516	10.0	0.23×10^{-10}	0.188×10^{1}	0.01
2	12.1	0.04727	5.0	0.11×10^{-12}	0.1038×10^{2}	0.001
3	12.50	0.04883	10.0	0.21×10^{-11}	0.839×10^{2}	0.006
4	14.50	0.2265	5.0	0.34×10^{-9}	0.395×10^{2}	0.002

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Splines Regularization Solution

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Fig. 1

Fig. 2

Fig. 3

Fig. 4